Some recent developments in strain-gradient elasticity and their applications to fracture mechanics

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joint work with Giulio Sciarra

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Nonlocal and strain-gradient elasticity

For nonlocal elasticity:

$$S(x) = \int_{\mathcal{D}} k(x,\, y) E(y)\, dy, \tag{1}$$

for some suitable kernel k.

Strain-gradient (*second-gradient*) elasticity is a particular case of (1) for kernels which decay sufficiently fast.

Suppose $k(x, y) \to 0$ as $\|y - x\| > \ell$; then for ℓ sufficiently small and differentiable strain/stress fields from (1) one obtains (Eringen, 1983)

$$(S - \ell^2 \Delta S)(x) = \mathbb{C}(x) E(x),$$

or

$$S(x) = \mathbb{C}(x) E(x) + \ell^2 \Delta E(x)$$

both distinguished features of strain-gradient theories.

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Outline

Despite being the simplest form of nonlocal elasticity much remains to be done in terms of:

- basic analysis problems (existence theorems, notion of quasi-convexity ...)
- analytical solutions of classical elasticity problems;
- measurement and identification of relevant constitutive parameters;
- numerical implementation.

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Relevant basic references

- Mindlin Micro-structure in linear elasticity. *Arch Rat Mech Anal* 16, 1964
- Sokolowski Theory of couple-stresses in bodies with constrained rotations. In *CISM Courses and Lectures* 26, 1970
- Germain La méthode des puissances virtuelles en mécanique des milieux continus: Théorie du second gradient. *J Mécanique* 12, 1973
- Eringen On differential equations of nonlocal elasticity. *J Appl Phys* 54, 1983

PS: (Aifantis, *IJSS* 2011) reviews the relevant anglo-saxon literature.

Stored elastic energy

Denote:

$$
\begin{aligned}
\text{placement:} &\quad x_\alpha := \chi_\alpha(X_i),\ X \in \mathcal{D}\\
\text{deformation:} &\quad F_{\alpha i} := \chi_{\alpha,i},\\
\text{rotation and stretch:} &\quad F_{\alpha i} = R_{\alpha j}\, U_{ji},\ R \in \mathrm{Orth}^+,\ U \in \mathrm{Sym}\\
\text{strain:} &\quad E_{ij} := (U_{hi}\, U_{hj} - \delta_{ij})/2,
\end{aligned}
$$

The stored elastic energy ψ is assumed to depend on both the deformation F and its gradient ∇F. The request for ψ to be objective means that:

$$\psi(F, \nabla F) = \psi(QF, Q\,\nabla F), \quad \forall\, Q \in \mathrm{Orth}$$

The objectivity condition with $Q_{\beta\alpha} = R_{\alpha\beta}$ implies

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\begin{aligned}
\psi(F_{\alpha i},\, F_{\alpha i,j}) &= \psi(Q_{\beta\alpha}F_{\alpha i},\, Q_{\beta\alpha}F_{\alpha i,j}) =\\
&= \psi(\delta_{\beta j}U_{ji},\, R_{\alpha\beta}\, R_{\alpha l,j}\, U_{li} + U_{ki,j}),
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where both $R^\top \nabla R$ and ∇U appear.

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Stored elastic energy

Fortune&Vallée (2001) have solved the nonlinear compatibility equations

$$\operatorname{curl}(R\,U) = 0$$

to obtain explicitly $R^\top \nabla R$ in terms of ∇U, namely

$$R_{\alpha f} R_{\alpha l,k} = \frac{\varepsilon_{flm}}{\det U} \left(U_{ml} \left(\operatorname{curl} U\right)_{nl} - \frac{1}{2} U_{ij} \left(\operatorname{curl} U\right)_{ij} \delta_{mn} \right) U_{nk}$$

where ε_{flm} is the Levi-Civita alternator. Hence we conclude that every objective stored energy must be written in the form:

$$\psi(F, \nabla F) = \hat{\psi}(U, \nabla U) = \tilde{\psi}(E, \nabla E)$$

Remark: this result readily extends to continua with gradient of arbitrary order!

$$\psi(F, \nabla F, \nabla\nabla F, \ldots) = \tilde{\psi}(E, \nabla E, \nabla\nabla E, \ldots)$$

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Every third order tensor, say $E_{ij,k}$, symmetric with respect to its first two indices can be decomposed:

$$E_{ij,k} = \widetilde{E}_{ijk} + \frac{1}{3}\left(\varepsilon_{jkl}\,\widehat{E}_{li} + \varepsilon_{ikl}\,\widehat{E}_{lj}\right),$$

where

$$\widetilde{E}_{ijk} = \frac{E_{ij,k} + E_{jk,i} + E_{ki,j}}{3}, \quad \text{completely symmetric (10 comp.)}$$

$$\widehat{E}_{li} = \varepsilon_{ljk}\,E_{ij,k}, \quad \text{deviatoric (8 comp.)}$$

For infinitesimal deformations, the strain gradient components $\widehat{E}$:

$$\widehat{E}_{li} \equiv \omega_{l,i}, \quad \omega = \operatorname{curl} u,$$

These are the only strain-gradient components used in "couple-stress" or Cosserat theories where $\psi = \psi(E, \operatorname{grad}\omega)$.

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Corresponding to $\psi(E, \nabla E)$, one obtains

$$S_{ij} = \frac{\partial \psi}{\partial E_{ij}}, \qquad P_{ijk} = \frac{\partial \psi}{\partial E_{ij,k}},$$

as the second Piola-Kirchhoff stress and hyperstress.

The associated internal working

$$W_{\text{int}} = \int_{\mathcal{D}} \left(S_{ij}\, \dot{E}_{ij} + P_{ijh}\, \dot{E}_{ij,h} \right) = \ldots$$

corresponds, integrating by parts, to model more refined contact actions:

$$\ldots = \int_{\mathcal{D}^*} b_\alpha \dot{\chi}_\alpha + \int_{\partial\mathcal{D}^*} t_\alpha \dot{\chi}_\alpha + \int_{\partial\mathcal{D}^*} \tau_\alpha \frac{\partial \dot{\chi}_\alpha}{\partial n} + \int_{\partial\partial\mathcal{D}^*} f_\alpha \dot{\chi}_\alpha = W_{\text{ext}}$$

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Principle of Virtual Working

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The resulting balance equations are expressed as (Germain, 1973):

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\begin{aligned}
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F_{\alpha i}\,(S_{ij} - P_{ijk,k})\,n_j - \left(Q_{Bj}F_{\alpha i}P_{ijk}n_k\right)_{,B} &= J_S t_\alpha, && \text{on } \partial\mathcal{D},\\
F_{\alpha i}P_{ijk}n_k n_j &= J_S \tau_\alpha && \text{on } \partial\mathcal{D},\\
[\![Q_{Bj}F_{\alpha i}P_{ijk}n_k\nu_B]\!] &= J_L f_\alpha && \text{on } \partial\partial\mathcal{D},
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The quantity:

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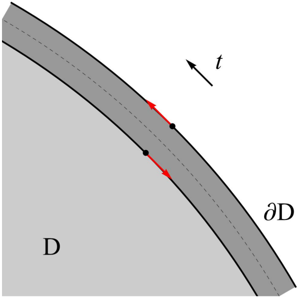
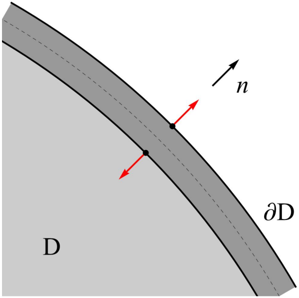
The boundary is actually composed by two-layers:

$$\int_{\partial\mathcal{D}^*} \tau_\alpha \frac{\partial \dot{\chi}_\alpha}{\partial n} = \int_{\partial\mathcal{D}^*} \tau_t \frac{\partial \dot{\chi}_t}{\partial n} + \int_{\partial\mathcal{D}^*} \tau_n \frac{\partial \dot{\chi}_n}{\partial n}$$



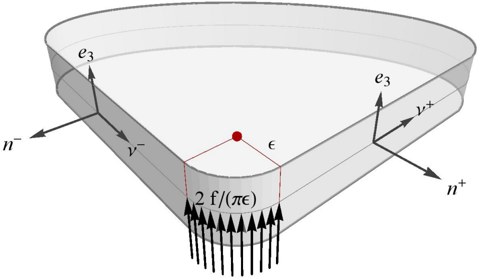
"Couple-stress" theories
Sokolowski(1970)



Complete strain-gradient theories
Germain (1973)

... with possible edges $\partial\partial\mathcal{D}^*$

$$W_{\text{ext}} = \dots + \int_{\partial\partial\mathcal{D}^*} f_\alpha \dot{\chi}_\alpha$$



For instance:

$$f_i = [\![P_{iBk} n_k \nu_B]\!] = \\ = P_{iBk}(n_k^+ \nu_B^+ - n_k^- \nu_B^-)$$

with $n^\pm$ and $\nu^\pm$ the normal and the Darboux tangent-normal vectors.

The only hyperstresses which does not imply edge tractions are in the form:

$$\mathbf{P} = \pi \otimes \mathbf{I}, \qquad P_{ijk} = \pi_i\, \delta_{jm}\, \delta_{mk},$$

for some vector field $\pi = \pi(E, \nabla E)$. (PodioGuidugli&Vianello, 2010)

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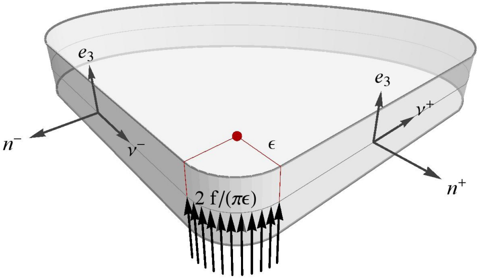

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The simplest case is to assume $\psi(E, \nabla E)$ to be a quadratic form (neoHookean).
The Piola-Kirchhoff stress and hyperstress are then linear in E and ∇E:

$$S_{ij} = \frac{\partial \psi(E, \nabla E)}{\partial E_{ij}} = \mathbb{C}_{ijhk} E_{hk} + \mathbb{H}_{ijhkm} E_{hk,m},$$

$$P_{ijh} = \frac{\partial \psi(E, \nabla E)}{\partial E_{ij,h}} = \mathbb{H}_{ijhkm} E_{km} + \mathbb{G}_{ijhkmn} E_{km,n}.$$

Using results of (Suiker&Chang, 2000) for high-order isotropic tensors, we obtain:

$$\begin{aligned}
\mathbb{C}_{ijkl} = \quad & \lambda\, \delta_{ij}\delta_{kl} + \mu\, (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad \mathbb{H}_{ijklp} = 0 \\
\mathbb{G}_{ijklpq} = \quad & c_2\, (\delta_{ij}\delta_{kl}\delta_{pq} + \delta_{ij}\delta_{kp}\delta_{lq} + \delta_{ik}\delta_{jq}\delta_{lp} + \delta_{iq}\delta_{jk}\delta_{lp}) + \\
& c_3\, (\delta_{ij}\delta_{kq}\delta_{lp}) + c_5\, (\delta_{ik}\delta_{jl}\delta_{pq} + \delta_{ik}\delta_{jp}\delta_{lq} + \delta_{il}\delta_{jk}\delta_{pq} + \delta_{ip}\delta_{jk}\delta_{lq}) + \\
& c_{11}\, (\delta_{il}\delta_{jp}\delta_{kq} + \delta_{ip}\delta_{jl}\delta_{kq}) + \\
& c_{15}\, (\delta_{il}\delta_{jq}\delta_{kp} + \delta_{ip}\delta_{jq}\delta_{kl} + \delta_{iq}\delta_{jl}\delta_{kp} + \delta_{iq}\delta_{jp}\delta_{kl})
\end{aligned}$$

The simplest case is to assume $\psi(E, \nabla E)$ to be a quadratic form (neoHookean). The Piola-Kirchhoff stress and hyperstress are then linear in E and ∇E:

$$S_{ij} = \frac{\partial \psi(E, \nabla E)}{\partial E_{ij}} = \mathbb{C}_{ijhk} E_{hk} + \mathbb{H}_{ijhkm} E_{hk,m},$$

$$P_{ijh} = \frac{\partial \psi(E, \nabla E)}{\partial E_{ij,h}} = \mathbb{H}_{ijhkm} E_{km} + \mathbb{G}_{ijhkmn} E_{km,n}.$$

Using results of (Suiker&Chang, 2000) for high-order isotropic tensors, we obtain:

$$\mathbb{C}_{ijkl} = \lambda\, \delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \quad \mathbb{H}_{ijklp} = 0$$

$$\begin{aligned}\mathbb{G}_{ijklpq} = {} & c_2\left(\delta_{ij}\delta_{kl}\delta_{pq} + \delta_{ij}\delta_{kp}\delta_{lq} + \delta_{ik}\delta_{jq}\delta_{lp} + \delta_{iq}\delta_{jk}\delta_{lp}\right) + \\ & c_3\left(\delta_{ij}\delta_{kq}\delta_{lp}\right) + c_5\left(\delta_{ik}\delta_{jl}\delta_{pq} + \delta_{ik}\delta_{jp}\delta_{lq} + \delta_{il}\delta_{jk}\delta_{pq} + \delta_{ip}\delta_{jk}\delta_{lq}\right) + \\ & c_{11}\left(\delta_{il}\delta_{jp}\delta_{kq} + \delta_{ip}\delta_{jl}\delta_{kq}\right) + \\ & c_{15}\left(\delta_{il}\delta_{jq}\delta_{kp} + \delta_{ip}\delta_{jq}\delta_{kl} + \delta_{iq}\delta_{jl}\delta_{kp} + \delta_{iq}\delta_{jp}\delta_{kl}\right)\end{aligned}$$

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& c_3 \left(\delta_{ij}\delta_{kq}\delta_{lp}\right) + c_5 \left(\delta_{ik}\delta_{jl}\delta_{pq} + \delta_{ik}\delta_{jp}\delta_{lq} + \delta_{il}\delta_{jk}\delta_{pq} + \delta_{ip}\delta_{jk}\delta_{lq}\right) + \\
& c_{11} \left(\delta_{il}\delta_{jp}\delta_{kq} + \delta_{ip}\delta_{jl}\delta_{kq}\right) + \\
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For the energy ψ to be convex in both its arguments, since $\mathbb{H} = 0$ then, as usual:

$$\mu > 0, \qquad 3\lambda + 2\mu > 0.$$

The conditions for the c_2, c_3, c_5, c_{11} and c_{15} can be computed by using the Sylvester criterion[1] on the Voigth representation of $\mathbb{G}_{ijklpq}$, a 18×18 matrix.

A suitable decomposition of P and ∇E:

$$P = \{\widetilde{P}_{111}, \widetilde{P}_{122} + \widetilde{P}_{133}, \hat{P}_{32} - \hat{P}_{23}, ..., \widetilde{P}_{123}\},$$
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reduces to 3 the maximal dimension of the constitutively coupled blocks and renders the application of the Sylvester criterion feasible.

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$$\begin{pmatrix} \widetilde{P}_{111} \\ \widetilde{P}_{122} + \widetilde{P}_{133} \\ \widehat{P}_{32} - \widehat{P}_{23} \end{pmatrix} = \begin{pmatrix} \gamma_1 & 2\gamma_1 - \gamma_2 & \gamma_3 \\ 2\gamma_1 - \gamma_2 & 4\gamma_1 + \gamma_2 & 2\gamma_3 \\ \gamma_3 & 2\gamma_3 & \gamma_4 \end{pmatrix} \begin{pmatrix} \widetilde{E}_{111} \\ \widetilde{E}_{122} + \widetilde{E}_{133} \\ \widehat{E}_{32} - \widehat{E}_{23} \end{pmatrix}, \dots$$

$$\begin{pmatrix} \widehat{P}_{11} \\ \widehat{P}_{22} \\ \widehat{P}_{33} \end{pmatrix} = \gamma_5 \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} \widehat{E}_{11} \\ \widehat{E}_{22} \\ \widehat{E}_{33} \end{pmatrix}, \quad \widetilde{P}_{123} = 3\gamma_2 \widetilde{E}_{123},$$

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where $\gamma_1 = 2\,(c_{11} + 2c_{15}) + 4c_2 + c_3 + 4c_5$, $\gamma_2 = 4\,(c_{11} + 2c_{15})$, $\gamma_3 = \frac{2}{3}(4c_5 - 2c_2 - 2c_3)$, $\gamma_4 = \frac{8}{9}(3c_{11} - 3c_{15} - 4c_2 + 2c_3 + 2c_5)$, $\gamma_5 = \frac{4}{9}(c_{11} - c_{15})$.

These reduce to the "couple stress" relations derived by (Sokolowski, 1970) iff

$$\gamma_1 = \gamma_2 = \gamma_3 = 0, \quad \gamma_4 = 4\ell^2(1-\eta)\,\mu, \quad \gamma_5 = \frac{2}{3}\ell^2(1+\eta)\,\mu,$$

Hyperstress and strain-gradient decompositions

By standard graph theory algorithms (*MinimumBandwidthOrdering*, *MinCut*...)

$$\begin{pmatrix} \widetilde{P}_{111} \\ \widetilde{P}_{122}+\widetilde{P}_{133} \\ \widehat{P}_{32}-\widehat{P}_{23} \end{pmatrix} = \begin{pmatrix} \gamma_1 & 2\gamma_1-\gamma_2 & \gamma_3 \\ 2\gamma_1-\gamma_2 & 4\gamma_1+\gamma_2 & 2\gamma_3 \\ \gamma_3 & 2\gamma_3 & \gamma_4 \end{pmatrix} \begin{pmatrix} \widetilde{E}_{111} \\ \widetilde{E}_{122}+\widetilde{E}_{133} \\ \widehat{E}_{32}-\widehat{E}_{23} \end{pmatrix}, \ldots$$

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Convexity of stored energy

Finally, $\mathbb{G}$ is positive definite iff (dell'Isola, Sciarra, Vidoli, 2009):

$$\gamma_1 > 0, \quad 0 < \gamma_2 < 5\gamma_1, \quad \gamma_4 > \frac{5\gamma_3^2}{5\gamma_1 - \gamma_2}, \quad \gamma_5 > 0,$$

or in terms of the constitutive parameters c_i:

$$\begin{aligned} & c_{11} > 0, \quad -\frac{c_{11}}{2} < c_{15} < c_{11}, \quad 5c_3 + 4c_{11} > 2c_{15}, \\ & c_5 > \frac{c_3\,(3c_{11} + c_{15}) + 2\left(c_{11}^2 - 5c_2^2 - 6c_{15}c_2 - 2c_{15}^2 + c_{11}\,(2c_2 + c_{15})\right)}{4c_{15} - 10c_3 - 8c_{11}} \end{aligned} \tag{2}$$

Some results of (Unger& Aifantis, 2000) in antiplane second gradient elasticity were obtained assuming $(c_5 + c_{11} + c_{15}) > 0$ which is in contradiction with (2), and corresponds to a strictly concave energy.

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Even in linear isotropic material "couple-stress" are constitutively coupled to completely symmetric deformations!

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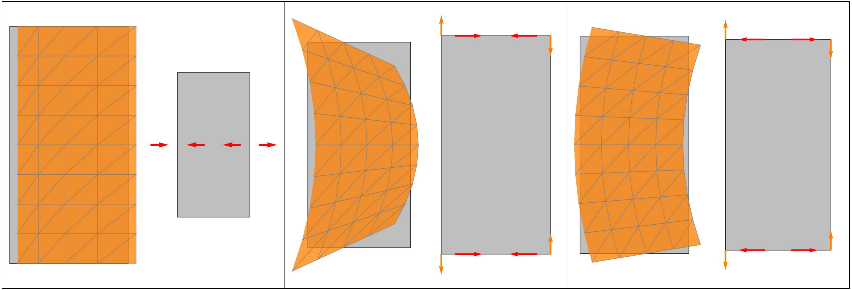

$\widetilde{E}_{111}$ and $\widetilde{P}_{111}$ $\qquad$ $\widetilde{E}_{122}$ and $\widetilde{P}_{122}$ $\qquad$ $\hat{E}_{32}$ and $\hat{P}_{32}$

Under plane strain hypothesis 2D

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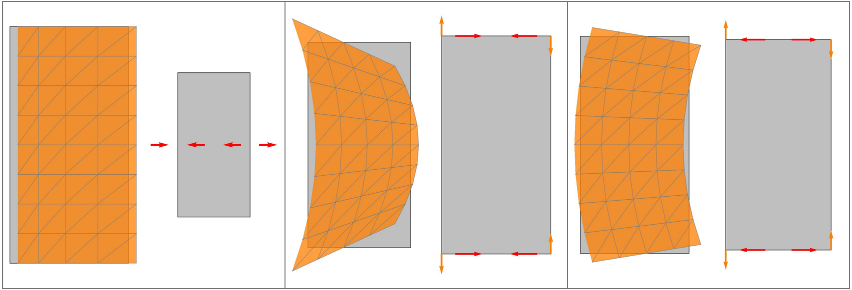

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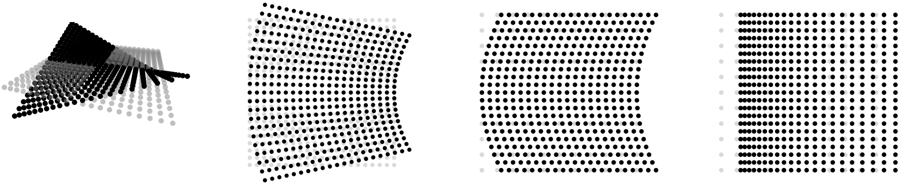
Characteristic lengths $[\gamma_i/\mu] = \ell^2$

For general isotropic strain-gradient materials:

$$\psi = \psi(E_{ij},\ E_{ij,k})$$



$$\ell_t^2 = \frac{2\gamma_2 + 9\gamma_5}{6\mu} \qquad \ell_b^2 = \frac{\gamma_4 + 6\gamma_5}{\mu} \qquad \ell_s^2 = \frac{16\gamma_1 + \ldots + 54\gamma_5}{144\mu} \qquad \ell_e^2 = \frac{\gamma_1}{\mu}$$

$+$ the coupling parameter γ_3

Characteristic lengths $\quad [\gamma_i/\mu] = \ell^2$

For isotropic couple-stress materials:

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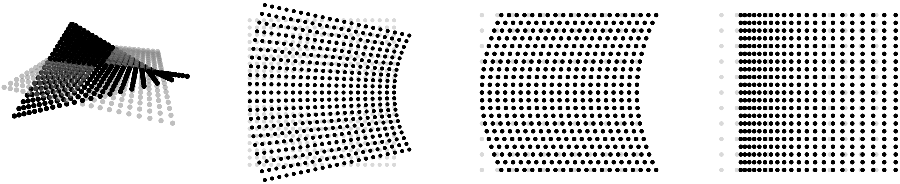


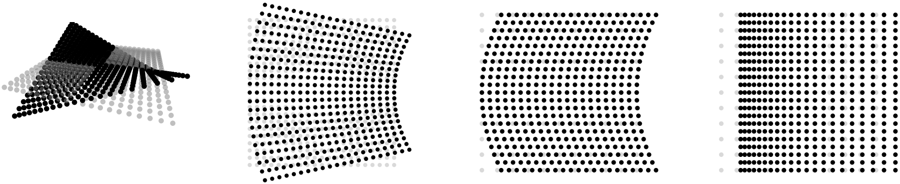
$$\ell_t^2 = \frac{3\gamma_5}{2\mu} \qquad \ell_b^2 = 16\,\ell_s^2 \qquad \ell_s^2 = \frac{\gamma_4 + 6\gamma_5}{16\,\mu} \qquad \ell_e^2 = 0$$

$+$ the coupling parameter $\gamma_3 = 0$

Characteristic lengths $\quad [\gamma_i/\mu] = \ell^2$

For isotropic Lazar-Maugin materials:

$$\psi = \lambda\, E_{ii}E_{jj}/2 + \mu E_{ij}E_{ij} + c\,\lambda\, E_{ii,m}E_{jj,m}/2 + c\,\mu\, E_{ij,m}E_{ij,m}$$



$\ell_t^2 = 2c$ $\qquad$ $\ell_b^2 = \dfrac{16\,c}{9}\,(3+\lambda/\mu)$ $\qquad$ $\ell_s^2 = c$ $\qquad$ $\ell_e^2 = c\,(2+\lambda/\mu)$

+ the coupling parameter $\gamma_3 = -\dfrac{4\lambda\, c}{3\mu}$ $\qquad$ $\left(\dfrac{\lambda}{\mu} = \dfrac{2\nu}{1-2\nu}\right)$

When strain-gradient effects are important?

$$\psi(E,\,\nabla E) = \frac{1}{2}\,\mathbb{C}\,E\cdot E + \frac{1}{2}\,\mathbb{G}\,\nabla E\cdot\nabla E$$

When strain-gradient effects are important?

$$\psi(E, \nabla E) = \quad \psi_1(E) \quad + \frac{1}{2}\, \mathbb{G}\, \nabla E \cdot \nabla E$$

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In which processes $\dfrac{\psi_2(\nabla E)}{\psi_1(E)} \geq 1$?

When strain-gradient effects are important?

A simple one-dimensional analysis call for displacement fields with:

- either **high spatial oscillations**: porous, granular, microstructured materials

$$u = \bar{u}\,\sin(k\,x) \quad \Rightarrow \quad \frac{\psi_2}{\psi_1} \simeq \frac{k^2\|\mathbb{G}\|}{\|\mathbb{C}\|} = (k\,\ell)^2$$

- or **localization phenomena**: stress concentrations, fracture, boundary layers

$$u = \bar{u}\,(x - x_0)^\alpha \quad \Rightarrow \quad \frac{\psi_2}{\psi_1} \simeq \frac{(\alpha-1)^2\|\mathbb{G}\|}{(x-x_0)^2\|\mathbb{C}\|} = \frac{(\alpha-1)^2\ell^2}{(x-x_0)^2}$$

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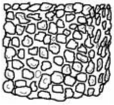
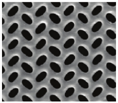

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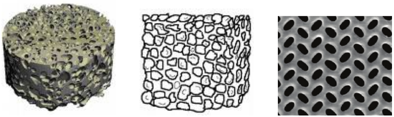

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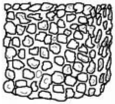
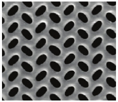

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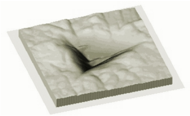


When "couple stress" are sufficient?

- Torsion of a prismatic strain-gradient cylinder; for a circular section the torsional stiffness is:

$$K_t = \mu I_P + \mu \, \ell_t^2 \, A = \mu \, A \left(\frac{R^2}{2} + \ell_t^2 \right),$$

being R the cross-section radius and ℓ_t the torsional characteristic length.

- (Radi, 2008) solved the mode III fracture under the antiplane hypothesis using the standard couple-stress theory

$$\mu \, \Delta w - \mu \, \ell^2 \, \Delta\Delta w = 0$$

His results are extended to the complete second-gradient case by simply renaming the constants: all relevant effects are already included.

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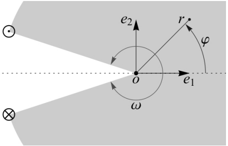

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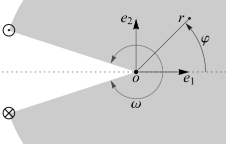

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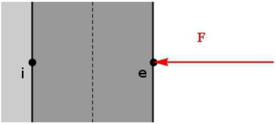
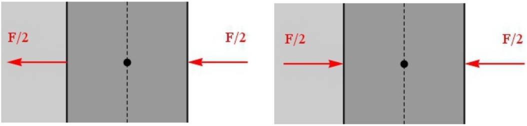
When "couple stress" are not sufficient?

The Flamant-Boussinesq problem: a simple contact problem at the micro scale

$$F \cdot v_e = F \cdot \frac{v_e + v_i}{2} + F \cdot \frac{v_e - v_i}{2} =$$





$$= F \cdot \dot{\chi} \qquad + \qquad \frac{F}{2} \frac{\partial \dot{\chi}_n}{\partial n} \ell$$

Membrane piercing, material nanoindentation, surface tension problems...

When "couple stress" are not sufficient?

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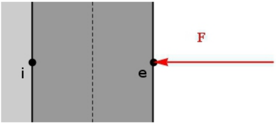

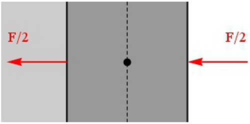

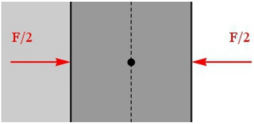

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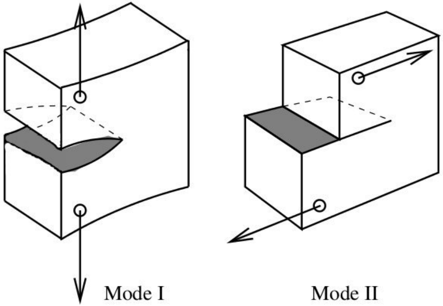
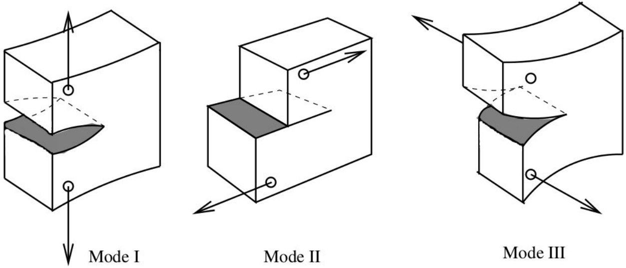


The three classical opening modes for a general material with ℓ_t, ℓ_b, ℓ_s, ℓ_e and γ_3



For Lazar-Maugin materials
(Grentzelou & Georgiadis, JMPS 2009)

For couple-stress materials
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Griffith laws

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GRIFFITH LAWS

1. $L \overset{t}{\nearrow}$: the crack can only grow;
2. $-\dfrac{\partial \mathcal{E}}{\partial L}(t, L(t)) \leq G$: the energy release rate is bounded from above by the toughness G;
3. $\left(\dfrac{\partial \mathcal{E}}{\partial L}(t, L(t)) + G \right) \dot{L} = 0$: the crack will not grow unless the energy release rate is critical.

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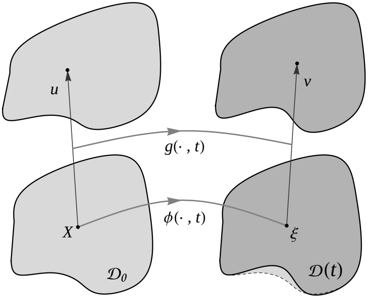

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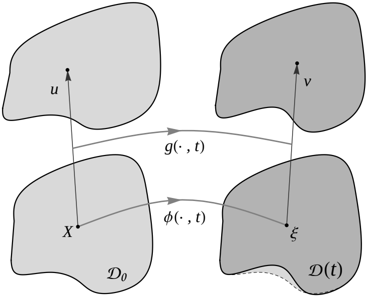

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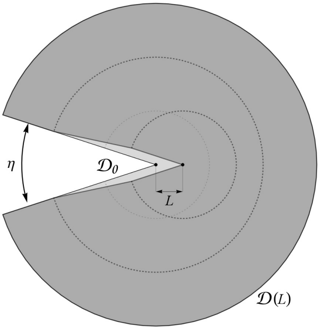
Since $\psi = \psi(E, \nabla E)$ one obtains:

$$\dot{\mathcal{E}} = \left.\frac{\partial \mathcal{E}}{\partial t}\right|_{t \to 0} = \int_{\partial \mathcal{D}} \left[\psi \, \dot{\phi}_l \, n_l + t_l \left(\dot{g}_l - u_{l,m} \dot{\phi}_m \right) + \tau_l \left(\dot{g}_l - u_{l,m} \dot{\phi}_m \right)_{,q} n_q \right] + \int_{\partial \partial \mathcal{D}} \left[f_l \left(\dot{g}_l - u_{l,m} \dot{\phi}_m \right) \right]$$

Energy Release Rate for crack-opening

For a map ϕ to describe crack opening choose:

$$\phi(X,\ L) = \begin{cases} X + L\, e_1, & \text{if } \|X - o\| \leq r_0, \\ X + \left(1 - \dfrac{\|X - o\| - r_0}{r_1(L) - r_0}\right) L\, e_1, & \text{if } r_0 < \|X - o\| \leq r_1(L) := r_0 + \alpha\, L \\ X, & \text{if } r_1(L) < \|X - o\| \end{cases}$$



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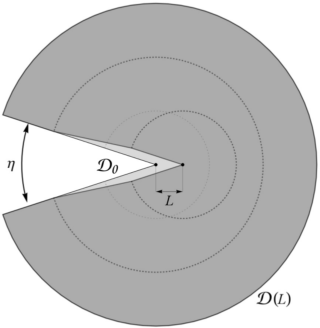
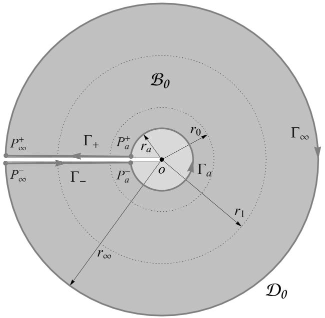


This correspond to:

$$\dot\phi = e_1, \quad \nabla\dot\phi = 0, \quad \text{for } r \le r_1,$$
$$\dot\phi = 0, \quad \nabla\dot\phi = 0, \quad \text{for } r > r_1,$$

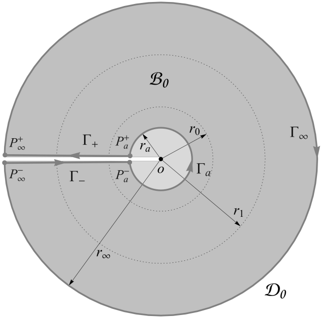
For each $\mathcal{B}_0 \subset \mathcal{D}_0$ with inner radius $r_a < r_0$ and outer radius $r_b > r_1$



Generalized J-integral

For each $\mathcal{B}_0 \subset \mathcal{D}_0$ with inner radius $r_a < r_0$ and outer radius $r_b > r_1$

$$\dot{\mathcal{E}}_{\mathcal{B}_0} = \int_{\Gamma_a} (\psi\, n_1 - t_l\, u_{l,1} - \tau_l\, u_{l,1q}\, n_q) - (f_l\, u_{l,1})\big|_{P_a^+} + (f_l\, u_{l,1})\big|_{P_a^-} =: J(\overline{\Gamma}_a)$$



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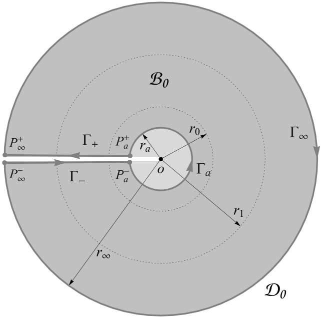

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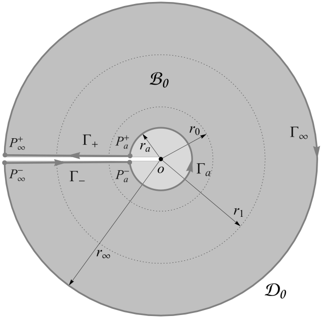

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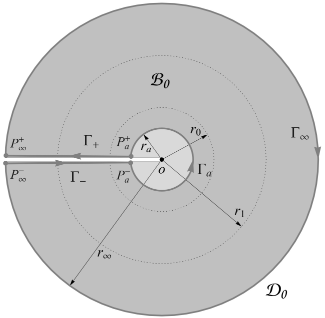

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- The limit is well-posed since: $\dfrac{\partial}{\partial r_a} J(\overline{\Gamma}_a) = 0$ for $r_a < r_0$

$\overline{\Gamma} \neq \Gamma$: does it matter?

For LM materials Gourgiotis & Georgiadis have found the solution of mode I and II

$$
\begin{aligned}
v_r = \quad & r^{\frac{3}{2}} \left[A_1 \left((3-8\nu)\cos\frac{\theta}{2} + \frac{3(16\nu-11)}{32\nu-41}\cos\frac{3\theta}{2} \right) + A_2 \left(\frac{3(11-16\nu)}{32\nu-41}\cos\frac{3\theta}{2} + \cos\frac{5\theta}{2} \right) \right. \\
& \left. + B_1 \sin\frac{\theta}{2} + B_2 \left(\frac{12}{37-32\nu}\sin\frac{\theta}{2} + \frac{3(11-16\nu)}{32\nu-37}\sin\frac{3\theta}{2} + \sin\frac{5\theta}{2} \right) \right] \\
v_\theta = \quad & r^{\frac{3}{2}} \left[A_1 \left((9-8\nu)\sin\frac{\theta}{2} + \frac{3(13-16\nu)}{32\nu-41}\sin\frac{3\theta}{2} \right) + A_2 \left(\frac{3(16\nu-13)}{32\nu-41}\sin\frac{3\theta}{2} - \sin\frac{5\theta}{2} \right) \right. \\
& \left. - B_1 \cos\frac{\theta}{2} + B_2 \left(\frac{3(13-16\nu)}{32\nu-37}\cos\frac{3\theta}{2} + \cos\frac{5\theta}{2} \right) \right]
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with $A_2 = A_1(32\nu - 35)/6$.

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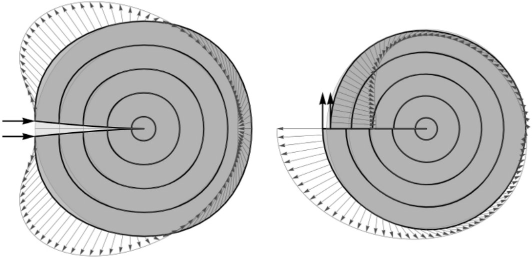

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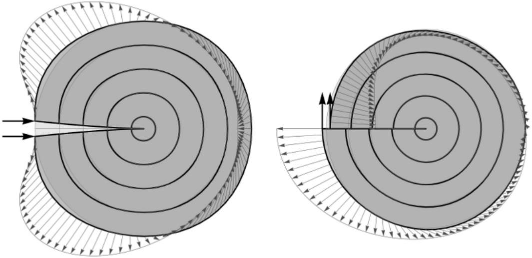


$$J = \begin{pmatrix} d(\nu) & c_1(\nu) & c_2(\nu) \\ c_1(\nu) & e_{11}(\nu) & e_{12}(\nu) \\ c_2(\nu) & e_{12}(\nu) & e_{22}(\nu) \end{pmatrix} \begin{pmatrix} A_1 \\ B_1 \\ B_2 \end{pmatrix} \cdot \begin{pmatrix} A_1 \\ B_1 \\ B_2 \end{pmatrix}$$

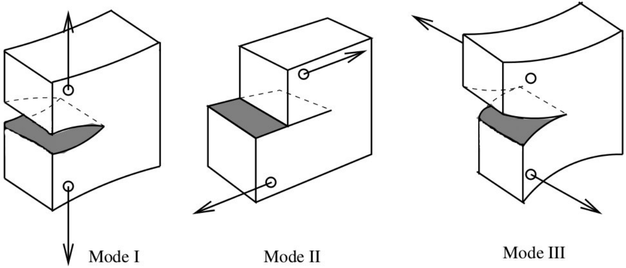
The J-integral is not a quadratic diagonal form of the amplitude factors!
Responsible for the coupling out-of-diagonal terms are the edge-forces contributions.

What we have studied?

Having established the correct expression for the energy release rate:

$$\dot{\mathcal{E}}_{\mathcal{D}_0} = \lim_{r \to 0} \left[\int_\Gamma (\psi\, n_1 - t_l\, u_{l,1} - \tau_l\, u_{l,1q}\, n_q) - (f_l\, u_{l,1})_{P+} + (f_l\, u_{l,1})_{P-} \right]$$

we study the classical opening modes for a general material (ℓ_t, ℓ_b, ℓ_s, ℓ_e, γ_3)



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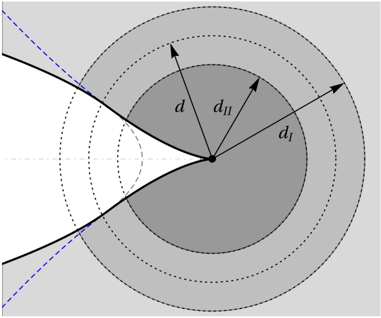
Asymptotic solutions



if $r > d_I$ then $\psi_1(E)$ is dominant

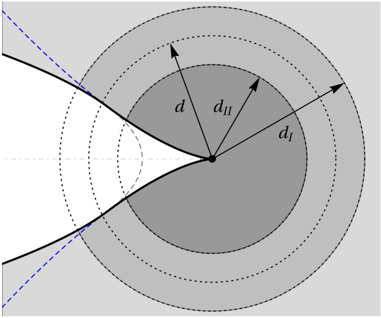
$$u \propto \sqrt{r} \quad \text{(FAR FIELD)}$$

if $r < d_{II}$ then $\psi_2(\nabla E)$ is dominant

$$u \propto r^{3/2}$$

For each mode:

- estimate the distance d at which $\psi_1(E) \simeq \psi_2(\nabla E)$
- describe the detailed solution inside $r < d_{II}$



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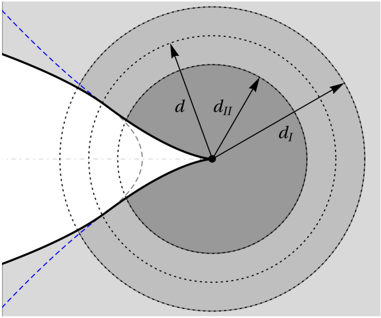

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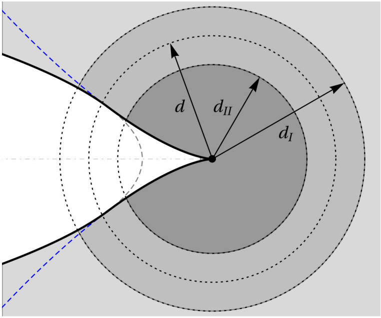

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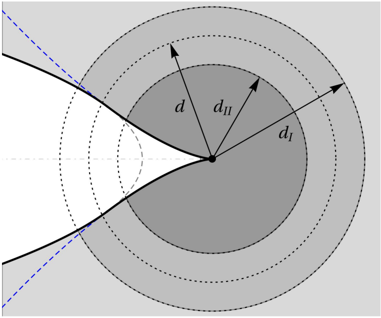

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Bulk equation for antiplane mode: $u = \{0,\, 0,\, w\}$

Bulk equations for plane modes: $u = \{v_r,\, v_\theta,\, 0\}$

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$$\mu\,\Delta w + \mu\,\ell_s^2\,\Delta\Delta w = 0$$

Bulk equations for plane modes: $u = \{v_r,\, v_\theta,\, 0\}$

$$\mu\Delta v + (\lambda+\mu)\,\nabla(\mathrm{div}\,v) - \mu\,\Delta\left[\ell_e^2\,\Delta v + (\ell_e^2 - \ell_s^2)\,\mathrm{curl}\,\mathrm{curl}\,v\right] = 0$$

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Asymptotic solutions are found as:

$$w = r^\alpha W(\theta), \qquad v_r = r^\alpha V_r(\theta), \qquad v_\theta = r^\alpha V_\theta(\theta),$$

solving differential boundary problems for the functions W, V_r, V_θ in $[-\pi, \pi]$:

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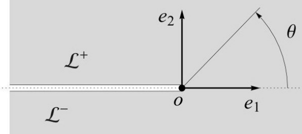

Antiplane mode

The asymptotic strain-gradient solution is:

$$w(r,\,\theta) = C_{I\!I\!I}\, r^{3/2}\,\left(\frac{3\,\sin\theta/2}{16\,(\ell_s/\ell_t)^2 - 3} - \sin\frac{3\theta}{2}\right),$$

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Matching $w(\theta = \pi)$ and $w_f(\theta = \pi)$ and imposing $\psi_1 \simeq \psi_2$ we estimate the radius of validity d/ℓ_s as function of $\ell_t/\ell_s \in [0,\,2]$:

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while the standard far-field solution is $w_f = K_{\mathit{III}}\sqrt{r}\,\sin\dfrac{\theta}{2}$.

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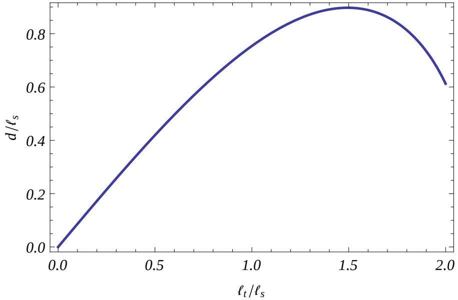

For most materials:

$$d \simeq 0.5\ell_s$$

In agreement with (Radi, IJSS 2008) estimate obtained by Wiener-Hopf technique

Plane strain modes

Imposing only $t = 0$ and $\tau = 0$ on the lips...

...the asymptotic strain-gradient solution is both symmetric:

$$v_r^{\mathrm{I}} = r^{3/2}\left[C_2\left(-2\cos\theta/2 + (2/5 - k_6)\cos 5\theta/2\right) + C_4\left(-k_2\cos 3\theta/2 + k_7\cos 5\theta/2\right)\right],$$

$$v_\theta^{\mathrm{I}} = r^{3/2}\left[C_2\left((2/5 - 6k_6)\sin\theta/2 - (2/5 - k_6)\sin 5\theta/2\right) + \right.$$
$$\left. + C_4\left(k_8\sin\theta/2 - 2\sin 3\theta/2 - k_7\sin 5\theta/2\right)\right],$$

and skew-symmetric:

$$v_r^{\mathrm{II}} = r^{3/2}\left[C_1\left(k_1\sin\theta/2 + k_2\sin 3\theta/2 + k_3\sin 5\theta/2\right) + C_3\left(k_4\sin\theta/2 + k_5\sin 5\theta/2\right)\right],$$

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for $k_1 \ldots\ k_8$ assigned functions of the material characteristic lengths $\ell_e, \ell_b, \ell_s, \gamma_3$.

Two constants for the symmetric and two for the skew-symmetric mode!

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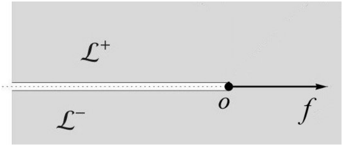
Two constants for the symmetric and two for the skew-symmetric mode!

When evaluating the edge force f in the crack tip o one obtains:

$$f_i = [\![P_{iBk} n_k \nu_B]\!] \quad \Rightarrow \quad f_1 = \alpha\, C_2 + \beta\, C_4,\ f_2 = f_3 = 0$$



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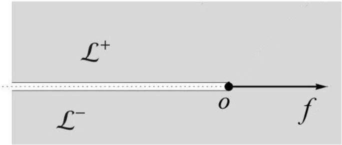

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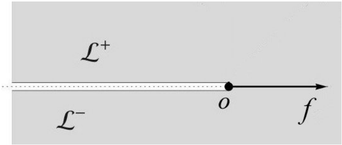

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Matching with the standard far-field solution of mode I

$$u_f = K_I\sqrt{r}\,\{(5-8\nu)\cos(\theta/2) - \cos(3\theta/2),\ (8\nu-7)\sin(\theta/2) + \sin(3\theta/2),\ 0\}$$

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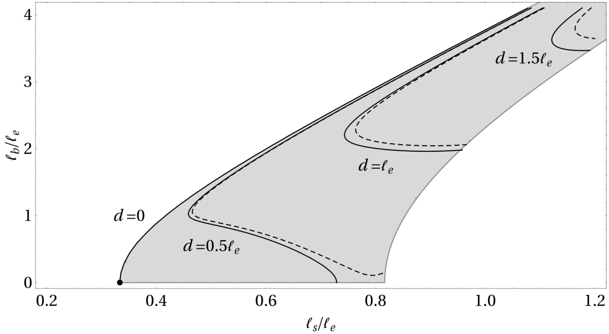
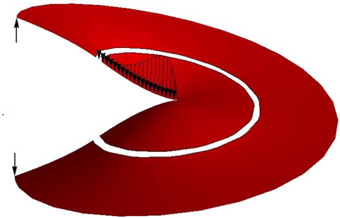
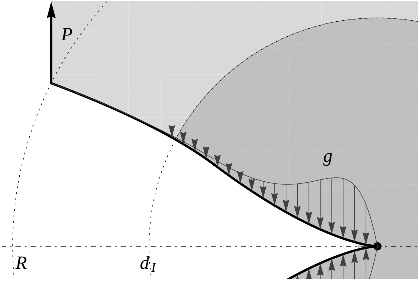
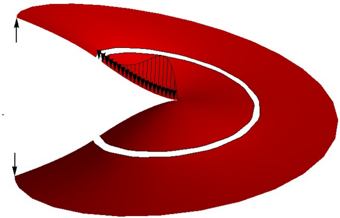
and imposing $\psi_1 \simeq \psi_2$, we estimate the radius d/ℓ_e as function of ℓ_b, ℓ_s and ν

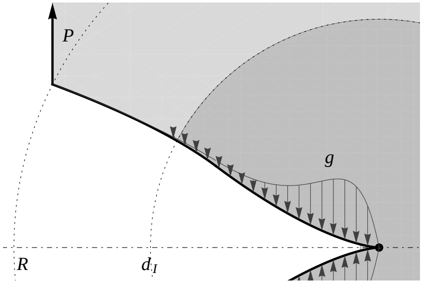


Strain-gradient materials and cohesive forces



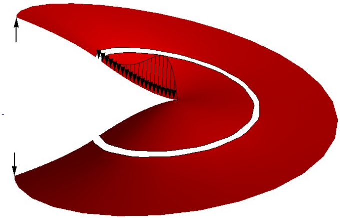


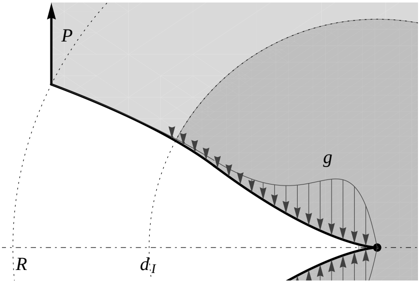




Barenblatt COD solution:

$$w_B(r) = \frac{2}{\pi\mu}\left[\left(\frac{P}{\sqrt{R}} + \int_0^\infty \frac{g(s)}{\sqrt{s}}\,ds\right)\sqrt{r} + \left(\frac{P}{R^{3/2}} + \int_0^\infty \frac{g(s)}{3\,s^{3/2}}\,ds\right) r^{3/2} + \ldots\right]$$

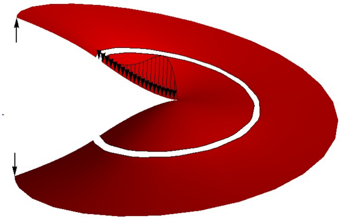


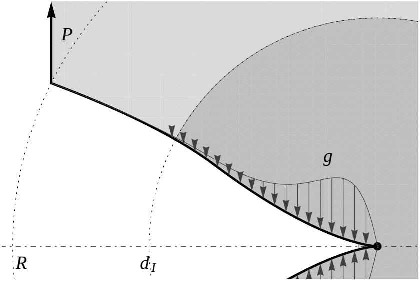


Barenblatt COD solution:

$$w_B(r) = \frac{2}{\pi\mu}\left(-\frac{1}{R}\int_0^\infty \frac{\hat{g}(s)}{\sqrt{s}}\,ds + \int_0^\infty \frac{\hat{g}(s)}{3\,s^{3/2}}\,ds\right) r^{3/2} + \ldots \qquad \text{with } \hat{g}(s) = \hat{g}(P)(s)$$

Strain-gradient materials and cohesive forces





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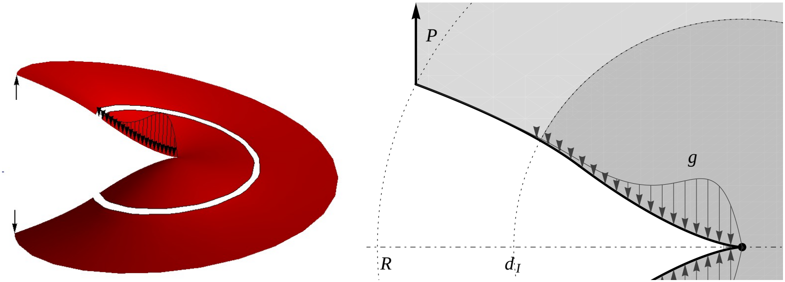

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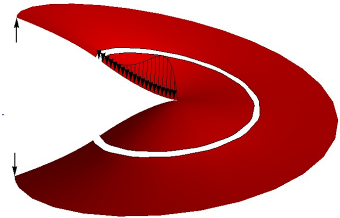
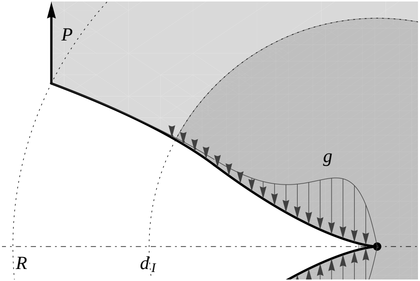
Strain-Gradient COD solution:

$$w_{SG}(r) = \lim_{\theta\to\pi} w = C_{I\!I\!I}\,\frac{32\,\ell_s^2}{3\ell_t^2 - 16\ell_s^2}\; r^{3/2} + \ldots$$





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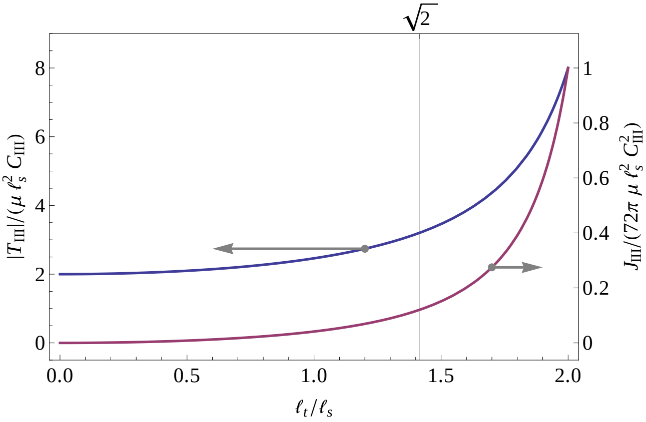
Strain-Gradient materials can be seen as "equivalent" to cohesive forces ...

Equivalent cohesive force and energy release for antiplane mode

$$\frac{2}{3\pi\mu}\left(\int_0^\infty \frac{g_3(s)}{s^{3/2}}\,ds\right) = C_{\mathit{III}}\,\frac{32\,\ell_s^2}{3\ell_t^2-16\ell_s^2} =: T_{\mathit{III}} \qquad J_{\mathit{III}} = \frac{72\pi\;\ell_s^2\,\ell_t^2\;\left(8\,\ell_s^2-\ell_t^2\right)}{\left(16\,\ell_s^2-3\,\ell_t^2\right)^2}\,C_{\mathit{III}}^2$$



For LM materials $\ell_t/\ell_s = \sqrt{2}$

Equivalent cohesive force and energy release for inplane opening

For purely symmetric far-field data (only $C_I \neq 0$), the equivalent cohesive force $T_I \ldots$

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$$
\begin{aligned}
T_I \; &= 8192\,\mu\,\ell_e^2\,C_I\,\left[\frac{\ell_s^2}{\ell_e^2}\left(2304\frac{\ell_s^4}{\ell_e^4} - 32\left(9\frac{\ell_b^2}{\ell_e^2} + 80\right)\frac{\ell_s^2}{\ell_e^2} + \left(3\frac{\ell_b^2}{\ell_e^2} + 16\right)^2\right)\right] / \\
&\left[\left(3\left(45\frac{\ell_b^4}{\ell_e^4} - 32\left(135\frac{\ell_b^2}{\ell_e^2} + 872\right)\frac{\ell_s^4}{\ell_e^4} + 960\frac{\ell_b^2}{\ell_e^2} - 2304\right)\frac{\ell_s^2}{\ell_e^2} + \left(16 - 3\frac{\ell_b^2}{\ell_e^2}\right)^2 + 34560\frac{\ell_s^6}{\ell_e^6}\right) * \right. \\
&\qquad \left. * \left(9\frac{\ell_b^2}{\ell_e^2} + 16\left(1 - 9\frac{\ell_s^2}{\ell_e^2}\right)\right)\right].
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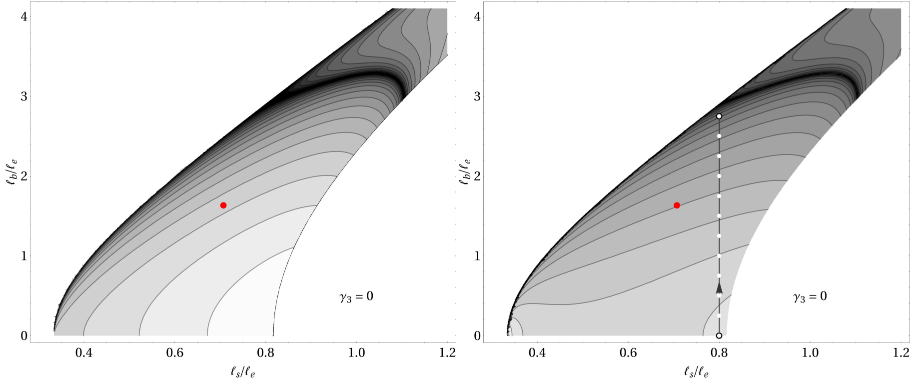
&\left[\left(3\left(45\frac{\ell_b^4}{\ell_e^4}-32\left(135\frac{\ell_b^2}{\ell_e^2}+872\right)\frac{\ell_s^4}{\ell_e^4}+960\frac{\ell_b^2}{\ell_e^2}-2304\right)\frac{\ell_s^2}{\ell_e^2}+\left(16-3\frac{\ell_b^2}{\ell_e^2}\right)^2+34560\frac{\ell_s^6}{\ell_e^6}\right)*\right. \\
&\qquad\left. *\left(9\frac{\ell_b^2}{\ell_e^2}+16\left(1-9\frac{\ell_s^2}{\ell_e^2}\right)\right)\right].
\end{aligned}
$$

... and the energy release rate J_I

$$
\begin{aligned}
J_I \; &= 1152\pi\ell_e^2\ell_s^2\left(3\ell_b^2-8\gamma_3-16\left(\ell_e-3\ell_s\right)\left(\ell_e-\ell_s\right)\right)\left(-3\ell_b^2+8\gamma_3+16\left(\ell_e+\ell_s\right)\left(\ell_e+3\ell_s\right)\right)* \\
&\left(2304\ell_s^6\left(81\ell_b^2-72\gamma_3+896\ell_e^2\right)+16\ell_s^4\left(48\ell_e^2\left(104\gamma_3-261\ell_b^2\right)+27\left(8\gamma_3-3\ell_b^2\right)\left(9\ell_b^2+8\gamma_3\right)+\right.\right. \\
&\quad\left.-71168\ell_e^4\right)+\ell_s^2\left(256\ell_e^4\left(51\ell_b^2+296\gamma_3\right)-32\ell_e^2\left(8\gamma_3-3\ell_b^2\right)\left(33\ell_b^2+8\gamma_3\right)+9\left(8\gamma_3-3\ell_b^2\right)^2*\right. \\
&\quad\left.\left.\left(3\ell_b^2+8\gamma_3\right)+65536\ell_e^6\right)+\ell_e^2\left(3\ell_b^2-8\gamma_3-16\ell_e^2\right)^2\left(9\ell_b^2+24\gamma_3+16\ell_e^2\right)-995328\ell_s^8\right)/ \\
&\left[\left(9\ell_b^2+24\gamma_3+16\left(\ell_e^2-9\ell_s^2\right)\right)\left(32\ell_s^4\left(-135\ell_b^2+360\gamma_3-872\ell_e^2\right)+3\ell_s^2\left(320\ell_e^2\left(3\ell_b^2-8\gamma_3\right)+\right.\right.\right. \\
&\quad\left.\left.\left.+5\left(8\gamma_3-3\ell_b^2\right)^2-2304\ell_e^4\right)+\ell_e^2\left(-3\ell_b^2+8\gamma_3+16\ell_e^2\right)^2+34560\ell_s^6\right)^2\right]
\end{aligned}
$$

Equivalent cohesive force and energy release for inplane opening



For LM materials $T_I = 128\ell_e^2 C_I \dfrac{1-\nu}{9-24\nu}$; for couple-stress $T_I = 0$.

For LM materials $J_I = 128\pi\ell_e^2 \dfrac{(1-2\nu)(7-4\nu)}{(3-8\nu)^2}$; for couple-stress $J_I = 0$.

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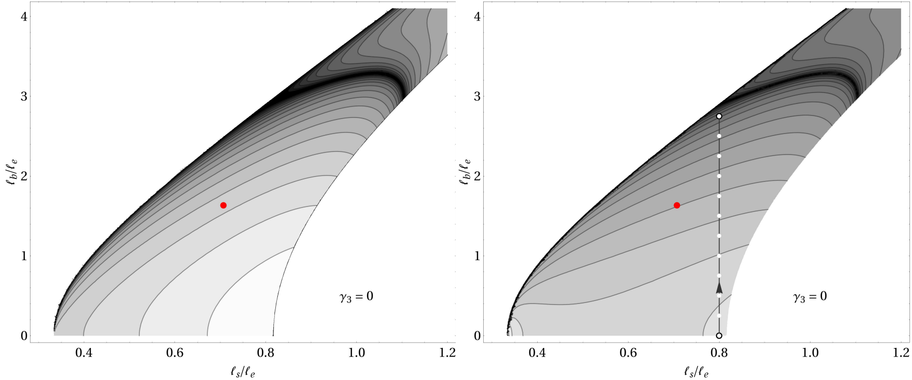

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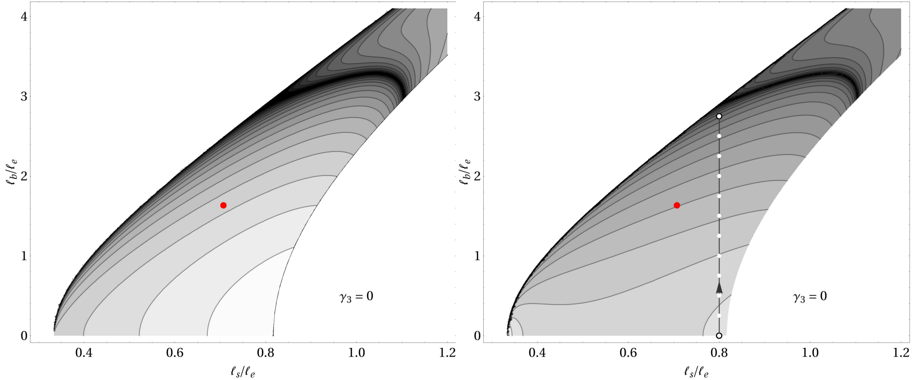

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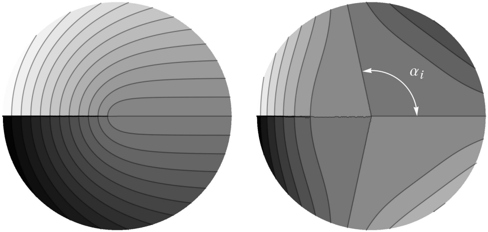
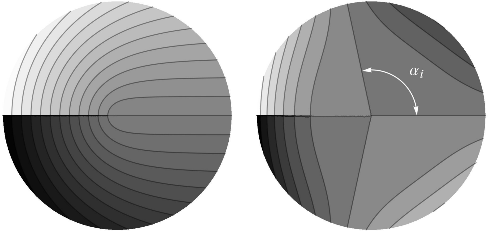
Observable deformations for antiplane mode



$r > d_I > d$ $\qquad$ $r < d_{II} < d$



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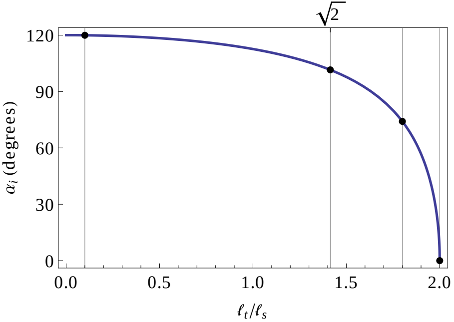
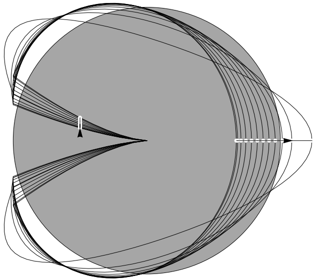
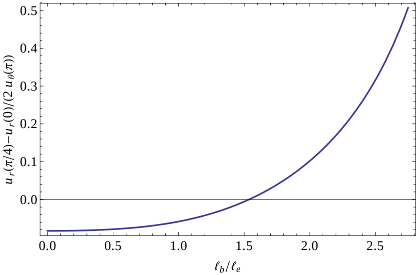


The inversion angle α_i is a monotone function of the ratio ℓ_t/ℓ_s!

Observable deformations for inplane opening

Deformations at critical value of the energy release rate $\left|\dfrac{\partial \mathcal{E}}{\partial L}\right| = \bar{G}$





This "ovalization" effect can be measured; its intensity is a monotone function of ℓ_b/ℓ_e!

Current work and future directions

- Identification from geometry to strain-gradient moduli

$$\Rightarrow \quad (\ell_e, \ell_b, \dots)$$

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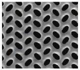

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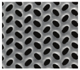

$$\Rightarrow \quad (\ell_e, \ell_b, \ldots)$$

- Cohesive forces and strain-gradient materials (bi- and tri-linear cohesive models)

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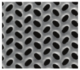
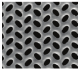


$$\Rightarrow \quad (\ell_e, \ell_b, \dots)$$

- Cohesive forces and strain-gradient materials (bi- and tri-linear cohesive models)
- Numerical implementations (FE, NURBS, dedicated SG elements)

Current work and future directions

- Identification from geometry to strain-gradient moduli



$$\Rightarrow \quad (\ell_e, \ell_b, \ldots)$$

- Cohesive forces and strain-gradient materials (bi- and tri-linear cohesive models)
- Numerical implementations (FE, NURBS, dedicated SG elements)
- Experimental tests to measure the material characteristic lengths

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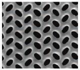

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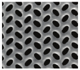

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$$\alpha_i = 2 \cos^{-1} \left(\frac{2}{\sqrt{16 - 3(\ell_t/\ell_s)^2}} \right)$$

Direct measurement of analytical effects

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