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Finite Elements for Large Strains – A double mixed (M^2) Formulation

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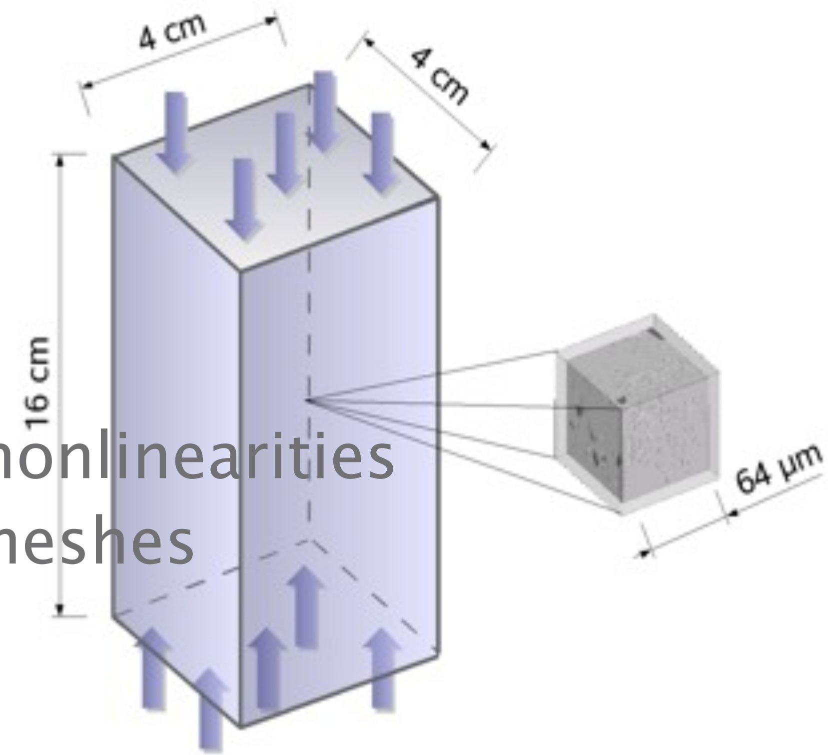
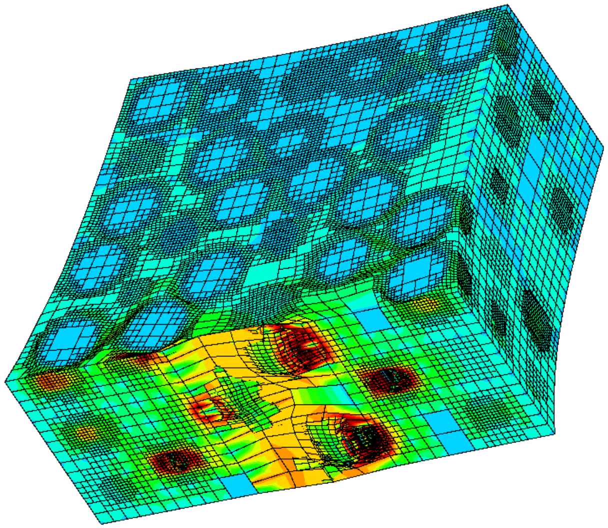
Motivation

Development of user friendly elements

- robustness
- simple treatment of
  - incompressible materials
  - complex geometries
  - geometrical and material nonlinearities
  - coarse and non-uniform meshes
- efficiency
- simple implementation

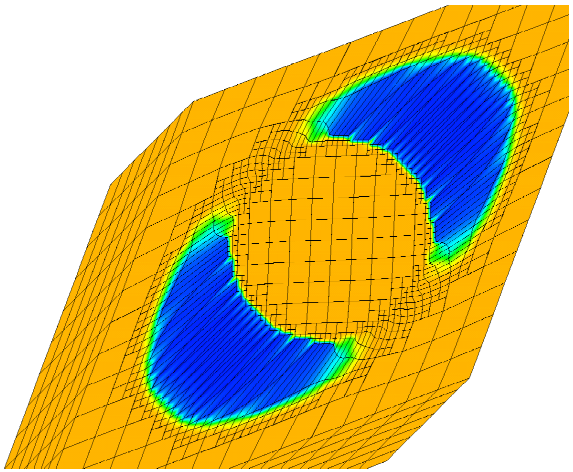


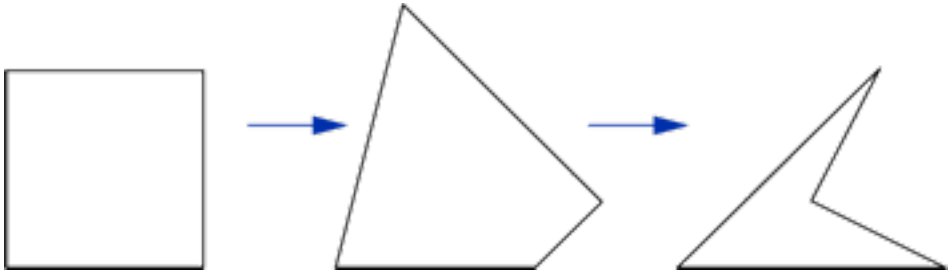
Numerical simulation of incompressible materials is still challenging within the context of large deformations



Challenges:

- Locking in incompressibility
- Large deformations
- Non-convex element shapes
- Robustness





Deformation Measures

$$\boldsymbol{F}(\boldsymbol{X}) = \operatorname{Grad}\boldsymbol{\varphi}(\boldsymbol{X})$$

$$(\boldsymbol{F}, \operatorname{cof}\boldsymbol{F}, \det\boldsymbol{F})$$

$$d\boldsymbol{x} = \boldsymbol{F}\, d\boldsymbol{X}, \quad \boldsymbol{n}\, da = \operatorname{cof}[\boldsymbol{F}]\, \boldsymbol{N}\, dA \quad \text{and} \quad dv = \det[\boldsymbol{F}]\, dV$$

$$\operatorname{cof}[\boldsymbol{F}] = \begin{bmatrix} F_{22}F_{33} - F_{32}F_{23} & F_{31}F_{23} - F_{21}F_{33} & F_{21}F_{32} - F_{31}F_{22} \\ F_{32}F_{13} - F_{12}F_{33} & F_{11}F_{33} - F_{31}F_{13} & F_{31}F_{12} - F_{11}F_{32} \\ F_{12}F_{23} - F_{22}F_{13} & F_{21}F_{13} - F_{11}F_{23} & F_{11}F_{22} - F_{21}F_{12} \end{bmatrix}$$

$$\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$$

Polyconvex Material

[Definition of Polyconvexity] $\boldsymbol{F} \mapsto W(\boldsymbol{F})$ is polyconvex if and only if there exists a function $P : \mathbb{R}^{3\times3} \times \mathbb{R}^{3\times3} \times \mathbb{R} \mapsto \mathbb{R}$ (in general non-unique) such that

$$W(\boldsymbol{F}) = P(\boldsymbol{F}, \mathrm{cof}\boldsymbol{F}, \det\boldsymbol{F}) \tag{10}$$

and $(\boldsymbol{F}, \mathrm{cof}\boldsymbol{F}, \det\boldsymbol{F}) \in \mathbb{R}^{19} \mapsto P(\boldsymbol{F}, \mathrm{cof}\boldsymbol{F}, \det\boldsymbol{F}) \in \mathbb{R}$ is convex for all points $\boldsymbol{X} \in \mathbb{R}^3$. $\square$

Potential

$$\Pi(\boldsymbol{x}) = \int_{\mathcal{B}} W(\boldsymbol{F}(\boldsymbol{x}))\, dV + \Pi^{ext}(\boldsymbol{x})$$

Hu–Washizu Functional

$$\Pi(\boldsymbol{x}, \boldsymbol{H}^c, \theta\,, p, \boldsymbol{B}) = \int_{\mathcal{B}} \mathcal{P}(\boldsymbol{F}, \boldsymbol{H}^c, \theta)\, dV + \int_{\mathcal{B}} p(J - \theta)\, dV + \int_{\mathcal{B}} \boldsymbol{B}(\mathrm{cof}\boldsymbol{F} - \boldsymbol{H}^c)\, dV$$

Hu–Washizu Functional

$$\Pi(\boldsymbol{x}, \boldsymbol{H}^c, \theta\,, p, \boldsymbol{B}) = \int_{\mathcal{B}} \mathcal{P}(\boldsymbol{C}, \boldsymbol{H}^c, \theta)\, dV + \int_{\mathcal{B}} p(J - \theta)\, dV + \int_{\mathcal{B}} \boldsymbol{B} \cdot (\mathrm{cof}\boldsymbol{C} - \boldsymbol{H}^c)\, dV$$

Hu-Washizu Functional

$$\Pi(\boldsymbol{x}, \boldsymbol{H}^c, \theta, p, \boldsymbol{B}) = \int_{\mathcal{B}} \mathcal{P}(\boldsymbol{C}, \boldsymbol{H}^c, \theta)\, dV + \int_{\mathcal{B}} p(J-\theta)\, dV + \int_{\mathcal{B}} \boldsymbol{B} \cdot (\mathrm{cof}\boldsymbol{C} - \boldsymbol{H}^c)\, dV$$

Variation

$$\delta_C \Pi = \int_{\mathcal{B}} \frac{1}{2} \delta \boldsymbol{C} \cdot \underbrace{2(\partial_C \mathcal{P} + \frac{p}{2J} \mathrm{cof}\boldsymbol{C} + \boldsymbol{B} : \partial_C \mathrm{cof}\boldsymbol{C})}_{\boldsymbol{S}}\, dV\,,$$

$$\delta_\theta \Pi = \int_{\mathcal{B}} \delta\theta(\partial_\theta \mathcal{P} - p)\, dV \quad \rightarrow \quad p = \partial_\theta \mathcal{P}\,,$$

$$\delta_p \Pi = \int_{\mathcal{B}} \delta p (J - \theta)\, dV \quad \rightarrow \quad J = \theta\,,$$

$$\delta_H \Pi = \int_{\mathcal{B}} \delta \boldsymbol{H}^c \cdot (\partial_H \mathcal{P} - \boldsymbol{B})\, dV \quad \rightarrow \quad \boldsymbol{B} = \partial_H \mathcal{P}\,,$$

$$\delta_B \Pi = \int_{\mathcal{B}} \delta \boldsymbol{B} \cdot (\mathrm{cof}\boldsymbol{C} - \boldsymbol{H}^c)\, dV \quad \rightarrow \quad \boldsymbol{H}^c = \mathrm{cof}\boldsymbol{C}$$

Hu-Washizu Functional

Choice of Strain Energy Function

$$\mathcal{P}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = \mathcal{P}_1(\boldsymbol{C}) + \mathcal{P}_2(\boldsymbol{H}^c) + \mathcal{P}_3(\theta)$$

Hu–Washizu Functional

Choice of Strain Energy Function

$$\mathcal{P}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = \mathcal{P}_1(\boldsymbol{C}) + \mathcal{P}_2(\boldsymbol{H}^c) + \mathcal{P}_3(\theta)$$

Linearization, Definitions

$$\mathbb{C}_1 = \partial^2_{CC}\mathcal{P}, \quad \mathbb{C}_2 = \partial^2_{H^cH^c}\mathcal{P}, \quad \mathbb{C}_3 = \partial^2_{\theta\theta}\mathcal{P}$$

$$\mathbb{P}^c := \partial_C \mathrm{cof}\boldsymbol{C} = (\mathrm{cof}\boldsymbol{C} \otimes \mathrm{cof}\boldsymbol{C} - \mathrm{cof}\boldsymbol{C} \boxtimes \mathrm{cof}\boldsymbol{C})/\det \boldsymbol{C}, \quad \Xi = \partial^2_{CC}\mathrm{cof}\boldsymbol{C}$$

$$(\boldsymbol{A} \boxtimes \boldsymbol{B}) : \boldsymbol{a} \otimes \boldsymbol{b} = \boldsymbol{A}\boldsymbol{a} \otimes \boldsymbol{B}\boldsymbol{b}$$

$$\overline{\mathbb{C}} = \frac{p}{2J}\mathbb{P}^c - \frac{p}{4J^3}\mathrm{cof}\boldsymbol{C} \otimes \mathrm{cof}\boldsymbol{C} + \boldsymbol{B} : \Xi$$

Linearization of Hu-Washizu Functional

$$
\begin{aligned}
\Delta\delta_C\Pi &= \int_{\mathcal{B}} \frac{1}{2}\Delta\delta\boldsymbol{C}\cdot\boldsymbol{S}\,dV + \int_{\mathcal{B}} \tfrac{1}{2}\delta\boldsymbol{C}\cdot[4\,(\mathbb{C}_1+\overline{\mathbb{C}}):\tfrac{1}{2}\Delta\boldsymbol{C}]\,dV \\
&+ \int_{\mathcal{B}} \tfrac{1}{2}\delta\boldsymbol{C}\cdot(\frac{1}{J}\mathrm{cof}\boldsymbol{C}\;\Delta p)\,dV + \int_{\mathcal{B}} \tfrac{1}{2}\delta\boldsymbol{C}\cdot 2\,(\mathbb{P}^c:\Delta\boldsymbol{B})\,dV \\
\Delta\delta_\theta\Pi &= \int_{\mathcal{B}} \delta\theta\;\mathbb{C}_3\;\Delta\theta\,dV - \int_{\mathcal{B}} \delta\theta\Delta p\,dV\,, \\
\Delta\delta_p\Pi &= \int_{\mathcal{B}} \delta p\,\frac{1}{J}\mathrm{cof}\boldsymbol{C}\cdot\tfrac{1}{2}\Delta\boldsymbol{C}\,dV - \int_{\mathcal{B}} \delta p\;\Delta\theta\,dV\,, \\
\Delta\delta_H\Pi &= \int_{\mathcal{B}} \delta\boldsymbol{H}^c\cdot(\mathbb{C}_2:\Delta\boldsymbol{H}^c)\,dV - \int_{\mathcal{B}} \delta\boldsymbol{H}^c\cdot\Delta\boldsymbol{B}\,dV \\
\Delta\delta_B\Pi &= \int_{\mathcal{B}} \delta\boldsymbol{B}\cdot(2\,\mathbb{P}^c:\tfrac{1}{2}\Delta\boldsymbol{C})\,dV - \int_{\mathcal{B}} \delta\boldsymbol{B}\cdot\Delta\boldsymbol{H}^c\,dV
\end{aligned}
$$

Finite Element Approximation

- a quadratic interpolation for $\boldsymbol{x}$, $\quad \boldsymbol{u} = \boldsymbol{N}\,\boldsymbol{d}_u$
- a linear interpolation for $\boldsymbol{H}^c$ and $\boldsymbol{B}$ $\quad \boldsymbol{H}^c = \boldsymbol{N}_c\,\boldsymbol{d}_c, \quad \boldsymbol{B} = \boldsymbol{N}_b\,\boldsymbol{d}_b$
- a constant interpolation for θ and p. $\quad \theta = \boldsymbol{N}_\theta\,\boldsymbol{d}_\theta \quad p = \boldsymbol{N}_p\,\boldsymbol{d}_p$

$$
\begin{aligned}
\Delta\delta_C\Pi \;=\;& \delta\boldsymbol{d}_u^T \left[\boldsymbol{k}^{geo} + \int_{\mathcal{B}} \mathbb{B}^T 4\,(\mathbb{C}_1 + \overline{\mathbb{C}})\,\mathbb{B}\,dV\right]\Delta\boldsymbol{d}_u \\
&+ \delta\boldsymbol{d}_u^T \left\{\int_{\mathcal{B}} \mathbb{B}^T \frac{1}{J}\mathrm{cof}\boldsymbol{C}\,\boldsymbol{N}_p\,dV\;\Delta\boldsymbol{d}_p + \int_{\mathcal{B}} \mathbb{B}^T 2\,\mathbb{P}^c\boldsymbol{N}_b\,dV\;\Delta\boldsymbol{d}_b\right\} \\
=\;& \delta\boldsymbol{d}_u^T \left\{\boldsymbol{k}_{uu}\,\Delta\boldsymbol{d}_u + \boldsymbol{k}_{up}\,\Delta\boldsymbol{d}_p + \boldsymbol{k}_{ub}\,\Delta\boldsymbol{d}_b\right\},
\end{aligned}
$$

$$\tfrac{1}{2}\delta\boldsymbol{C} = \mathbb{B}\,\delta\boldsymbol{d}_u \quad \text{and} \quad \tfrac{1}{2}\Delta\boldsymbol{C} = \mathbb{B}\,\Delta\boldsymbol{d}_u$$

Finite Element Approximation

$$
\begin{aligned}
\Delta\delta_\theta\Pi &= \delta\boldsymbol{d}_\theta^T \int_{\mathcal{B}} \boldsymbol{N}_\theta^T\, \mathbb{C}_3\, \boldsymbol{N}_\theta\, dV\ \Delta\boldsymbol{d}_\theta - \delta\boldsymbol{d}_\theta^T \int_{\mathcal{B}} \boldsymbol{N}_\theta^T \boldsymbol{N}_p\, dV\ \Delta\boldsymbol{d}_p \\
&= \delta\boldsymbol{d}_\theta^T\, \boldsymbol{k}_{\theta\theta}\, \Delta\boldsymbol{d}_\theta + \delta\boldsymbol{d}_\theta^T\, \boldsymbol{k}_{\theta p}\, \Delta\boldsymbol{d}_p\,, \\
\Delta\delta_p\Pi &= \delta\boldsymbol{d}_p^T \int_{\mathcal{B}} \boldsymbol{N}_p^T\, \frac{1}{J}\mathrm{cof}\boldsymbol{C}\, \mathbb{B}\, dV\ \Delta\boldsymbol{d}_u - \delta\boldsymbol{d}_p^T \int_{\mathcal{B}} \boldsymbol{N}_p^T\, \boldsymbol{N}_\theta\, dV\ \Delta\boldsymbol{d}_\theta \\
&= \delta\boldsymbol{d}_p^T\, \boldsymbol{k}_{pu}\, \Delta\boldsymbol{d}_u + \delta\boldsymbol{d}_p^T\, \boldsymbol{k}_{p\theta}\, \Delta\boldsymbol{d}_\theta\,, \\
\Delta\delta_H\Pi &= \delta\boldsymbol{d}_c^T \int_{\mathcal{B}} \boldsymbol{N}_c^T\, \mathbb{C}_2\, \boldsymbol{N}_c\, dV\ \Delta\boldsymbol{d}_c - \delta\boldsymbol{d}_c^T \int_{\mathcal{B}} \boldsymbol{N}_c^T\, \boldsymbol{N}_b\, dV\ \Delta\boldsymbol{d}_b \\
&= \delta\boldsymbol{d}_c^T\, \boldsymbol{k}_{cc}\, \Delta\boldsymbol{d}_c + \delta\boldsymbol{d}_c^T\, \boldsymbol{k}_{cb}\, \Delta\boldsymbol{d}_b\,, \\
\Delta\delta_B\Pi &= \delta\boldsymbol{d}_b^T \int_{\mathcal{B}} \boldsymbol{N}_b^T\, 2\,\mathbb{P}^c\, \mathbb{B}\, dV\ \Delta\boldsymbol{d}_u - \delta\boldsymbol{d}_b^T \int_{\mathcal{B}} \boldsymbol{N}_b^T\, \boldsymbol{N}_c\, dV\ \Delta\boldsymbol{d}_c \\
&= \delta\boldsymbol{d}_b^T\, \boldsymbol{k}_{bu}\, \Delta\boldsymbol{d}_u + \delta\boldsymbol{d}_b^T\, \boldsymbol{k}_{bc}\, \Delta\boldsymbol{d}_c
\end{aligned}
$$

Finite Element Approximation

$$
\begin{bmatrix} \Delta\delta_C\Pi \\ \Delta\delta_\theta\Pi \\ \Delta\delta_p\Pi \\ \Delta\delta_H\Pi \\ \Delta\delta_B\Pi \end{bmatrix} = \begin{bmatrix} \delta\boldsymbol{d}_u \\ \delta\boldsymbol{d}_\theta \\ \delta\boldsymbol{d}_p \\ \delta\boldsymbol{d}_c \\ \delta\boldsymbol{d}_b \end{bmatrix}^T \left[\begin{array}{c|c|c|c|c} \boldsymbol{k}_{uu} & 0 & \boldsymbol{k}_{up} & 0 & \boldsymbol{k}_{ub} \\ \hline 0 & \boldsymbol{k}_{\theta\theta} & \boldsymbol{k}_{\theta p} & 0 & 0 \\ \hline \boldsymbol{k}_{pu} & \boldsymbol{k}_{p\theta} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \boldsymbol{k}_{cc} & \boldsymbol{k}_{cb} \\ \hline \boldsymbol{k}_{bu} & 0 & 0 & \boldsymbol{k}_{bc} & 0 \end{array}\right] \begin{bmatrix} \Delta\boldsymbol{d}_u \\ \Delta\boldsymbol{d}_\theta \\ \Delta\boldsymbol{d}_p \\ \Delta\boldsymbol{d}_c \\ \Delta\boldsymbol{d}_b \end{bmatrix}
$$

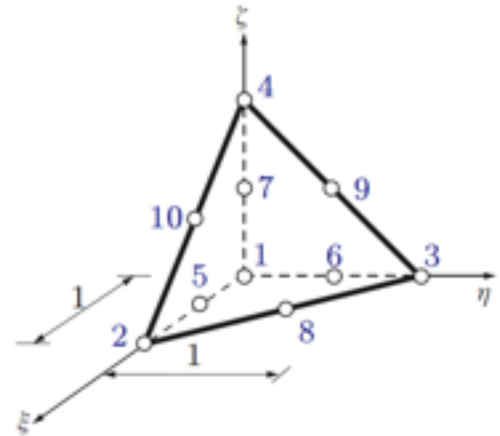
Newton Solution Procedure

$$
\begin{aligned}
\boldsymbol{k}_{uu}\Delta\boldsymbol{d}_u &+ \boldsymbol{k}_{up}\Delta\boldsymbol{d}_p + \boldsymbol{k}_{ub}\Delta\boldsymbol{d}_b &= -\boldsymbol{r}_u\\
\boldsymbol{k}_{\theta\theta}\Delta\boldsymbol{d}_\theta &+ \boldsymbol{k}_{\theta p}\Delta\boldsymbol{d}_p &= -\boldsymbol{r}_\theta\\
\boldsymbol{k}_{pu}\Delta\boldsymbol{d}_u &+ \boldsymbol{k}_{p\theta}\Delta\boldsymbol{d}_\theta &= -\boldsymbol{r}_p\\
\boldsymbol{k}_{cc}\Delta\boldsymbol{d}_c &+ \boldsymbol{k}_{cb}\Delta\boldsymbol{d}_b &= -\boldsymbol{r}_c\\
\boldsymbol{k}_{bu}\Delta\boldsymbol{d}_u &+ \boldsymbol{k}_{bc}\Delta\boldsymbol{d}_c &= -\boldsymbol{r}_b
\end{aligned}
$$

$$
\left[\boldsymbol{k}_{uu} + \boldsymbol{k}_{up}\widetilde{\boldsymbol{k}}_{pp}^{-1}\boldsymbol{k}_{pu} + \boldsymbol{k}_{ub}\widetilde{\boldsymbol{k}}_{cc}\boldsymbol{k}_{bu}\right]\Delta\boldsymbol{d}_u = -\left[\boldsymbol{r}_u + \boldsymbol{k}_{up}\widetilde{\boldsymbol{k}}_{pp}^{-1}\widetilde{\boldsymbol{r}}_p + \boldsymbol{k}_{ub}\boldsymbol{k}_{cb}^{-1}\widetilde{\boldsymbol{r}}_c\right]
$$

$$
\widetilde{\boldsymbol{k}}_{cc} = \boldsymbol{k}_{cb}^{-1}\boldsymbol{k}_{cc}\boldsymbol{k}_{bc}^{-1} \quad \text{and} \quad \widetilde{\boldsymbol{r}}_c = -\boldsymbol{r}_c + \boldsymbol{k}_{cc}\boldsymbol{k}_{bc}^{-1}\boldsymbol{r}_b\,.
$$

Mixed finite element formulation



Displacement field

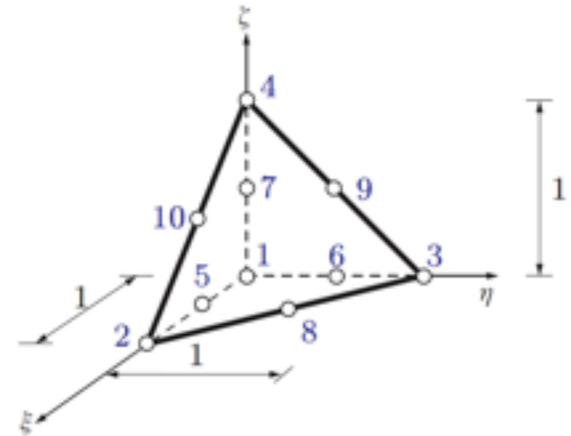
$$\boldsymbol{u} = \sum_{I=1}^{nel} N_I \, \boldsymbol{d}_u^I = \boldsymbol{N} \, \boldsymbol{d}_u$$

$$\boldsymbol{N} = \left[\begin{array}{cccc|cccc|cccc} N_1^u & N_2^u & \dots & N_{nel}^u & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & N_1^u & N_2^u & \dots & N_{nel}^u & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & N_1^u & N_2^u & \dots & N_{nel}^u \end{array} \right]_{3 \times 3 \times nel}$$

$$\boldsymbol{d}_u = \left[\begin{array}{cccc|cccc|ccc} d_{ux}^1 & d_{ux}^2 & \dots & d_{ux}^{nel} & d_{uy}^1 & d_{uy}^2 & \dots & d_{uy}^{nel} & d_{uz}^1 & \dots & d_{uz}^{nel} \end{array} \right]^T$$

$$nel = 10$$

Mixed finite element formulation



Displacement field

$$\boldsymbol{u} = \sum_{I=1}^{nel} N_I \, \boldsymbol{d}_u^I = \boldsymbol{N} \, \boldsymbol{d}_u$$

$$\boldsymbol{N} = \left[\begin{array}{cccc|cccc|cccc} N_1^u & N_2^u & \dots & N_{nel}^u & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & N_1^u & N_2^u & \dots & N_{nel}^u & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 & N_1^u & N_2^u & \dots & N_{nel}^u \end{array} \right]_{3\times 3\times nel}$$

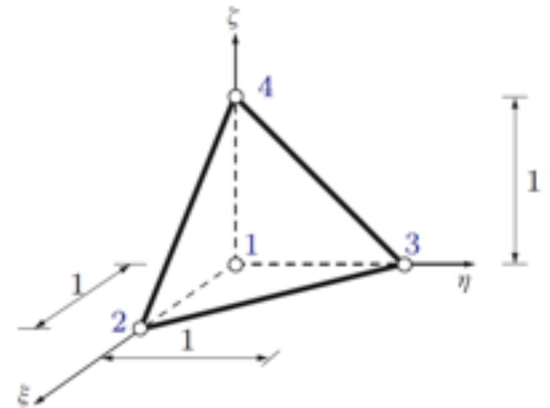
$$\boldsymbol{d}_u = \left[\begin{array}{cccc|cccc|ccc} d_{ux}^1 & d_{ux}^2 & \dots & d_{ux}^{nel} & d_{uy}^1 & d_{uy}^2 & \dots & d_{uy}^{nel} & d_{uz}^1 & \dots & d_{uz}^{nel} \end{array} \right]^T$$

$$nel = 10$$

Pressure Terms

$p_e = \bar{p}$ and $\theta_e = \bar{\theta}$ constant

Mixed finite element formulation



Approximation of co factor

$$\boldsymbol{H}^c = \boldsymbol{N}_c\, \boldsymbol{d}_c$$

$$\begin{bmatrix} H^c_{11} \\ H^c_{22} \\ H^c_{33} \\ H^c_{12} \\ H^c_{23} \\ H^c_{13} \end{bmatrix} = \left[\begin{array}{ccc|ccc|c|ccc} N^c_1 & \dots & N^c_{nelc} & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & N^c_1 & \dots & N^c_{nelc} & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \dots & N^c_1 & \dots & N^c_{nelc} \end{array}\right] \begin{bmatrix} d^1_{c11} \\ d^2_{c11} \\ d^3_{c11} \\ \dots \\ d^{(nelc-1)}_{c13} \\ d^{nelc}_{c13} \end{bmatrix}$$

$$nelc = 4.$$

Approximation of Lagrange Multiplier

$$\boldsymbol{B} = \boldsymbol{N}_b\, \boldsymbol{d}_b$$

Mixed finite element formulation

$$
\begin{aligned}
\boldsymbol{r}_u &= \int_{\mathcal{B}_e} \mathbf{B}^T \boldsymbol{S} \, dV = \mathbf{0}\,, \\
r_\theta &= \int_{\mathcal{B}_e} (\partial_\theta \mathcal{P} - \bar{p}) \, dV = 0\,, \\
r_p &= \int_{\mathcal{B}_e} (J - \bar{\theta}) \, dV = 0\,, \\
\boldsymbol{r}_c &= \int_{\mathcal{B}_e} \boldsymbol{N}_c{}^T (\partial_H \mathcal{P} - \boldsymbol{B}) \, dV = \mathbf{0}\,, \\
\boldsymbol{r}_b &= \int_{\mathcal{B}_e} \boldsymbol{N}_b{}^T (\mathrm{cof}\boldsymbol{C} - \boldsymbol{H}^c) \, dV = \mathbf{0}
\end{aligned}
$$

Mixed finite element formulation

$$
\begin{aligned}
\boldsymbol{r}_u &= \int\limits_{\mathcal{B}_e} \mathbf{B}^T \boldsymbol{S}\, dV = \mathbf{0}\,,\\
\cancel{r_\theta} &\cancel{= \int\limits_{\mathcal{B}_e} (\partial_\theta \mathcal{P} - \bar{p})\, dV = 0\,,}\\
\cancel{r_p} &\cancel{= \int\limits_{\mathcal{B}_e} (J - \bar{\theta})\, dV = 0\,,}\\
\boldsymbol{r}_c &= \int\limits_{\mathcal{B}_e} \boldsymbol{N}_c{}^T (\partial_H \mathcal{P} - \boldsymbol{B})\, dV = \mathbf{0}\,,\\
\boldsymbol{r}_b &= \int\limits_{\mathcal{B}_e} \boldsymbol{N}_b{}^T (\mathrm{cof}\boldsymbol{C} - \boldsymbol{H}^c)\, dV = \mathbf{0}
\end{aligned}
$$

$$
\bar{p} = \frac{1}{V_e} \int\limits_{\mathcal{B}_e} \partial_\theta \mathcal{P}\, dV \quad \text{and} \quad \bar{\theta} = \frac{1}{V_e} \int\limits_{\mathcal{B}_e} J\, dV = \frac{v_e}{V_e}
$$

Mixed finite element formulation

$$\delta \boldsymbol{d}_b^T \left(\int_{\mathcal{B}_e} \boldsymbol{N}_b^T \operatorname{cof} \boldsymbol{C} \, dV - \int_{\mathcal{B}_e} \boldsymbol{N}_b^T \boldsymbol{N}_c \, dV \, \boldsymbol{d}_c \right) = 0.$$

$$\boldsymbol{d}_c = (\boldsymbol{A}_c)^{-1} \boldsymbol{f}_c$$

$$\boldsymbol{f}_c := \int_{\mathcal{B}_e} \boldsymbol{N}_b^T \operatorname{cof} \boldsymbol{C} \, dV \quad \text{and} \quad \boldsymbol{A}_c := \int_{\mathcal{B}_e} \boldsymbol{N}_b^T \boldsymbol{N}_b \, dV$$

Block structure of $\boldsymbol{H}^c = \boldsymbol{N}_c \, \boldsymbol{d}_c$

$$\begin{bmatrix} \boldsymbol{M}^e & \boldsymbol{0} & \dots & \dots \\ \boldsymbol{0} & \boldsymbol{M}^e & \dots & \dots \\ \vdots & \vdots & \ddots & \vdots \\ \dots & \dots & \dots & \boldsymbol{M}^e \end{bmatrix} \left\{ \begin{array}{c} \boldsymbol{d}_{c\,11} \\ \boldsymbol{d}_{c\,22} \\ \vdots \\ \boldsymbol{d}_{c\,13} \end{array} \right\} = \left\{ \begin{array}{c} \boldsymbol{f}_{c11} \\ \boldsymbol{f}_{c22} \\ \vdots \\ \boldsymbol{f}_{c13} \end{array} \right\}$$

Mixed finite element formulations

Three-dimensional tetraeder

$$\boldsymbol{M}^e = \frac{V_e}{20}\begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix} \quad \Longrightarrow \quad (\boldsymbol{M}^e)^{-1} = \frac{4}{V_e}\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$$

Local Algorithm at element level

1) Update of basic variables: $\boldsymbol{d}_u \Leftarrow \boldsymbol{d}_u + \Delta \boldsymbol{d}_u$

2) 1st Pre-Gauss-loop:

2.1) Compute and store at each Gauss point:

$$F_{ij} = \delta_{ij} + \sum_{I=1}^{nel} N_{I,j}\, d_{u;i\,I}, \quad \mathrm{cof}\boldsymbol{C}, \quad \det\boldsymbol{F}$$

2.2) Calculate volume in the reference and actual configuration:

$$V_e = \int_{\mathcal{B}_e} dV \quad \text{and} \quad v_e = \int_{\mathcal{S}_e} dv$$

2.3) Compute local right hand side regarding $\boldsymbol{H}^c$:

$$\boldsymbol{f}_c = \int_{\mathcal{B}_e} \boldsymbol{N}_b^T \mathrm{cof}\boldsymbol{C}\, dV$$

3) Compute coefficient matrix, dofs regarding $\boldsymbol{H}_c$ and volume dilatation:

$\boldsymbol{M}^e$ following Equation (59)

$$\boldsymbol{d}_{cij} = (\boldsymbol{M}^e)^{-1} \boldsymbol{f}_{cij}$$

$$\bar{\theta} = v_e / V_e$$

Local algorithm at element level

4) 2nd Pre-Gauss-loop:

4.1) Compute and store at each Gausspoint: $\boldsymbol{H}^c = \boldsymbol{N}_c \, \boldsymbol{d}_c$

4.2) Compute local right hand side regarding $\boldsymbol{B}$:

$$\boldsymbol{f}_b \quad = \quad \int_{\mathcal{B}_e} \boldsymbol{N}_c^T (\partial_H \mathcal{P}) \, dV$$

5) Compute pressure $\bar{p} = \partial_\theta \mathcal{P}|_{\theta=\bar{\theta}}$

6) Compute $\boldsymbol{d}_{bij} = (\boldsymbol{M}^e)^{-1} \boldsymbol{f}_{bij}$

Local algorithm at element level

7) Final Gauss-loop:

7.1) Compute Lagrange multiplier $\boldsymbol{B} = \boldsymbol{N}_b \, \boldsymbol{d}_b$

7.2) Compute stress $\boldsymbol{S} = 2(\partial_C \mathcal{P} + \frac{1}{2}\,\bar{p}/J \operatorname{cof}\boldsymbol{C} + \boldsymbol{B} : \partial_C \operatorname{cof}\boldsymbol{C})$

7.3) Compute right hand side $\boldsymbol{r}_u = \int_{\mathcal{B}_e} \mathbb{B}^T \boldsymbol{S} \, dV$

7.4) Compute individual stiffness matrices:

$\boldsymbol{k}_{uu},\ \boldsymbol{k}_{\theta p},\ \boldsymbol{k}_{\theta\theta},\ \boldsymbol{k}_{bc},\ \boldsymbol{k}_{cc},\ \boldsymbol{k}_{bu},\ \boldsymbol{k}_{pu}$

8) Calculate static condensation matrices $\tilde{\boldsymbol{k}}_{pp},\ \tilde{\boldsymbol{k}}_{cc}$

9) Return overall residual vector $\boldsymbol{r} := \boldsymbol{r}_u$

and stiffness matrix $\boldsymbol{k} := \boldsymbol{k}_{uu} + \boldsymbol{k}_{up}(\widetilde{\boldsymbol{k}}_{pp})^{-1}\boldsymbol{k}_{pu} + \boldsymbol{k}_{ub}\widetilde{\boldsymbol{k}}_{cc}\boldsymbol{k}_{bu}$

Examples: isotropic strain energy function

$$\psi^{iso}(\boldsymbol{C}, \mathrm{cof}\boldsymbol{C}, \mathrm{det}\boldsymbol{C}) = \frac{\alpha}{2}\,(\mathrm{tr}\boldsymbol{C})^2 + \frac{\beta}{2}\,(\mathrm{tr}[\mathrm{cof}\boldsymbol{C}])^2 - \gamma \ln\left(\sqrt{\det \boldsymbol{C}}\right) + \epsilon_1(\mathrm{det}\boldsymbol{C}^{\epsilon_2} + \mathrm{det}\boldsymbol{C}^{-\epsilon_2} - 2)$$

Mixed form

$$\mathcal{P}^{iso}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = \frac{\alpha}{2}\,(\mathrm{tr}\boldsymbol{C})^2 + \frac{\beta}{2}\,(\mathrm{tr}\boldsymbol{H}^c)^2 - \gamma \ln(\theta) + \epsilon_1(\theta^{2\epsilon_2} + \theta^{-2\epsilon_2} - 2)$$

Examples: isotropic strain energy function

$$\psi^{iso}(\boldsymbol{C}, \mathrm{cof}\boldsymbol{C}, \det\boldsymbol{C}) = \frac{\alpha}{2}\,(\mathrm{tr}\boldsymbol{C})^2 + \frac{\beta}{2}\,(\mathrm{tr}[\mathrm{cof}\boldsymbol{C}])^2 - \gamma \ln\left(\sqrt{\det \boldsymbol{C}}\right) + \epsilon_1(\det\boldsymbol{C}^{\epsilon_2} + \det\boldsymbol{C}^{-\epsilon_2} - 2)$$

Mixed form

$$\mathcal{P}^{iso}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = \frac{\alpha}{2}\,(\mathrm{tr}\boldsymbol{C})^2 + \frac{\beta}{2}\,(\mathrm{tr}\boldsymbol{H}^c)^2 - \gamma \ln(\theta) + \epsilon_1(\theta^{2\epsilon_2} + \theta^{-2\epsilon_2} - 2)$$

Stress free inital configuration

$$\gamma = 6\,\alpha + 12\,\beta$$

Examples: isotropic strain energy function

$$\psi^{iso}(\boldsymbol{C},\mathrm{cof}\boldsymbol{C},\det\boldsymbol{C}) = \frac{\alpha}{2}(\mathrm{tr}\boldsymbol{C})^2+\frac{\beta}{2}(\mathrm{tr}[\mathrm{cof}\boldsymbol{C}])^2-\gamma\ln(\sqrt{\det\boldsymbol{C}})+\epsilon_1(\det\boldsymbol{C}^{\epsilon_2}+\det\boldsymbol{C}^{-\epsilon_2}-2)$$

Mixed form

$$\mathcal{P}^{iso}(\boldsymbol{C},\boldsymbol{H}^c,\theta) = \frac{\alpha}{2}(\mathrm{tr}\boldsymbol{C})^2+\frac{\beta}{2}(\mathrm{tr}\boldsymbol{H}^c)^2-\gamma\ln(\theta)+\epsilon_1(\theta^{2\epsilon_2}+\theta^{-2\epsilon_2}-2)$$

Stress free inital configuration

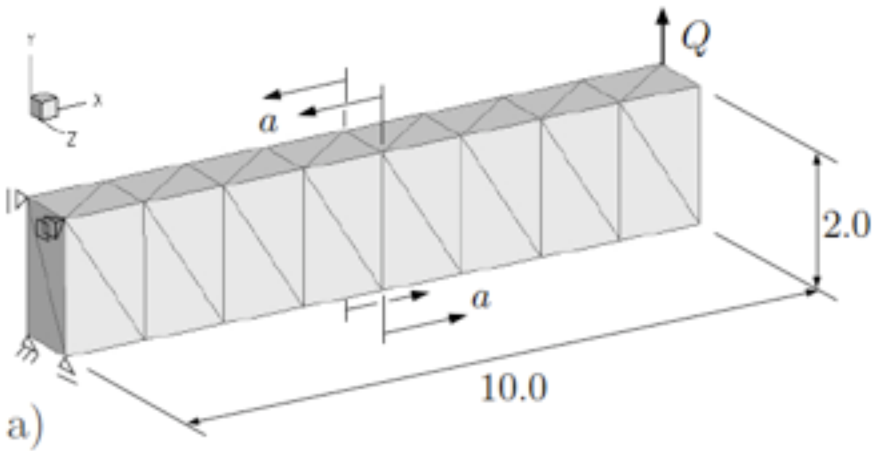
$$\gamma = 6\,\alpha + 12\,\beta$$

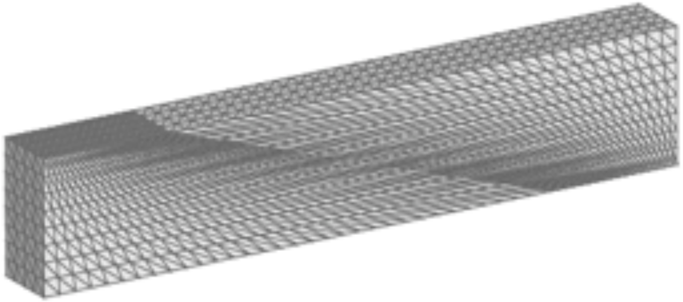
$$\partial_C\mathcal{P}^{iso} = \alpha\,(\mathrm{tr}\boldsymbol{C})\,\mathbf{1}, \quad \partial_{H^c}\mathcal{P}^{iso} = \beta\,(\mathrm{tr}\boldsymbol{H}^c)\,\mathbf{1}, \quad \partial_\theta\mathcal{P}^{iso} = -\frac{\gamma}{\theta}+2\epsilon_1\epsilon_2(\theta^{2\epsilon_2-1}-\theta^{-2\epsilon_2-1})$$

$$\partial^2_{CC}\mathcal{P}^{iso} = \alpha\,\mathbf{1}\otimes\mathbf{1}, \quad \partial^2_{H^cH^c}\mathcal{P}^{iso} = \beta\,\mathbf{1}\otimes\mathbf{1},$$

$$\partial^2_{\theta\theta}\mathcal{P}^{iso} = \frac{\gamma}{\theta^2}+2\epsilon_1\epsilon_2[2\epsilon_2(\theta^{2\epsilon_2-2}-\theta^{-2\epsilon_2-2})-\theta^{2\epsilon_2-2}+\theta^{-2\epsilon_2-2}]$$

Mesh distortion sensitivity: Cantilever beam





$$\alpha = 42.0\ \text{kPa}, \quad \beta = 84.0\ \text{kPa}, \quad \varepsilon_1 = 100\ \text{kPa} \quad \text{and} \quad \varepsilon_2 = 10$$

Mesh distortion sensitivity: Cantilever beam

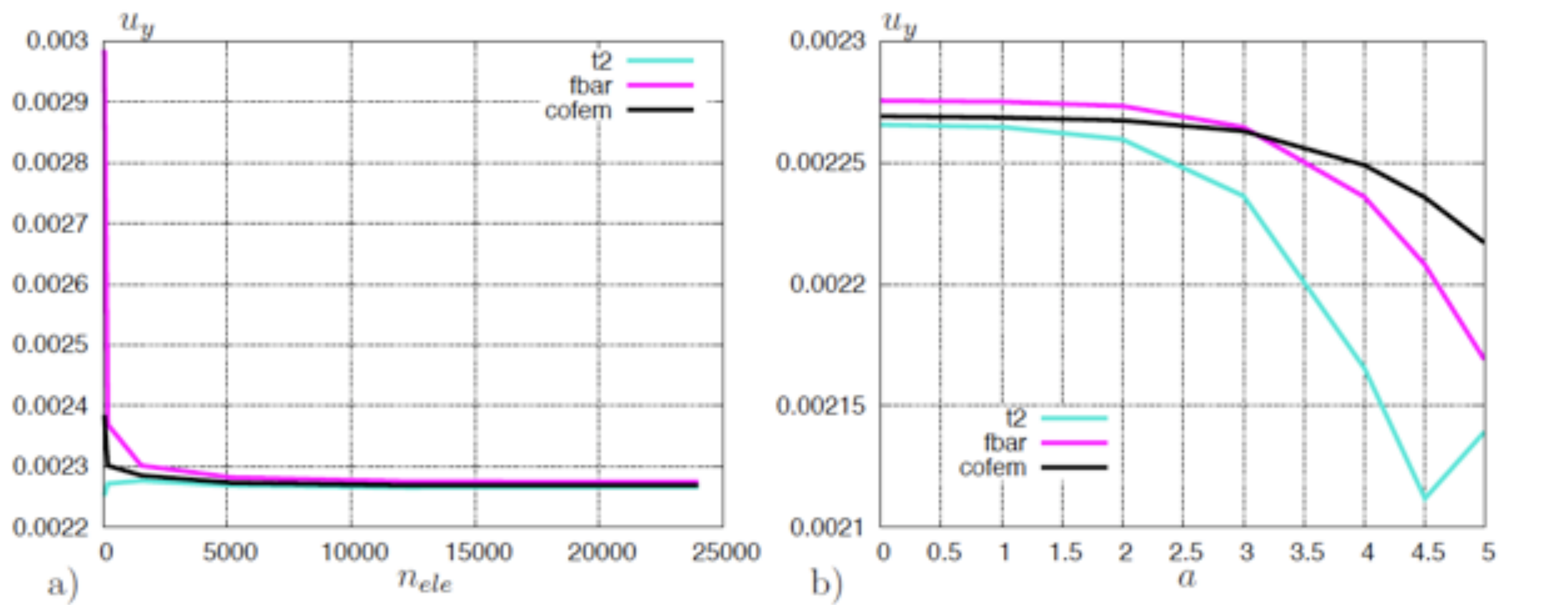


Figure 4: Vertical displacement at the loading point versus a) number of elements n_{ele} (for $a = 0$) and b) distortion a (for $n_{ele} = 12288$).

Mesh distortion sensitivity: Cantilever beam

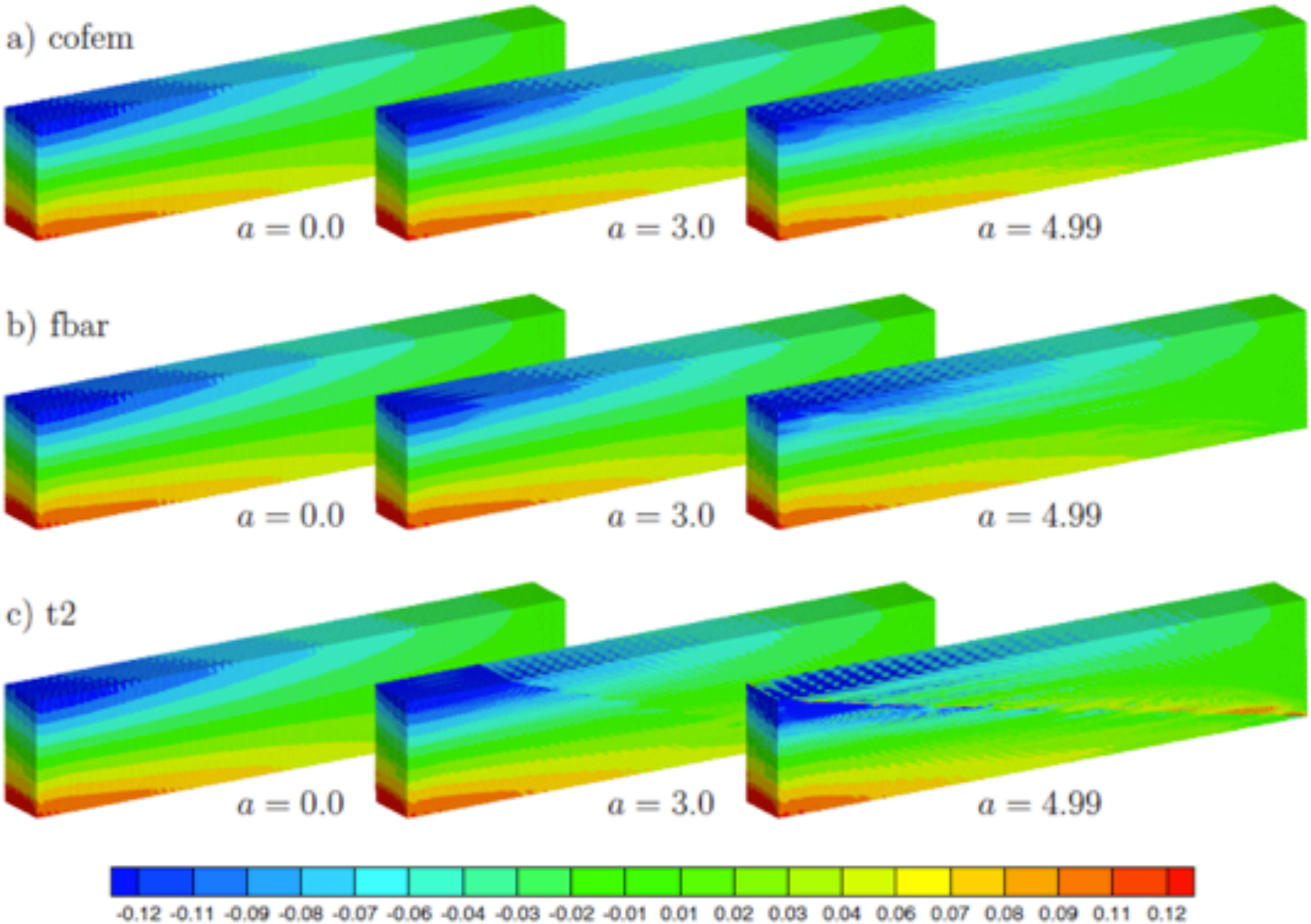


Figure 5: Distribution of normal Cauchy stress in x-direction σ_{11} for the a) cofem-element, b) fbar-element and c) t2-element at different distortion values.

Cook‘s membran problem

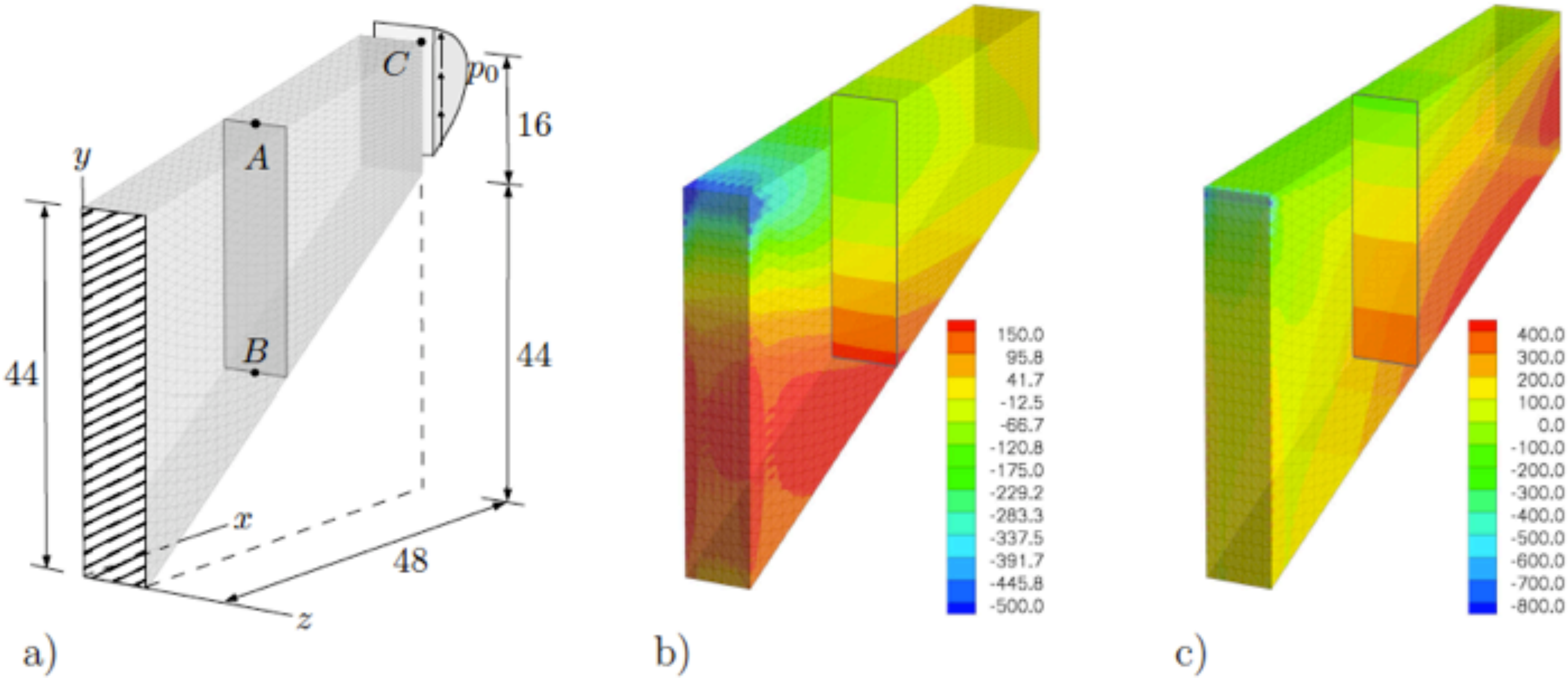


Figure 6: Illustration of the a) boundary value problem and the resulting distribution of normal stresses b) σ_{xx}- and c) σ_{yy}. The considered mesh consists of 32928 tetrahedral elements using the proposed formulation.

Cook‘s membran problem

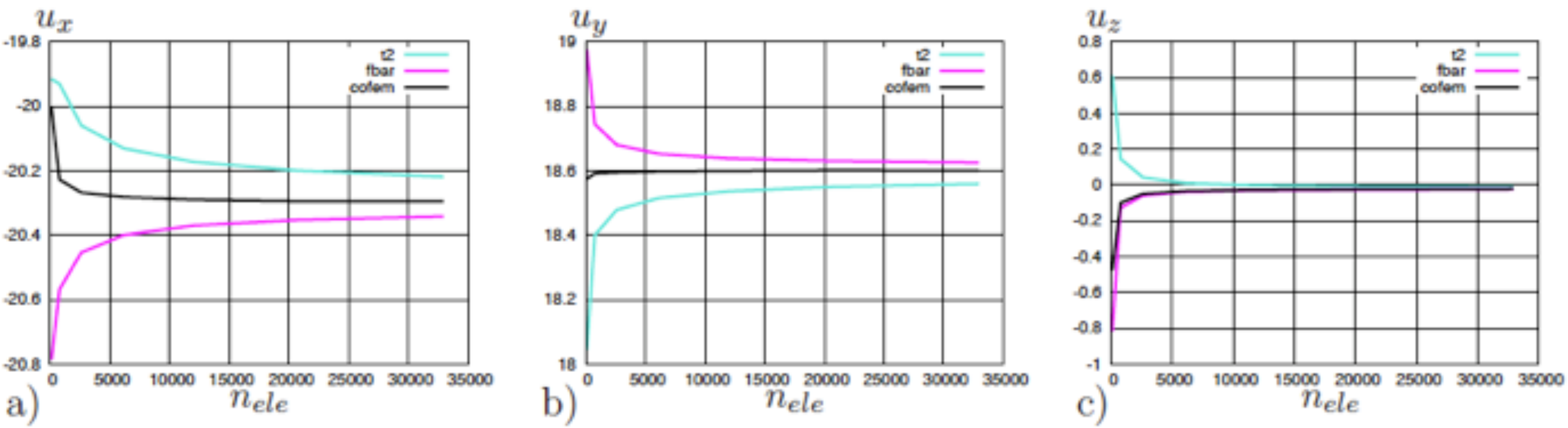


Figure 7: Displacements a) u_x and b) u_y and c) u_z at point C versus number of elements using the isotropic strain energy function $\mathcal{P}^{iso}$.

Cook‘s membran problem

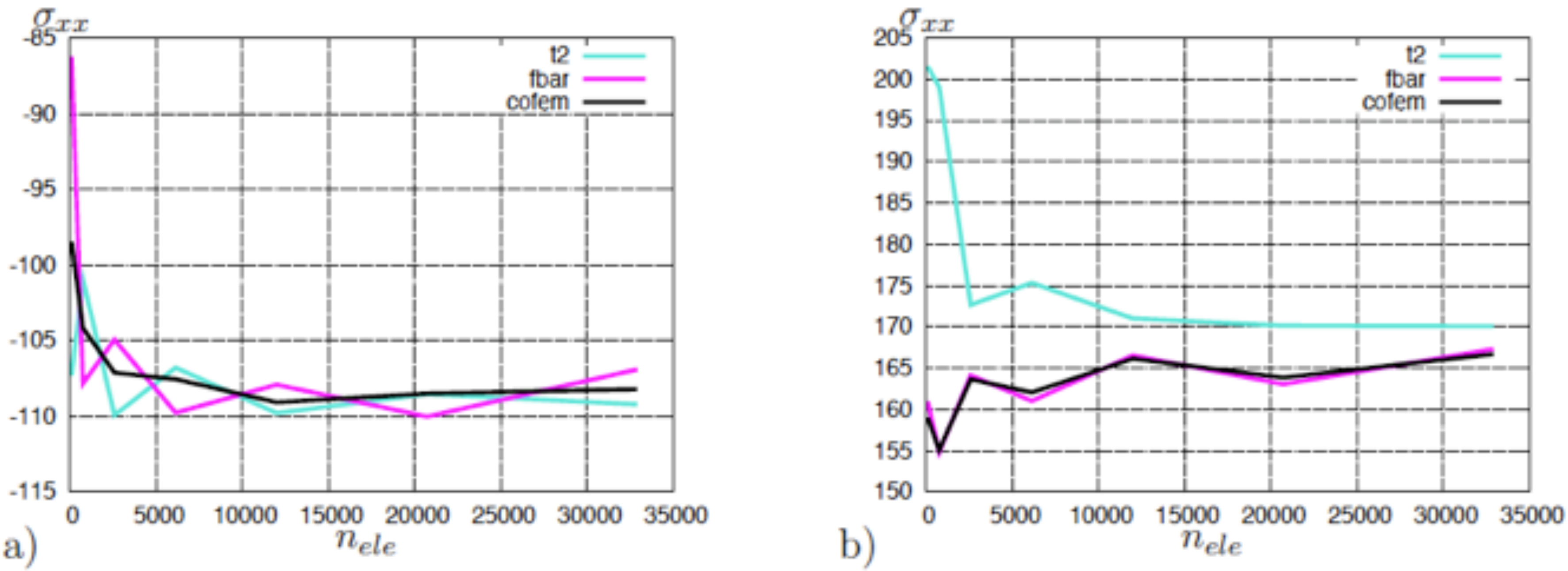


Figure 8: Stress values of σ_{xx} at a) point A and b) point B using the isotropic strain energy function $\mathcal{P}^{iso}$ and three different FE-formulations.

Cook's membran problem

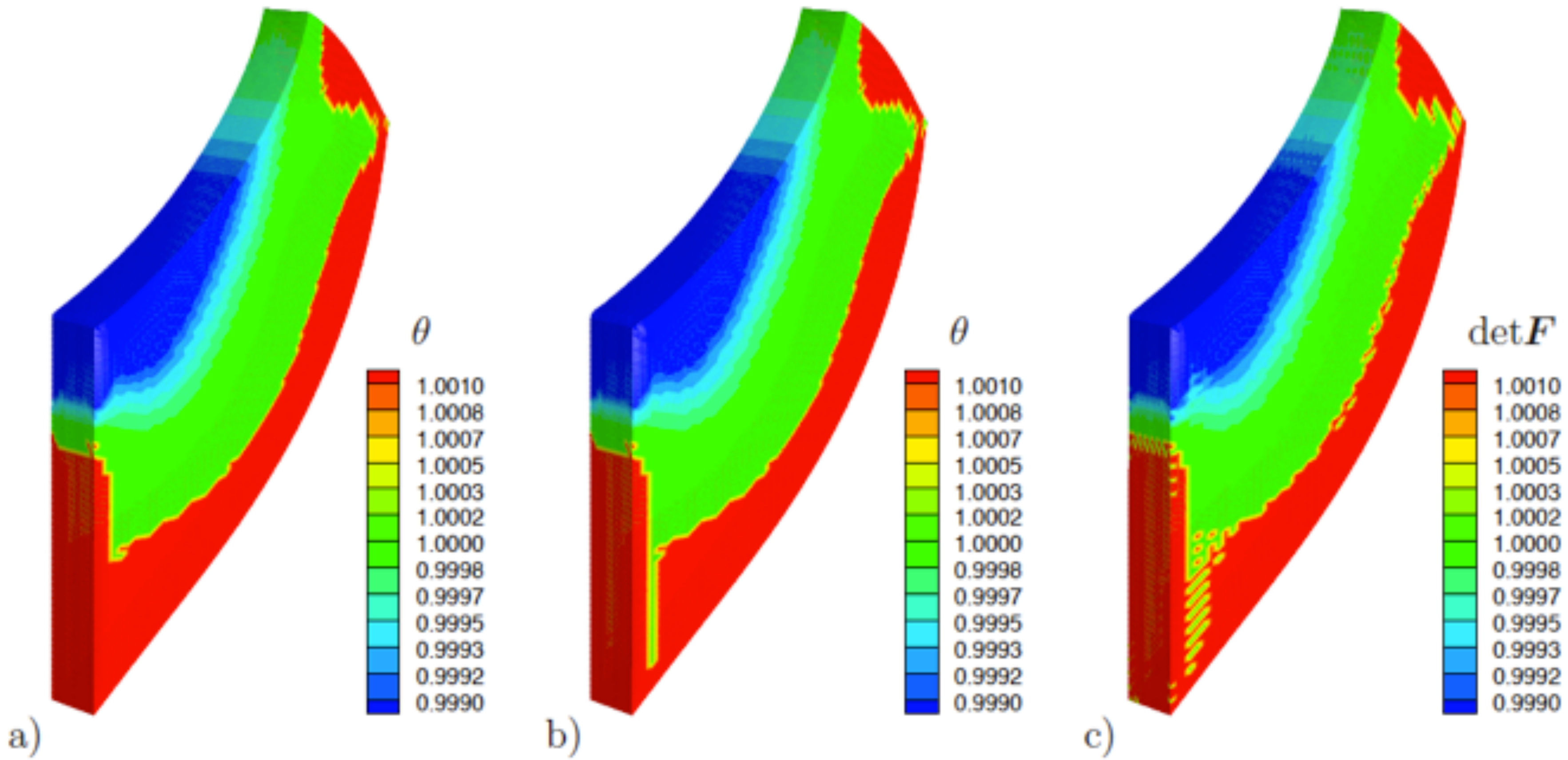


Figure 9: Distribution of θ/det$\boldsymbol{F}$ in the deformed configuration for the a) cofem-, b) fbar- and c) t2 formulation; the number of elements is $n_{ele} = 32928$.

Anisotropic strain energy function

$$\begin{aligned}\psi^{ti}(\boldsymbol{C}, \mathrm{cof}\boldsymbol{C}, \det\boldsymbol{C}) \;=\; & g_0 \Big[\frac{1}{g_C+1} (\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{g_C+1} \\ & + \frac{1}{g_H+1} (\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]\boldsymbol{M}])^{g_H+1} + \frac{1}{g_\theta} (\det\boldsymbol{C})^{-g_\theta} \Big]\end{aligned}$$

$$\boldsymbol{M} = \boldsymbol{a} \otimes \boldsymbol{a}$$

Schröder et al. 2009

Anisotropic strain energy function

$$\psi^{ti}(\boldsymbol{C}, \mathrm{cof}\boldsymbol{C}, \det\boldsymbol{C}) = g_0 \left[\frac{1}{g_C+1} (\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{g_C+1} + \frac{1}{g_H+1} (\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]\boldsymbol{M}])^{g_H+1} + \frac{1}{g_\theta} (\det\boldsymbol{C})^{-g_\theta} \right]$$

$$\boldsymbol{M} = \boldsymbol{a} \otimes \boldsymbol{a}$$

Schröder et al. 2009

Mixed form

$$\mathcal{P}^{ti}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = g_0 \left[\frac{1}{g_C+1} (\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{(g_C+1)} + \frac{1}{g_H+1} (\mathrm{tr}[\boldsymbol{H}^c\boldsymbol{M}])^{(g_H+1)} + \frac{1}{g_\theta} (\theta)^{-2\,g_\theta} \right]$$

Anisotropic strain energy function

$$\psi^{ti}(\boldsymbol{C}, \mathrm{cof}\boldsymbol{C}, \det\boldsymbol{C}) = g_0 \left[\frac{1}{g_C+1} (\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{g_C+1} + \frac{1}{g_H+1} (\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]\boldsymbol{M}])^{g_H+1} + \frac{1}{g_\theta} (\det\boldsymbol{C})^{-g_\theta} \right]$$

$$\boldsymbol{M} = \boldsymbol{a} \otimes \boldsymbol{a}$$

Schröder et al. 2009

Mixed form

$$\mathcal{P}^{ti}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = g_0 \left[\frac{1}{g_C+1} (\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{(g_C+1)} + \frac{1}{g_H+1} (\mathrm{tr}[\boldsymbol{H}^c\boldsymbol{M}])^{(g_H+1)} + \frac{1}{g_\theta} (\theta)^{-2\,g_\theta} \right]$$

$$\partial_C \mathcal{P}^{ti} = g_0\ (\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{g_C}\ \boldsymbol{M}, \quad \partial_{H^c} \mathcal{P}^{ti} = g_0\ (\mathrm{tr}[\boldsymbol{H}^c\boldsymbol{M}])^{g_H}\ \boldsymbol{M}$$

$$\partial_\theta \mathcal{P}^{ti} = -2\, g_0\, \theta^{(-2\,g_\theta - 1)}$$

Anisotropic strain energy function

$$\psi^{ti}(\boldsymbol{C}, \mathrm{cof}\boldsymbol{C}, \det\boldsymbol{C}) = g_0\left[\frac{1}{g_C+1}(\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{g_C+1} + \frac{1}{g_H+1}(\mathrm{tr}[\mathrm{cof}[\boldsymbol{C}]\boldsymbol{M}])^{g_H+1} + \frac{1}{g_\theta}(\det\boldsymbol{C})^{-g_\theta}\right]$$

$$\boldsymbol{M} = \boldsymbol{a}\otimes\boldsymbol{a}$$

Schröder et al. 2009

Mixed form

$$\mathcal{P}^{ti}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = g_0\left[\frac{1}{g_C+1}(\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{(g_C+1)} + \frac{1}{g_H+1}(\mathrm{tr}[\boldsymbol{H}^c\boldsymbol{M}])^{(g_H+1)} + \frac{1}{g_\theta}(\theta)^{-2g_\theta}\right]$$

$$\partial_C\mathcal{P}^{ti} = g_0\,(\mathrm{tr}[\boldsymbol{C}\boldsymbol{M}])^{g_C}\,\boldsymbol{M}, \quad \partial_{H^c}\mathcal{P}^{ti} = g_0\,(\mathrm{tr}[\boldsymbol{H}^c\boldsymbol{M}])^{g_H}\,\boldsymbol{M}$$

$$\partial_\theta\mathcal{P}^{ti} = -2\,g_0\,\theta^{(-2g_\theta-1)}$$

$$g_0 = 3000.0\ \mathrm{kPa}, \quad g_C = 4.0, \quad g_H = 8 \quad \text{and} \quad g_\theta = 1$$

Cook's membran problem

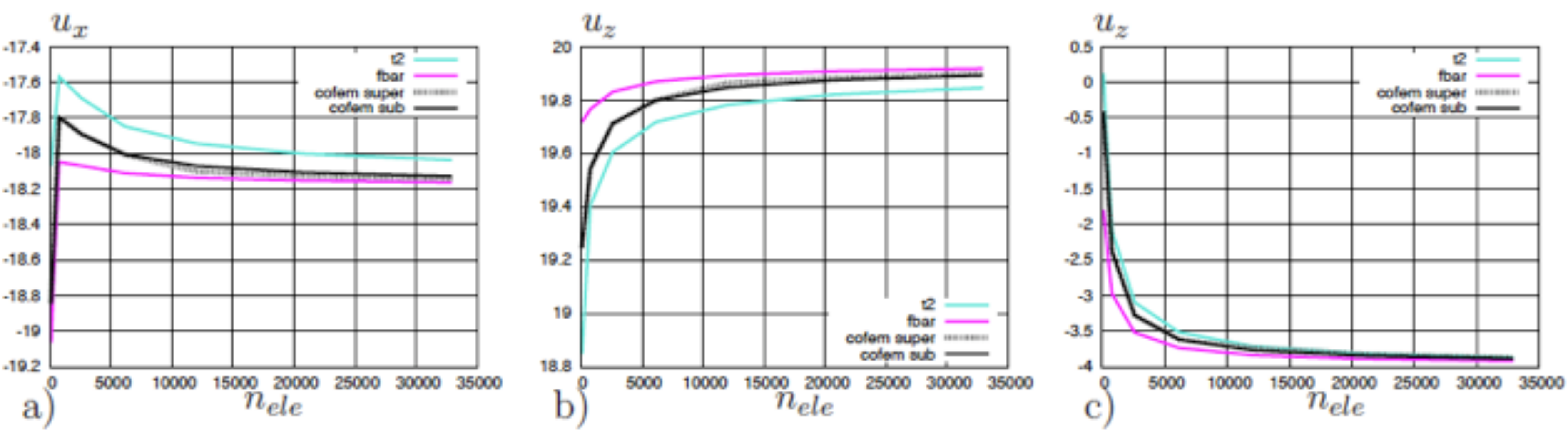


Figure 11: Displacements a) u_x and b) u_y and u_z at point C versus number of elements using the transversely isotropic strain energy function.

Cook's membran problem

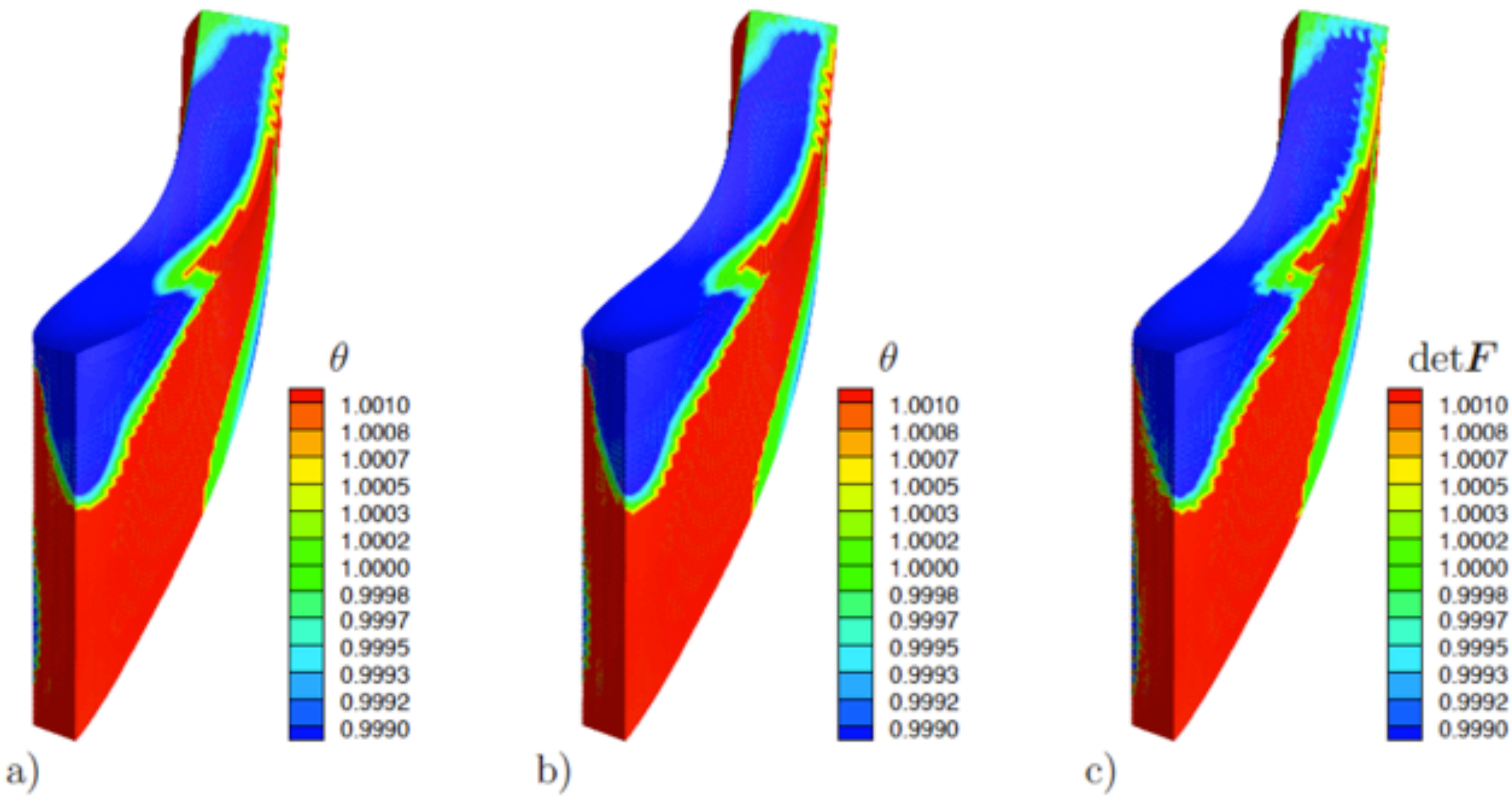


Figure 13: Distribution of $\theta/\det \boldsymbol{F}$ in the deformed configuration for the a) cofem-, b) fbar- and c) t2 formulation using the transversely isotropic strain energy function $\mathcal{P} := \mathcal{P}^{iso} + \mathcal{P}^{ti}$; the number of elements is $n_{ele} = 32928$.

Simulation of arterial walls

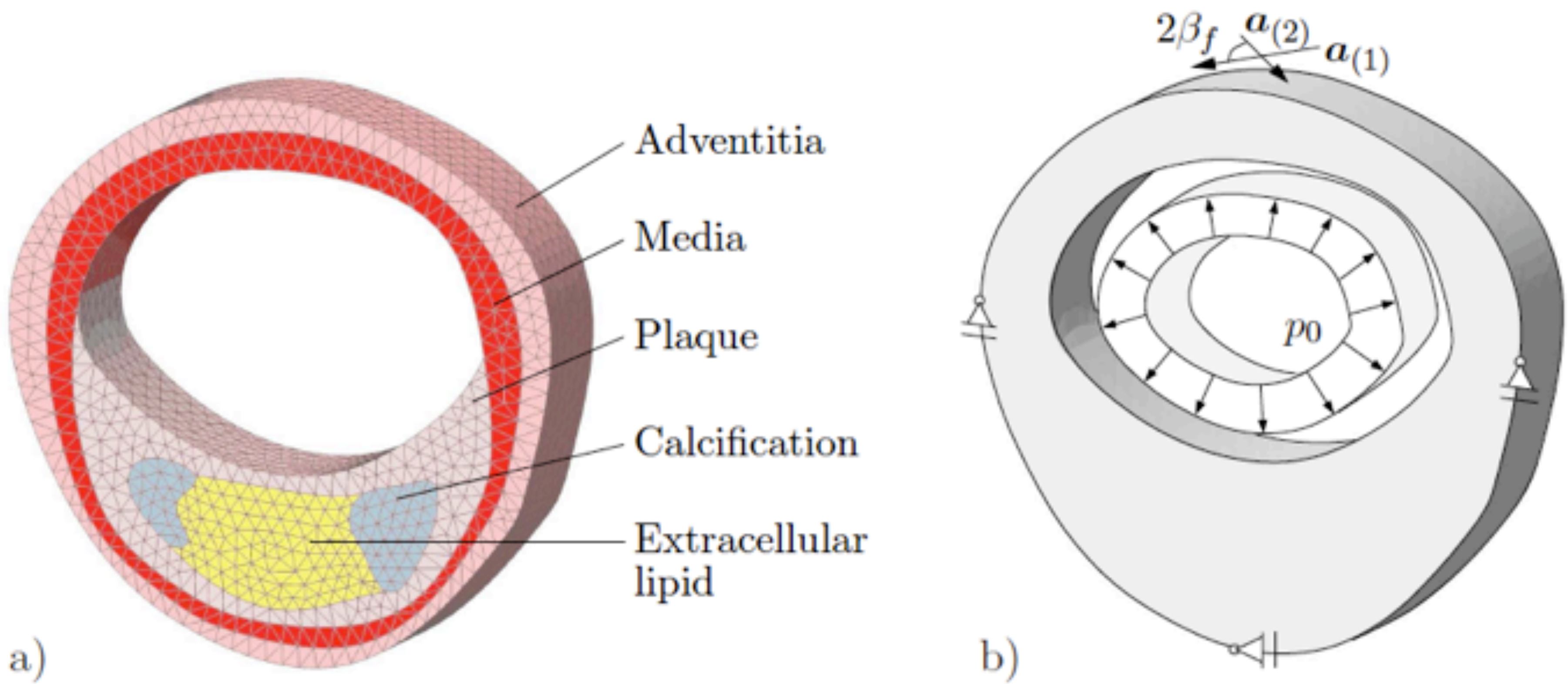


Figure 15: Illustration of the a) individual healthy and deceased tissue components and b) the boundary value problem; the considered mesh consists of 20860 tetrahedral elements.

Simulation of arterial walls

Isotropic material model for calcification and extracellular lipid

$$\mathcal{P}^{iso}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = c_1 \left(\frac{\mathrm{tr}\boldsymbol{C}}{(\det\boldsymbol{C})^{1/3}} - 3 \right) + \epsilon_1 \left(\theta^{2\epsilon_2} + \frac{1}{\theta^{2\epsilon_2}} - 2 \right)$$

Transversly isotropic material model for media and adventitia

$$\mathcal{P}^{ti}_{(a)}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = \alpha_1 \left\langle \mathrm{tr}\left[(\boldsymbol{H}^c)(\boldsymbol{1} - \boldsymbol{M}_{(a)}) \right] - 2 \right\rangle^{\alpha_2}$$

Complete strain energy function for media and adventitia

$$\mathcal{P}^{bio} = \mathcal{P}^{iso} + \sum_{a=1}^{2} \mathcal{P}^{ti}_{(a)}$$

Isotropic material model for plaque

$$\mathcal{P}^{MR}(\boldsymbol{C}, \boldsymbol{H}^c, \theta) = \beta_1 \,\mathrm{tr}\boldsymbol{C} + \eta_1 \,\mathrm{tr}(\boldsymbol{H}^c) + \delta_1 \,\theta^2 - \delta_2 \ln(\theta^2)$$

Schröder & Neff 2004, Balzani 2006, Brands et al. 2008

Simulation of arterial walls

Table 1: Material parameters for the individual tissues taken from Brands et al. (2008).

Layer	Model	Set	c_1 [kPa]	ε_1 [kPa]	ε_2 [-]	α_1 [kPa]	α_2 [-]	β_1 [kPa]	η_1 [kPa]	δ_1 [kPa]	β_f [°]
Media	$\mathcal{P}^{bio}$	1	17.5	100.0	50.0	5.0 E05	7.0	–	–	–	43.39
		2	17.5	499.8	2.4	30001.9	5.1	–	–	–	43.39
Adventitia	$\mathcal{P}^{bio}$	1	7.5	100.0	20.0	1.5 E10	20.0	–	–	–	49.0
		2	6.6	23.9	10.0	1503.0	6.3	–	–	–	49.0
Calcification	$\mathcal{P}^{iso}$	1,2	6800.0	50.0	10.0	–	–	–	–	–	–
Lipid	$\mathcal{P}^{iso}$	1,2	700.0	5250.0	10.0	–	–	–	–	–	–
Plaque	$\mathcal{P}^{MR}$	1,2	–	–	–	–	–	80.0	250.0	2000.0	–

Simulation of arterial walls

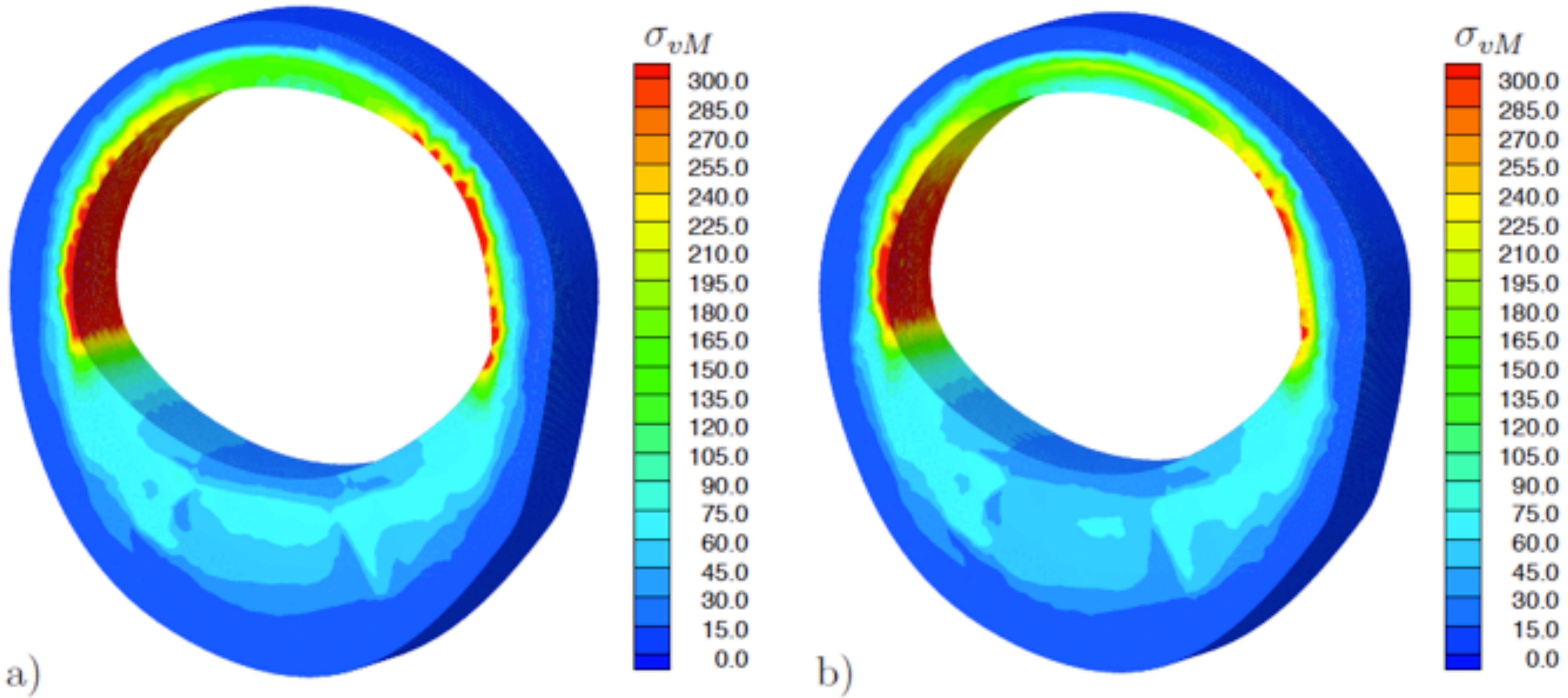


Figure 17: Distribution of the von Mises stresses σ_{vM} in the deformed configuration for the a) cofem- and b) fbar-formulation using parameter set 2.

Development of the EI9 element

- Motivation
- Split of the strain energy function
    - Different Treatment for homogenous and inhomogenous part
- Variational formulation
- Ansatz and Implementation
- Numerical Tests
    - non-uniformly meshed beam
    - incompressible block
    - surface buckling
- Conclusion and outlook

Split of the Strain Energy Function

Additive split into a homogeneous and an inhomogeneous part (Nadler & Rubin, 2003)

$$W\left(\boldsymbol{F}\right) = W_{\text{hom}}\left(\bar{\boldsymbol{F}}\right) + W_{\text{inh}}(\hat{\tilde{\boldsymbol{F}}})$$

Split of the deformation gradient:

homogeneous part

$$\boldsymbol{F} = \bar{\boldsymbol{F}} \cdot \hat{\tilde{\boldsymbol{F}}} \quad ,$$

$$\bar{\boldsymbol{F}} = \frac{1}{V} \int_{\Omega} \boldsymbol{F} \, \mathrm{d}\Omega$$

Homogeneous Part

Compressible Neo-Hooke Material

$$W_{\text{hom}}\left(\bar{\boldsymbol{F}}\right) = \frac{\mu}{2}\left(\bar{J}^{-2/3}\,\text{tr}(\bar{\boldsymbol{C}}) - 3\right) + \frac{K}{\beta^2}(\bar{J}^{-\beta} - 1 + \beta\ln(\bar{J}))$$

$$\begin{aligned} \bar{J} &= \det\left(\bar{\boldsymbol{F}}\right) \\ \bar{\boldsymbol{C}} &= \bar{\boldsymbol{F}}^T\bar{\boldsymbol{F}} \quad . \end{aligned}$$

$$\bar{\boldsymbol{P}} = \frac{\partial W_{\text{hom}}}{\partial \bar{\boldsymbol{F}}} = \mu\bar{J}^{-2/3}\left(\bar{\boldsymbol{F}} - \tfrac{1}{3}\text{tr}(\bar{\boldsymbol{C}})\bar{\boldsymbol{F}}^{-T}\right) + \frac{K}{\beta}\left(1 - \bar{J}^{-\beta}\right)\bar{\boldsymbol{F}}^{-T}$$

Inhomogeneous Part (1)

Linear Elastic Material:

$$W_{\text{inh}}(\hat{\tilde{\boldsymbol{F}}}) = \frac{1}{2}\left(\hat{\tilde{\boldsymbol{F}}} - \boldsymbol{1}\right) : \mathbb{C} : \left(\hat{\tilde{\boldsymbol{F}}} - \boldsymbol{1}\right)$$

$$\hat{\tilde{\boldsymbol{P}}} = (K - \tfrac{2}{3}\mu)\left(\text{tr}(\hat{\tilde{\boldsymbol{F}}}) - 3\right)\boldsymbol{1} + \mu\left(\hat{\tilde{\boldsymbol{F}}} + \hat{\tilde{\boldsymbol{F}}}^T - 2\,\boldsymbol{1}\right)$$

with

$$\mathbb{C}_{iklm} = (K - \tfrac{2}{3}\mu)\,\delta_{ik}\delta_{lm} + \mu\,(\delta_{il}\delta_{km} + \delta_{im}\delta_{kl})$$

(Nadler & Rubin, 2003)

Inhomogeneous Part (2)

Multiplicative Split of the inhomogeneous part of the displacement gradient

$$\hat{\tilde{\boldsymbol{F}}} = \tilde{\boldsymbol{F}} + \hat{\boldsymbol{H}} \qquad \text{with} \qquad \tilde{\boldsymbol{F}} = \bar{\boldsymbol{F}}^{-1} \cdot \boldsymbol{F}(\boldsymbol{x})$$

leading to

$$\boldsymbol{F} = \bar{\boldsymbol{F}} \cdot \left(\bar{\boldsymbol{F}}^{-1} \cdot \boldsymbol{F}(\boldsymbol{x}) + \hat{\boldsymbol{H}} \right) = \boldsymbol{F}(\boldsymbol{x}) + \bar{\boldsymbol{F}} \cdot \hat{\boldsymbol{H}}$$

Ansatz for inhomogeneous part of $\boldsymbol{H}$ leading to $\hat{\boldsymbol{H}}$

$$M_1 = (1 - \xi^2) \quad M_2 = (1 - \eta^2) \quad M_3 = (1 - \zeta^2)$$

(Wilson, Taylor, Doherty & Ghaboussi, 1973)

Potential and Variation

Hu–Washizu

$$\Pi\left(\boldsymbol{x},\hat{\boldsymbol{H}},\boldsymbol{P}\right)=\int\limits_{\Omega}\left[W_{\mathrm{hom}}(\bar{\boldsymbol{F}})+W_{\mathrm{inh}}(\hat{\tilde{\boldsymbol{F}}})-\boldsymbol{P}:\bar{\boldsymbol{F}}\cdot\hat{\boldsymbol{H}}\right]\mathrm{d}\Omega-P_{ext}$$

Potential and Variation

Hu–Washizu

$$\Pi\left(\boldsymbol{x},\hat{\boldsymbol{H}},\boldsymbol{P}\right)=\int\limits_{\Omega}\left[W_{\mathrm{hom}}(\bar{\boldsymbol{F}})+W_{\mathrm{inh}}(\hat{\tilde{\boldsymbol{F}}})-\boldsymbol{P}:\bar{\boldsymbol{F}}\cdot\hat{\boldsymbol{H}}\right]\mathrm{d}\Omega-P_{ext}$$

$$\begin{aligned}
\int\limits_{\Omega}\delta\bar{\boldsymbol{F}}:\frac{\partial W_{\mathrm{hom}}}{\partial\bar{\boldsymbol{F}}}\,\mathrm{d}\Omega+\int\limits_{\Omega}\delta\tilde{\boldsymbol{F}}:\frac{\partial W_{\mathrm{inh}}}{\partial\hat{\tilde{\boldsymbol{F}}}}\,\mathrm{d}\Omega-\int\limits_{\Omega}\boldsymbol{P}:\delta\bar{\boldsymbol{F}}\cdot\hat{\boldsymbol{H}}\,\mathrm{d}\Omega-\delta P_{ext} &= 0\\
\int\limits_{\Omega}\delta\hat{\boldsymbol{H}}:\frac{\partial W_{\mathrm{inh}}}{\partial\hat{\tilde{\boldsymbol{F}}}}\,\mathrm{d}\Omega-\int\limits_{\Omega}\boldsymbol{P}:\bar{\boldsymbol{F}}\cdot\delta\hat{\boldsymbol{H}}\,\mathrm{d}\Omega &= 0\\
\int\limits_{\Omega}\delta\boldsymbol{P}:\bar{\boldsymbol{F}}\cdot\hat{\boldsymbol{H}}\,\mathrm{d}\Omega &= 0
\end{aligned}$$

Potential and Variation

Hu–Washizu

$$\Pi\left(\boldsymbol{x},\hat{\boldsymbol{H}},\boldsymbol{P}\right)=\int\limits_{\Omega}\left[W_{\text{hom}}(\bar{\boldsymbol{F}})+W_{\text{inh}}(\hat{\tilde{\boldsymbol{F}}})-\boldsymbol{P}:\bar{\boldsymbol{F}}\cdot\hat{\boldsymbol{H}}\right]\mathrm{d}\Omega-P_{ext}$$

$$\begin{aligned}
\int\limits_{\Omega}\delta\bar{\boldsymbol{F}}:\frac{\partial W_{\text{hom}}}{\partial\bar{\boldsymbol{F}}}\,\mathrm{d}\Omega+\int\limits_{\Omega}\delta\tilde{\boldsymbol{F}}:\frac{\partial W_{\text{inh}}}{\partial\hat{\tilde{\boldsymbol{F}}}}\,\mathrm{d}\Omega-\int\limits_{\Omega}\boldsymbol{P}:\delta\bar{\boldsymbol{F}}\cdot\hat{\boldsymbol{H}}\,\mathrm{d}\Omega-\delta P_{ext}&=0\\
\int\limits_{\Omega}\delta\hat{\boldsymbol{H}}:\frac{\partial W_{\text{inh}}}{\partial\hat{\tilde{\boldsymbol{F}}}}\,\mathrm{d}\Omega-\int\limits_{\Omega}\boldsymbol{P}:\bar{\boldsymbol{F}}\cdot\delta\hat{\boldsymbol{H}}\,\mathrm{d}\Omega&=0\\
\int\limits_{\Omega}\delta\boldsymbol{P}:\bar{\boldsymbol{F}}\cdot\hat{\boldsymbol{H}}\,\mathrm{d}\Omega&=0
\end{aligned}$$

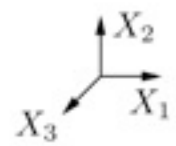
with

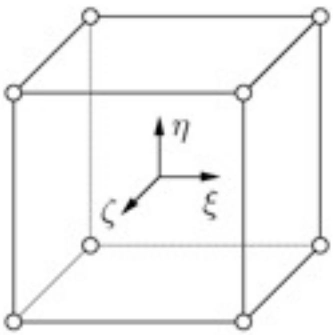
$$\delta\tilde{\boldsymbol{F}}=-\bar{\boldsymbol{F}}^{-1}\cdot\delta\bar{\boldsymbol{F}}\cdot\bar{\boldsymbol{F}}^{-1}\cdot\boldsymbol{F}(\boldsymbol{x})+\bar{\boldsymbol{F}}^{-1}\cdot\delta\boldsymbol{F}(\boldsymbol{x})$$

Ansatz and Implementation

$$\boldsymbol{x}^h = \sum_{I=1}^{N} N_I \boldsymbol{x}_I = \sum_{I=1}^{N} N_I (\boldsymbol{X}_I + \boldsymbol{u}_I)$$

$$N_I(\xi, \eta, \zeta) = \frac{1}{8}(1 + \xi\xi_I)(1 + \eta\eta_I)(1 + \zeta\zeta_I)$$





$$\boldsymbol{F}^h = \begin{pmatrix} F^h_{11} \\ F^h_{22} \\ F^h_{33} \\ F^h_{12} \\ F^h_{21} \\ F^h_{23} \\ F^h_{32} \\ F^h_{13} \\ F^h_{31} \end{pmatrix} = \sum_{I=1}^{8} \boldsymbol{B}_I \boldsymbol{x}_I \quad , \quad \boldsymbol{B}_I = \begin{pmatrix} N_{I,X} & 0 & 0 \\ 0 & N_{I,Y} & 0 \\ 0 & 0 & N_{I,Z} \\ N_{I,Y} & 0 & 0 \\ 0 & N_{I,X} & 0 \\ 0 & N_{I,Z} & 0 \\ 0 & 0 & N_{I,Y} \\ N_{I,Z} & 0 & 0 \\ 0 & 0 & N_{I,X} \end{pmatrix}$$

$$\bar{\boldsymbol{F}}^h = \frac{1}{\Omega_e} \int_{\Omega_e} \boldsymbol{F}^h \, \mathrm{d}\Omega = \sum_{I=1}^{8} \frac{1}{\Omega_e} \int_{\Omega_e} \boldsymbol{B}_I \, \mathrm{d}\Omega \, \boldsymbol{x}_I = \sum_{I=1}^{8} \bar{\boldsymbol{B}}_I \, \boldsymbol{x}_I$$

$$\hat{\boldsymbol{H}}^h = \begin{pmatrix} \hat{H}^h_{11} \\ \hat{H}^h_{22} \\ \hat{H}^h_{33} \\ \hat{H}^h_{12} \\ \hat{H}^h_{21} \\ \hat{H}^h_{23} \\ \hat{H}^h_{32} \\ \hat{H}^h_{13} \\ \hat{H}^h_{31} \end{pmatrix} = \sum_{L=1}^{4} \boldsymbol{G}_L \boldsymbol{\alpha}_L \quad , \quad \boldsymbol{G}_L = \begin{pmatrix} M_{L,X} & 0 & 0 \\ 0 & M_{L,Y} & 0 \\ 0 & 0 & M_{L,Z} \\ M_{L,Y} & 0 & 0 \\ 0 & M_{L,X} & 0 \\ 0 & M_{L,Z} & 0 \\ 0 & 0 & M_{L,Y} \\ M_{L,Z} & 0 & 0 \\ 0 & 0 & M_{L,X} \end{pmatrix} \quad \boldsymbol{G}_4 = \begin{pmatrix} M_{4,X} & M_{4,Y} & M_{4,Z} \\ M_{4,X} & M_{4,Y} & M_{4,Z} \\ M_{4,X} & M_{4,Y} & M_{4,Z} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

with

$$M_1 = (1-\xi^2) \quad M_2 = (1-\eta^2) \quad M_3 = (1-\zeta^2) \quad M_4 = \xi\eta\zeta$$

weak form

$$\bigcup_{e=1}^{n_e} \left(\sum_{I=1}^{8} \delta \boldsymbol{u}_I^T \bar{\boldsymbol{B}}_I^T \bar{\boldsymbol{P}}^h \Omega_e + \sum_{I=1}^{8} \delta \boldsymbol{u}_I^T \int_{\Omega_e} \tilde{\boldsymbol{B}}_I^T \hat{\tilde{\boldsymbol{P}}}^h \, d\Omega \right) - \delta P^h_{\text{ext}} = 0$$

$$\sum_{K=1}^{4} \delta \boldsymbol{\alpha}_K^T \int_{\Omega_e} \boldsymbol{G}_K^T \hat{\tilde{\boldsymbol{P}}}^h \, d\Omega = 0$$

Residual and tangent stiffness matrix

$$\boldsymbol{R}_I^u = \bar{\boldsymbol{B}}_I^T \bar{\boldsymbol{P}}^h \Omega_e + \int_{\Omega_e} \tilde{\boldsymbol{B}}_I^T \hat{\tilde{\boldsymbol{P}}}^h \, \mathrm{d}\Omega - \boldsymbol{P}_I^{ext}$$

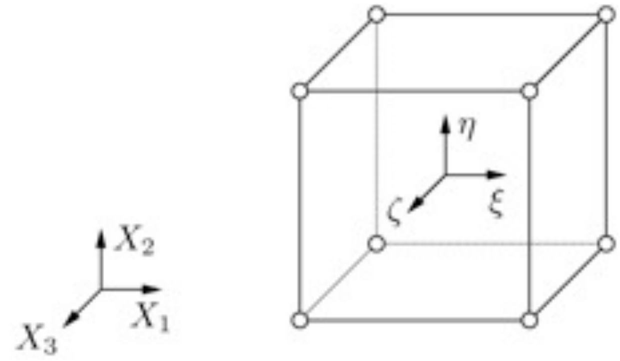
$$\boldsymbol{R}_K^\alpha = \int_{\Omega_e} \boldsymbol{G}_K^T \hat{\tilde{\boldsymbol{P}}}^h \, \mathrm{d}\Omega$$

$$\boldsymbol{K}_{IJ}^{uu} = \bar{\boldsymbol{B}}_I^T \bar{\mathbb{D}} \bar{\boldsymbol{B}}_J \Omega_e + \int_{\Omega_e} \tilde{\boldsymbol{B}}_I^T \mathbb{C} \tilde{\boldsymbol{B}}_J \, \mathrm{d}\Omega + \boldsymbol{K}_{IJ}^{uu\,geo}$$

$$\boldsymbol{K}_{IL}^{u\alpha} = \int_{\Omega_e} \tilde{\boldsymbol{B}}_I^T \mathbb{C} \boldsymbol{G}_L \, \mathrm{d}\Omega$$

$$\boldsymbol{K}_{KJ}^{\alpha u} = \int_{\Omega_e} \boldsymbol{G}_K^T \mathbb{C} \tilde{\boldsymbol{B}}_J \, \mathrm{d}\Omega$$

$$\boldsymbol{K}_{KL}^{\alpha\alpha} = \int_{\Omega_e} \boldsymbol{G}_K^T \mathbb{C} \boldsymbol{G}_L \, \mathrm{d}\Omega$$



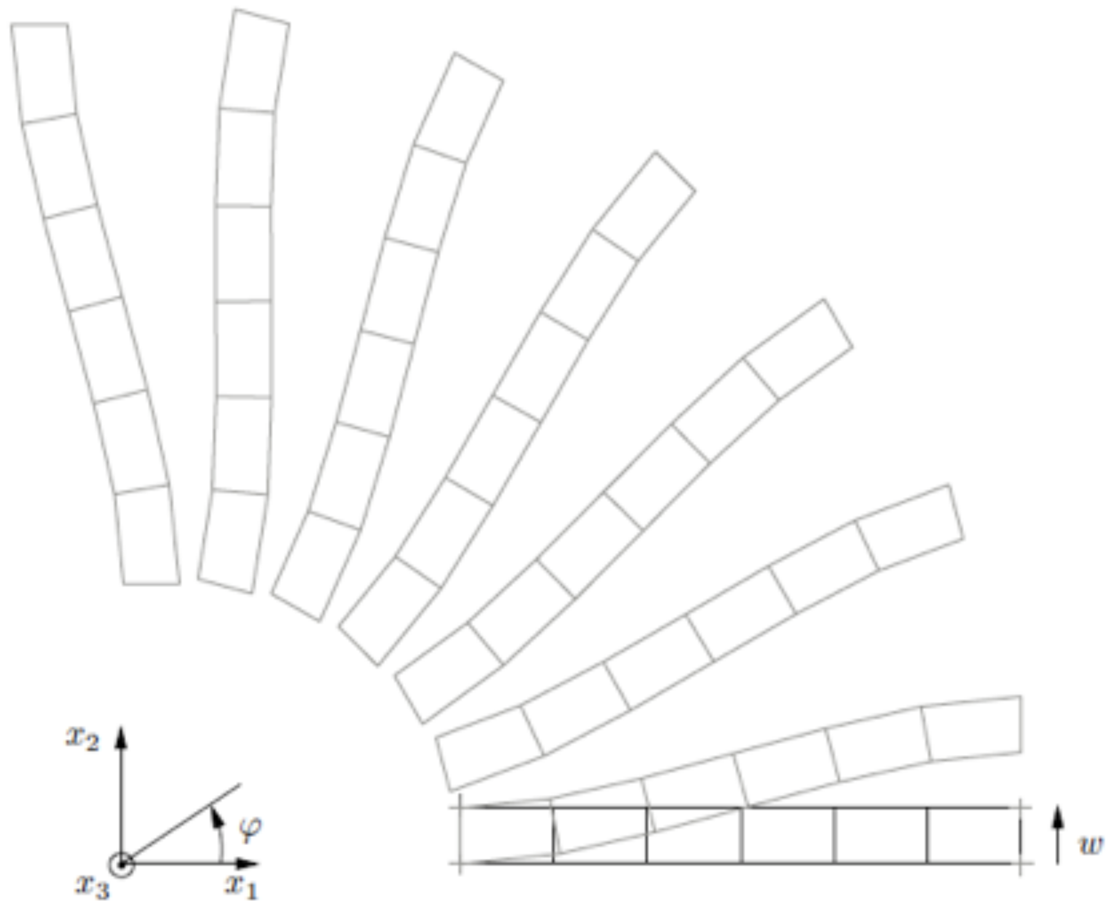
$$\delta \boldsymbol{u}_I \cdot \boldsymbol{K}_{IJ}^{uu\,geo} \cdot \Delta \boldsymbol{u}_J = \int_{\Omega_e} \Delta\delta\tilde{\boldsymbol{F}} : \mathbb{C} : \left(\hat{\tilde{\boldsymbol{F}}} - \boldsymbol{1} \right) \mathrm{d}\Omega$$

Equation system on element level

$$\boldsymbol{K}^{uu} \Delta \boldsymbol{u} + \boldsymbol{K}^{u\alpha} \Delta \boldsymbol{\alpha} = -\boldsymbol{R}^u$$
$$\boldsymbol{K}^{\alpha u} \Delta \boldsymbol{u} + \boldsymbol{K}^{\alpha\alpha} \Delta \boldsymbol{\alpha} = -\boldsymbol{R}^\alpha$$

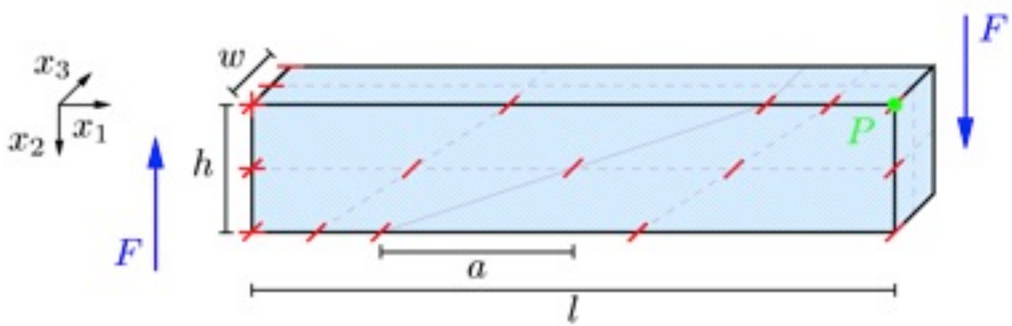
(Simo, Armero & Taylor, 1993)

Numerical Example: Objectivity Test



Numerical Example: Beam (1)

System

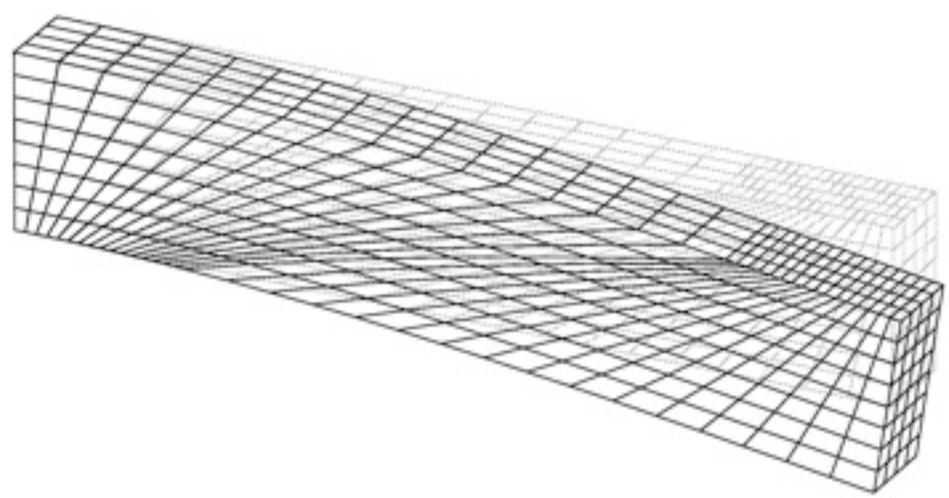


Geometry
$l = 10\,\mathrm{mm}$
$h = 2\,\mathrm{mm}$
$w = 1\,\mathrm{mm}$
$a = 3\,\mathrm{mm}$

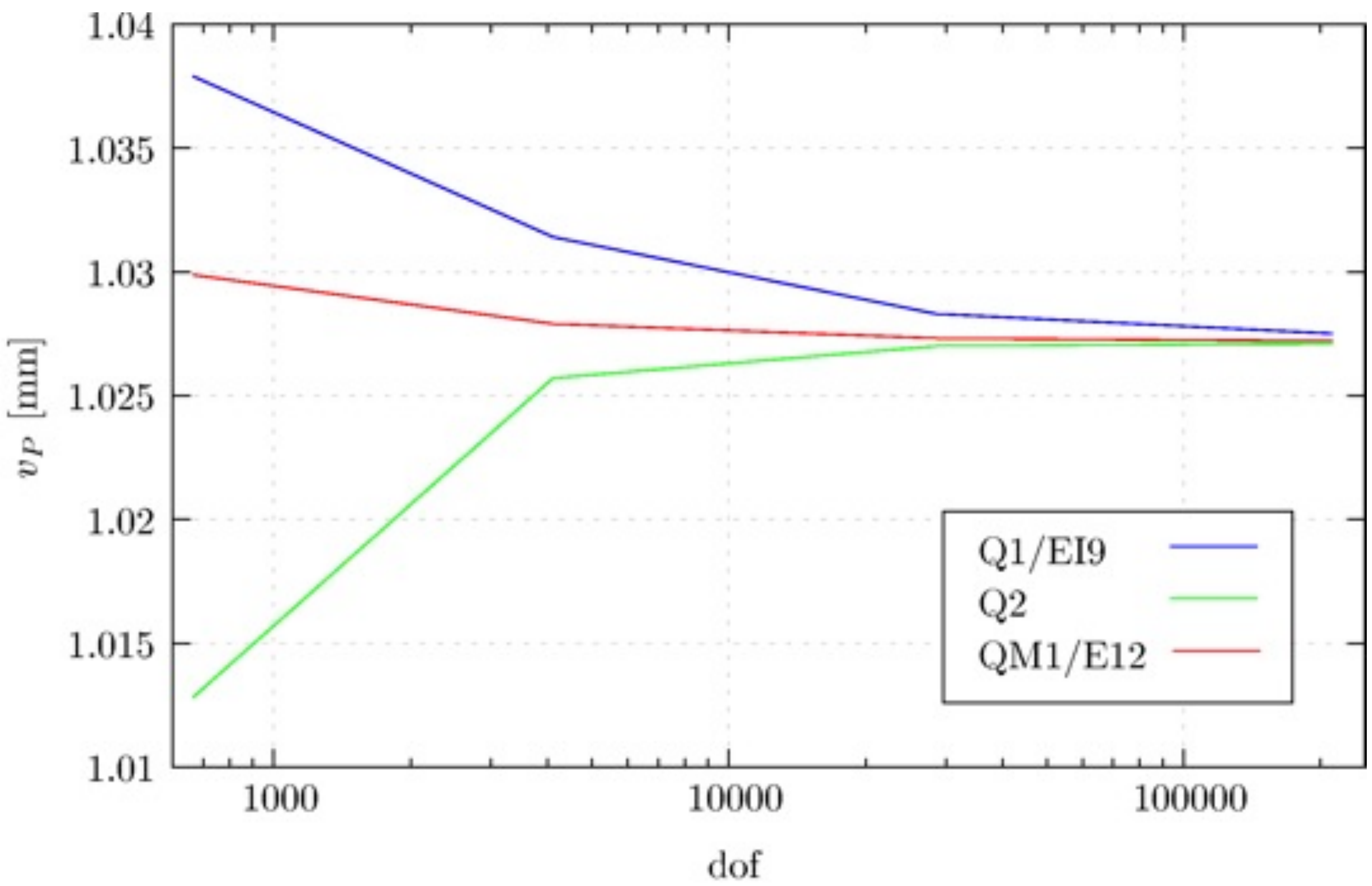
Load
$F = 6\,\mathrm{N}$

Material
$\Lambda = 600\,\mathrm{MPa}$
$\mu = 600\,\mathrm{MPa}$

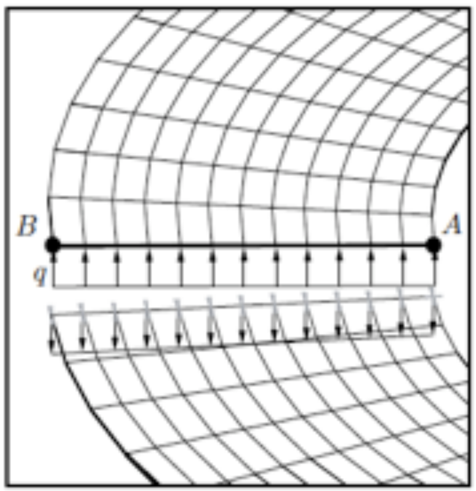
Deformed System

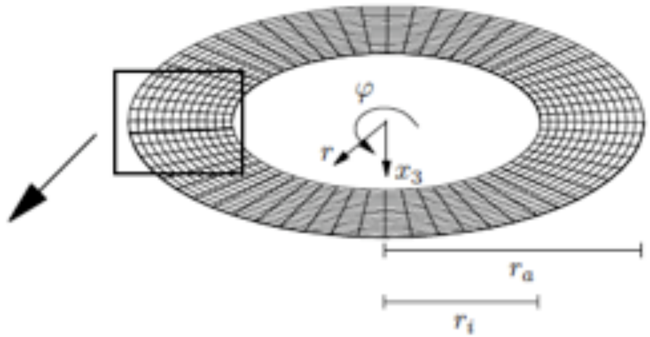


Numerical Example: Beam (2)



Numerical Example: Thin Plate Ring

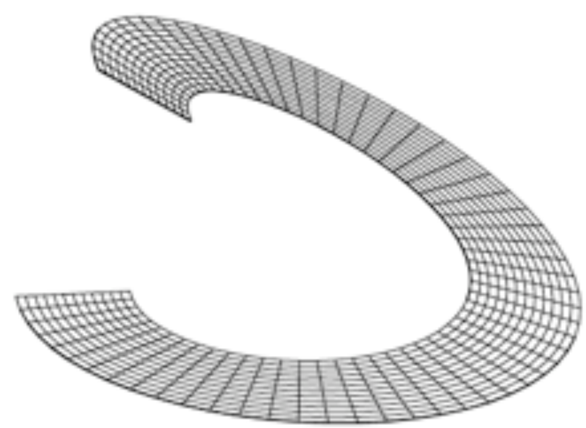




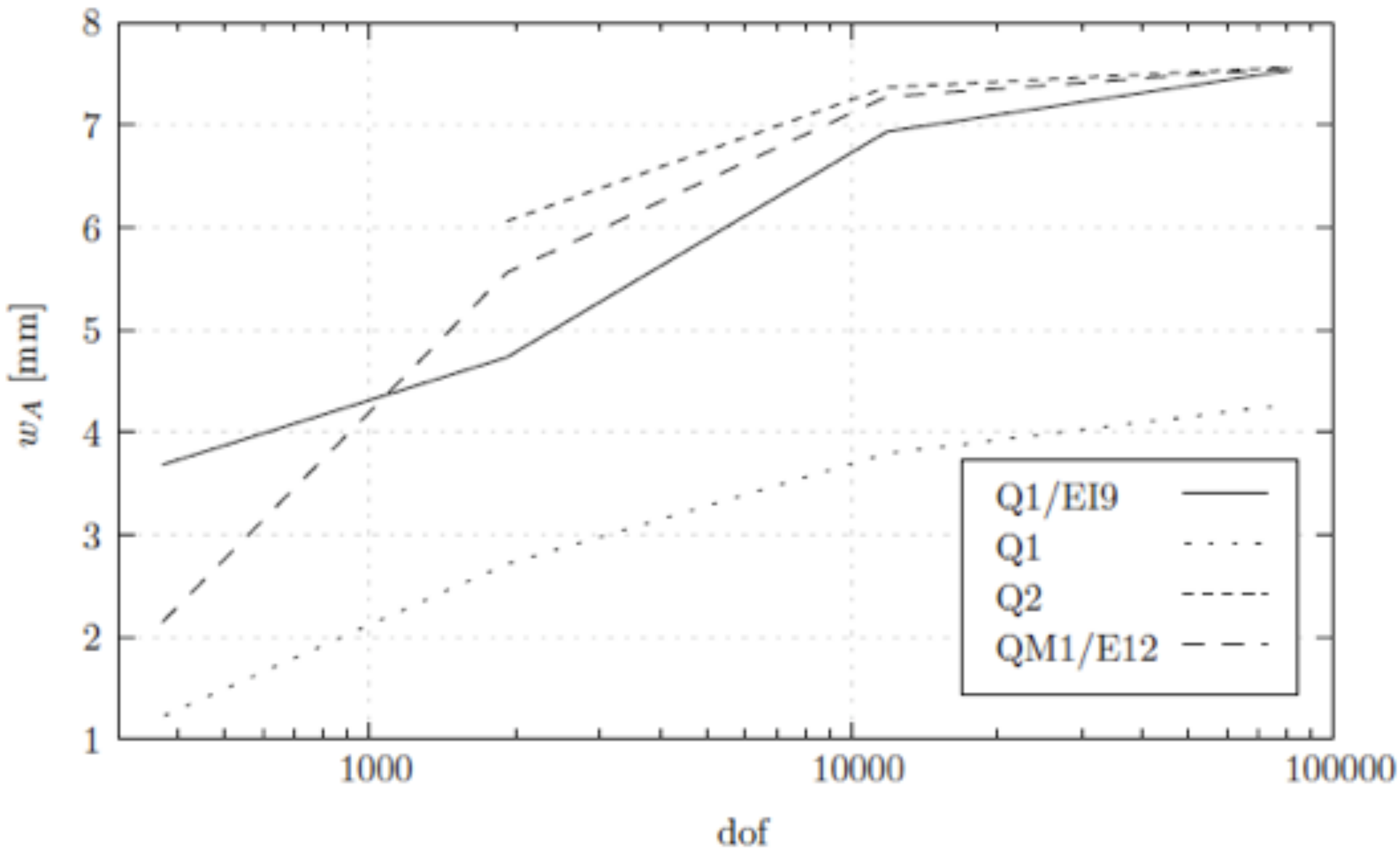
Geometry
$r_a = 6\,\mathrm{mm}$
$r_i = 10\,\mathrm{mm}$
$h = 0.03\,\mathrm{mm}$

Material
$K = 7000000\,\mathrm{MPa}$
$\mu = 10500000\,\mathrm{MPa}$
$\beta = -2$

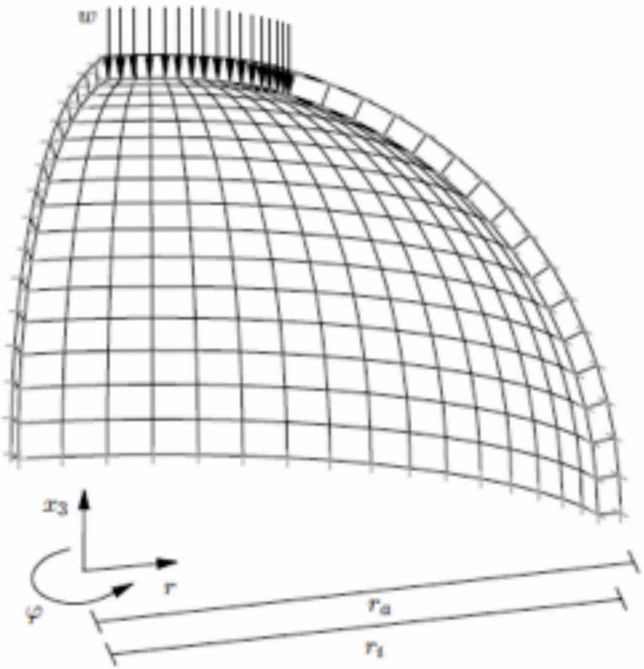
Load
$q = 6.67\,\mathrm{MPa}$



Numerical Example: Thin Plate Ring (2)



Numerical Example: Spherical Shell



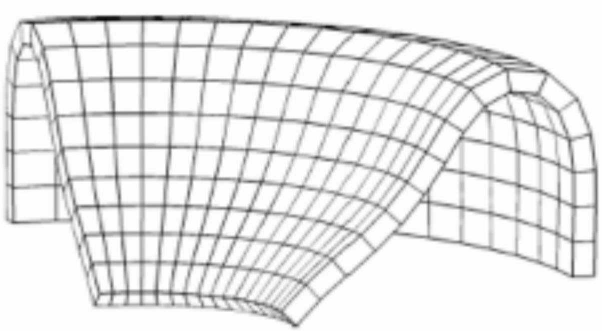
Geometry
$r_a = 9.75\,\mathrm{mm}$
$r_i = 10.25\,\mathrm{mm}$

Load
$w = 10\,\mathrm{mm}$

Material
$K = 175000\,\mathrm{MPa}$
$\mu = 80769.23\,\mathrm{MPa}$
$\beta = -2$

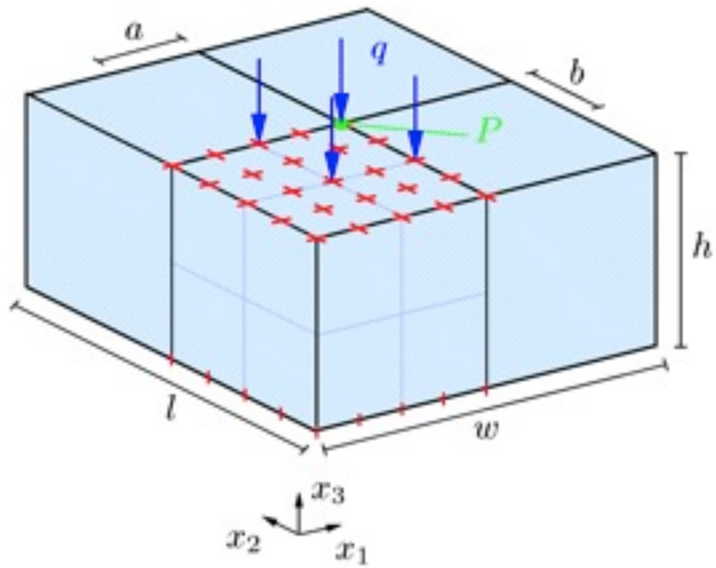
Degrees of freedom	Q1/EI9	Q1	Q2	QM1/E12
1615	5767.66	9319.42		5529.04
9471	5479.85	6508.51	5485.56	5379.84
62335	5377.19	5646.24	5350.54	5348.75
445695	5347.95	5416.33	5338.94	5340.71

Figure 13. Spherical shell: System, load and material data



Numerical Example: Block (1)

System

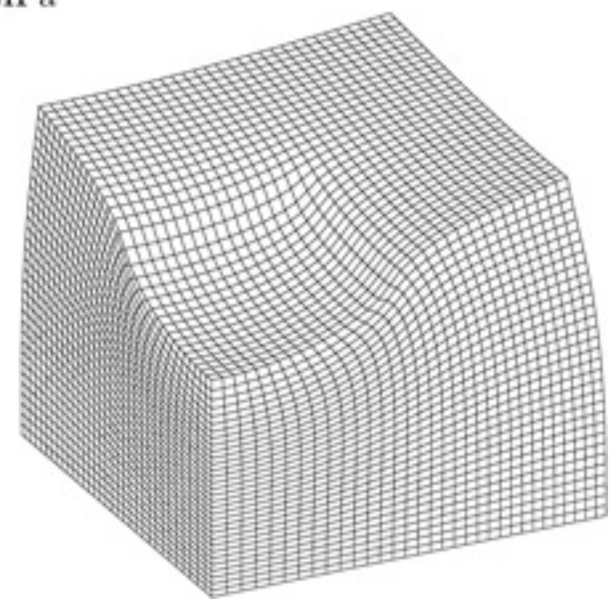


Geometry
$h = 50\,\mathrm{mm}$
$w = 100\,\mathrm{mm}$
$l = 100\,\mathrm{mm}$
$a = 25\,\mathrm{mm}$
$b = 25\,\mathrm{mm}$

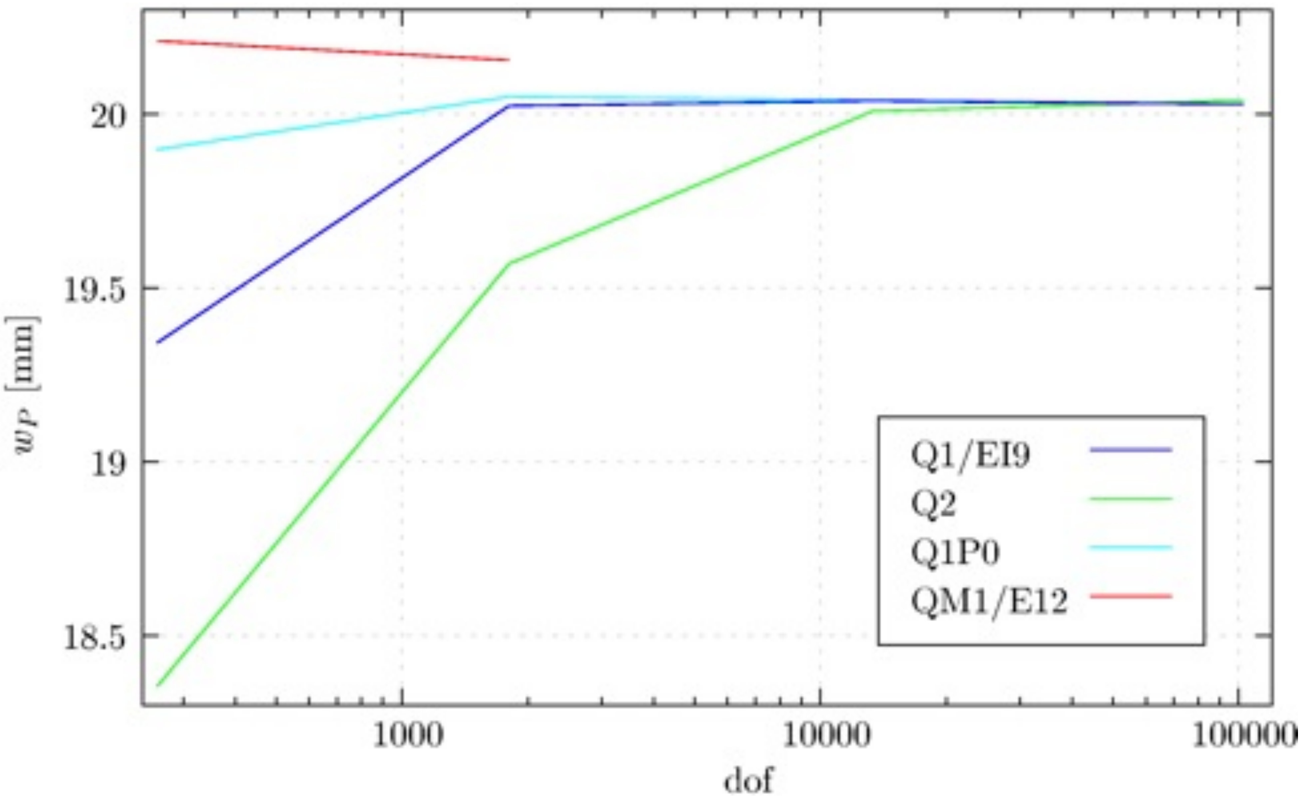
Load
$q = 3\,\mathrm{MPa}$

Material
$\Lambda = 499.92568\,\mathrm{MPa}$
$\mu = 1.61148\,\mathrm{MPa}$

Deformed System

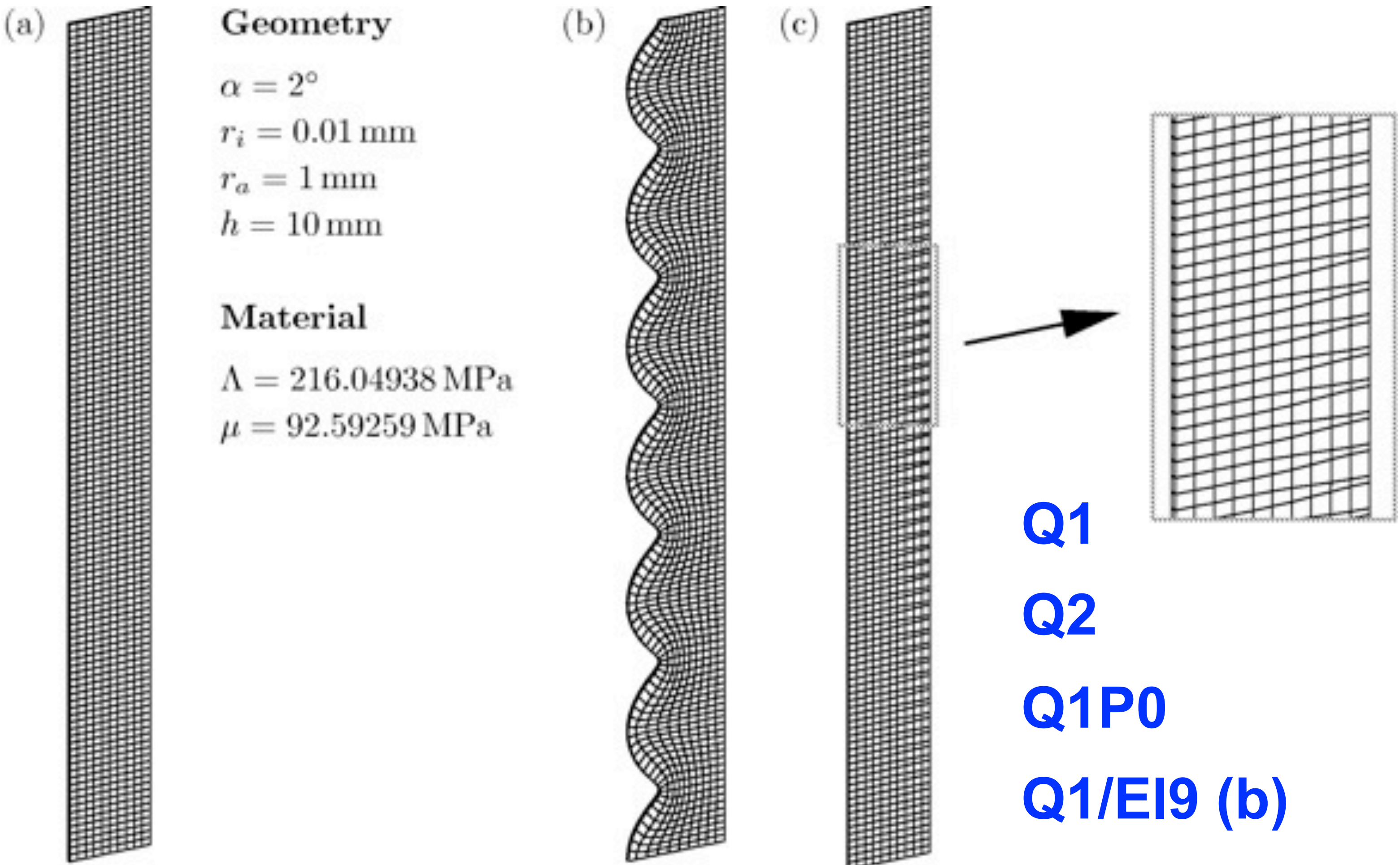


Numerical Example: Block (2)



Degrees of freedom	Q1/EI9	Q1/EI12	Q2	Q1P0	QM1/E12
1800	4	4	4	5	14
102432	6	6	6	7	

Numerical Example: Surface Buckling



Q1

Q2

Q1P0

Q1/EI9 (b)

QM1/E12 (c)

Conclusion and Outlook

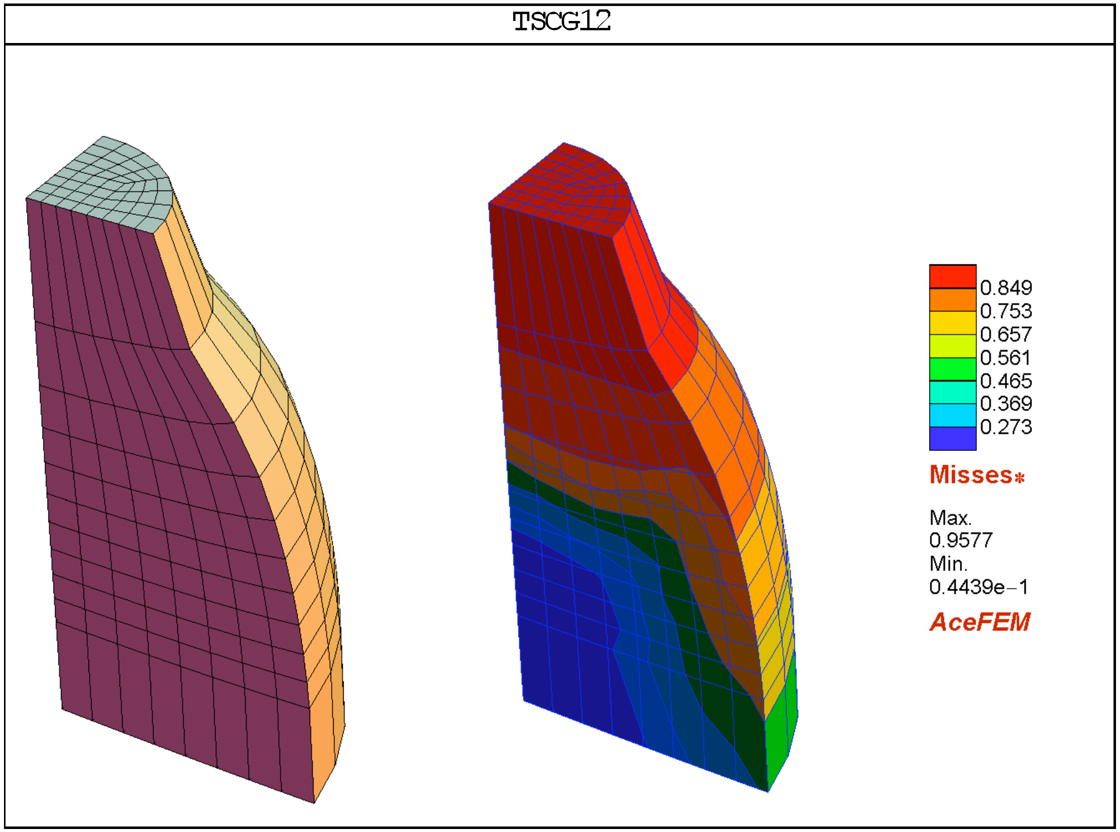
Advantages of Q1/EI9

- robust, no hourglassing
- no adjustment of any parameters
- works for coarse and unaligned meshes
- handles incompressible material
- has good bending properties

Outlook

- Inplementation of inelastic constitutive models (more complex than for standard elements)
- Solution by TSCG12 element

TSCG12 Element for plasticity



using 12 enhanced modes and an expansion of shape function derivatives using Taylor series

Korelc, Wriggers, Soric (2010)