

**Università degli Studi di Pavia**  
**Dipartimento di Meccanica Strutturale**

**7th November, 2008**

# **On the solution of the de Saint-Venant's problem and its limitations**

**Giuseppe Vairo**



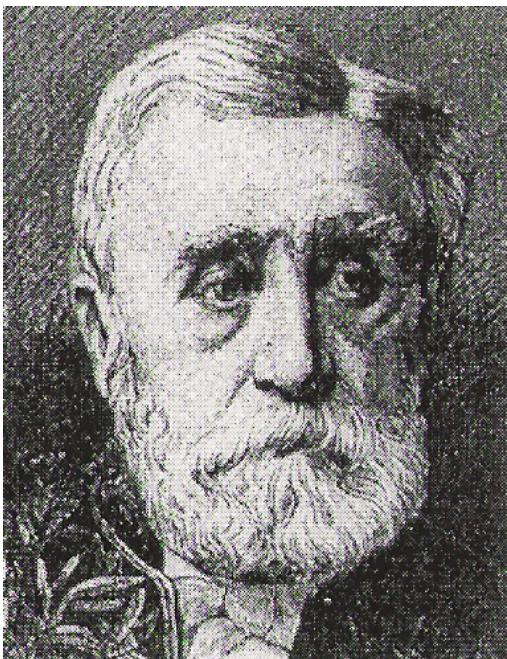
Università degli Studi di Roma “Tor Vergata”  
Dipartimento di Ingegneria Civile

# Adhémar Jean Claude Barrè de Saint Venant (1797-1886)



*an Engineer and Scientist with conservative manners  
which produced progressist results*  
(Benvenuto)

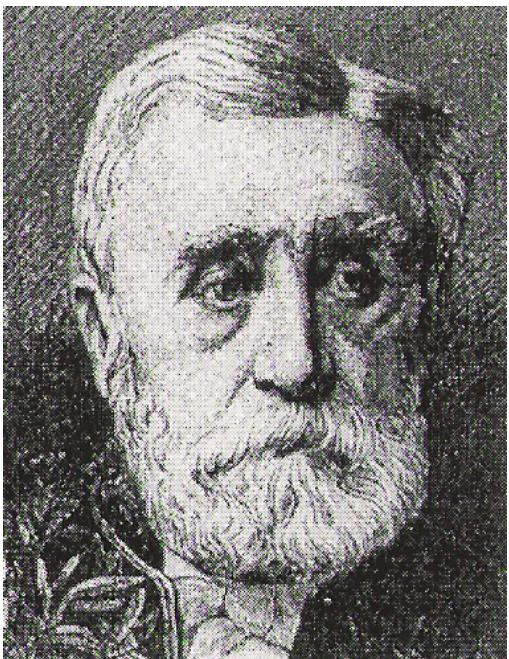
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- 1814:** expelled from the Ecole Polytechnique
  - 1815-1822:** industrial assistant for gunpowder
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  - 1838-1844:** hydraulique period
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*... the use of the mathematics will stop attracting reproaches if one places them in their real limits. The pure calculus is simply a tool... Mechanics add some physical principles, experimentally well-founded, but leave the validation task for the results to particular experiences . Results of mathematics and calculus are not unfailing “oracles”... but they are simply indications, precious, but only indications which reduce the field of the “instinctive evaluation”...*

## Adhémar Jean Claude Barrè de Saint Venant (1797-1886)

De Saint-Venant, A. Barré, Mémoire sur la torsion des prismes. Mémoires des Savants étrangers Acad. Sci. Paris 14 (1855), 223-560.

De Saint-Venant, A. Barré, Mémoire sur la flexion des prismes. J. Math. Liouville 1 (1856), 89-189.

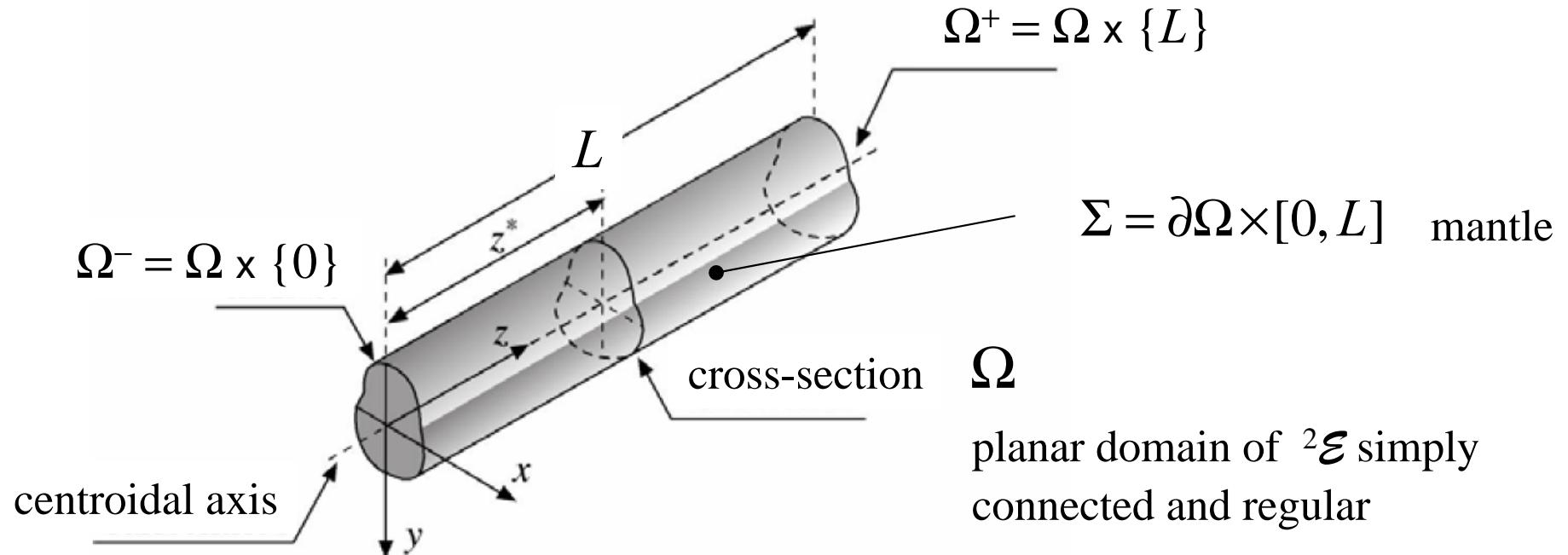
He resolved in a not-properly general way a problem addressed by Lamè (1846-1858): the problem of the elastic equilibrium for right prisms acted upon only by tractions on their end-sections

That solution, although not general, gives a fundamental contribution to engineering technical problems

# Problem statement

Beam

$$\mathcal{B} = \Omega \times [0, L]$$



## Problem statement

- ✓  $\mathcal{B}$  is free (not constrained)
- ✓ it is not acted upon by volume forces and tractions on  $\Sigma$
- ✓ it is acted upon only by tractions on  $\Omega^{+/-}$  (denoted by  $\mathbf{p}^{+/-}$ )

## Problem statement

- ✓  $\mathcal{B}$  is free (not constrained)
- ✓ it is not acted upon by volume forces and tractions on  $\Sigma$
- ✓ it is acted upon only by tractions on  $\Omega^{+/-}$  (denoted by  $\mathbf{p}^{+/-}$ )

- ✓  $\mathcal{B}$  is in equilibrium

$$\int_{\Omega} \mathbf{p}^+ + \mathbf{p}^- da = \mathbf{0}$$
$$\int_{\Omega^+} \mathbf{r} \times \mathbf{p}^+ da + \int_{\Omega^-} \mathbf{r} \times \mathbf{p}^- da = \mathbf{0}$$

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0} \quad \text{in } \mathcal{B}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n}_{\Sigma} = \mathbf{0} \quad \text{su } \Sigma$$

$$\boldsymbol{\sigma} \cdot \mathbf{n}^{+/-} = \mathbf{p}^{+/-} \quad \text{su } \Omega^{+/-}$$

## Problem statement

- ✓ The beam is assumed to be homogeneous and comprising isotropic linearly-elastic material

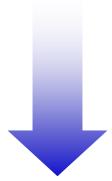
$$\boxed{\boldsymbol{\sigma} = 2G \boldsymbol{\varepsilon} + \lambda \operatorname{tr}(\boldsymbol{\varepsilon}) \mathbf{I}}$$
$$\boxed{\boldsymbol{\varepsilon} = 1/(2G) \boldsymbol{\sigma} - v / E \operatorname{tr}(\boldsymbol{\sigma}) \mathbf{I}}$$

$$G = \frac{E}{2(1+v)}, \quad \lambda = \frac{vE}{(1+v)(1-2v)}$$

- ✓ Compatibility

$$\boxed{\operatorname{Rot} \operatorname{Rot} \boldsymbol{\varepsilon} = \mathbf{0}}$$

Existence



Fichera, ...

Uniqueness



Kirchhoff

**Fichera G.**, Existence Theorem in Elasticity. In: S Flugge (ed.): *Handbuch der Physik*, Bd VIa/2. Springer-Verlag, Berlin 1972.

# The problem with weak boundary conditions

$$\sigma \cdot \mathbf{n}_\Sigma = 0 \quad \text{su } \Sigma$$

~~$$\sigma \cdot \mathbf{n}^{+/-} = p^{+/-} \quad \text{su } \Omega^{+/-}$$~~



$$N^{+/-} = \int_{\Omega} p_z^{+/-} da$$

$$T_x^{+/-} = \int_{\Omega} p_x^{+/-} da$$

$$T_y^{+/-} = \int_{\Omega} p_y^{+/-} da$$

$$M_x^{+/-} = \int_{\Omega} y p_z^{+/-} da$$

$$M_y^{+/-} = - \int_{\Omega} x p_z^{+/-} da$$

$$M_t^{+/-} = \int_{\Omega} (x p_y^{+/-} - y p_x^{+/-}) da$$

generalized internal forces

$$N(z) = \int_{\Omega} \sigma_z da$$

$$T_x(z) = \int_{\Omega} \tau_{xz} da$$

$$T_y(z) = \int_{\Omega} \tau_{yz} da$$

$$M_x(z) = \int_{\Omega} y \sigma_z da$$

$$M_y(z) = - \int_{\Omega} x \sigma_z da$$

$$M_t(z) = \int_{\Omega} (x \tau_{yz} - y \tau_{xz}) da$$

## The problem with weak boundary conditions

$$\begin{aligned}\int_{\Omega} \mathbf{p}^+ + \mathbf{p}^- da &= \mathbf{0} \\ \int_{\Omega^+} \mathbf{r} \times \mathbf{p}^+ da + \int_{\Omega^-} \mathbf{r} \times \mathbf{p}^- da &= \mathbf{0}\end{aligned}$$



$$\begin{aligned}N(z) = N^+ = N^- &= \text{cost}, & T_x(z) = T_x^+ = T_x^- &= \text{cost}, \\ T_y(z) = T_y^+ = T_y^- &= \text{cost}, & M_t(z) = M_t^+ = M_t^- &= \text{cost} \\ M_x(z) = M_x^- + zT_y, & & M_y(z) = M_y^- - zT_x &\end{aligned}$$

## The problem with weak boundary conditions

$$\int_{\Omega} \mathbf{p}^+ + \mathbf{p}^- da = \mathbf{0}$$
$$\int_{\Omega^+} \mathbf{r} \times \mathbf{p}^+ da + \int_{\Omega^-} \mathbf{r} \times \mathbf{p}^- da = \mathbf{0}$$



$$N(z) = N^+ = N^- = \text{cost}, \quad T_x(z) = T_x^+ = T_x^- = \text{cost},$$
$$T_y(z) = T_y^+ = T_y^- = \text{cost}, \quad M_t(z) = M_t^+ = M_t^- = \text{cost}$$
$$M_x(z) = M_x^- + zT_y, \quad M_y(z) = M_y^- - zT_x$$

There exist infinite traction distributions  $\mathbf{p}^{+/-}$  with the same resultants, therefore the problem with weak boundary conditions have not a unique solution

A solution for the problem with weak boundary conditions is the unique solution for the primary problem if and only if internal stress distributions satisfy local limit equilibrium on  $\Omega^{+/-}$ : in general this is not verified.

Nevertheless, a solution of the problem with weak boundary conditions can be seen as an approximate solution for the primary problem, because of the de Saint-Venant's principle.

# de Saint-Venant's principle

Let the right cylinder  $\mathcal{B}$  be loaded only on the end-section  $\Omega^+$  (that is  $\mathbf{p}^- = \mathbf{0}$ ) with a self-equilibrated distribution (that is  $N^+ = T^+ = M^+ = 0$ ). Let  $\mathbf{p}^+$  be sufficiently regular (e.g., square summable on  $\Omega$ ). Let  $\boldsymbol{\sigma}$  and  $\boldsymbol{\epsilon}$  the solution of the primary de Saint-Venant's problem (local limit equilibrium satisfied on the end-sections). Let be  $\mathcal{B}_z = \Omega \times [0, z]$  and let  $U(z)$  be the strain energy related to  $\mathcal{B}_z$ :

$$U(z) = \int_{\mathcal{B}_z} \frac{1}{2} \boldsymbol{\sigma} \cdot \boldsymbol{\epsilon} \, dv$$

Then, for every constant  $0 < l < L$  there exists a constant  $C > 0$  (generally depending on  $L, \Omega, E, \nu$ ) such that

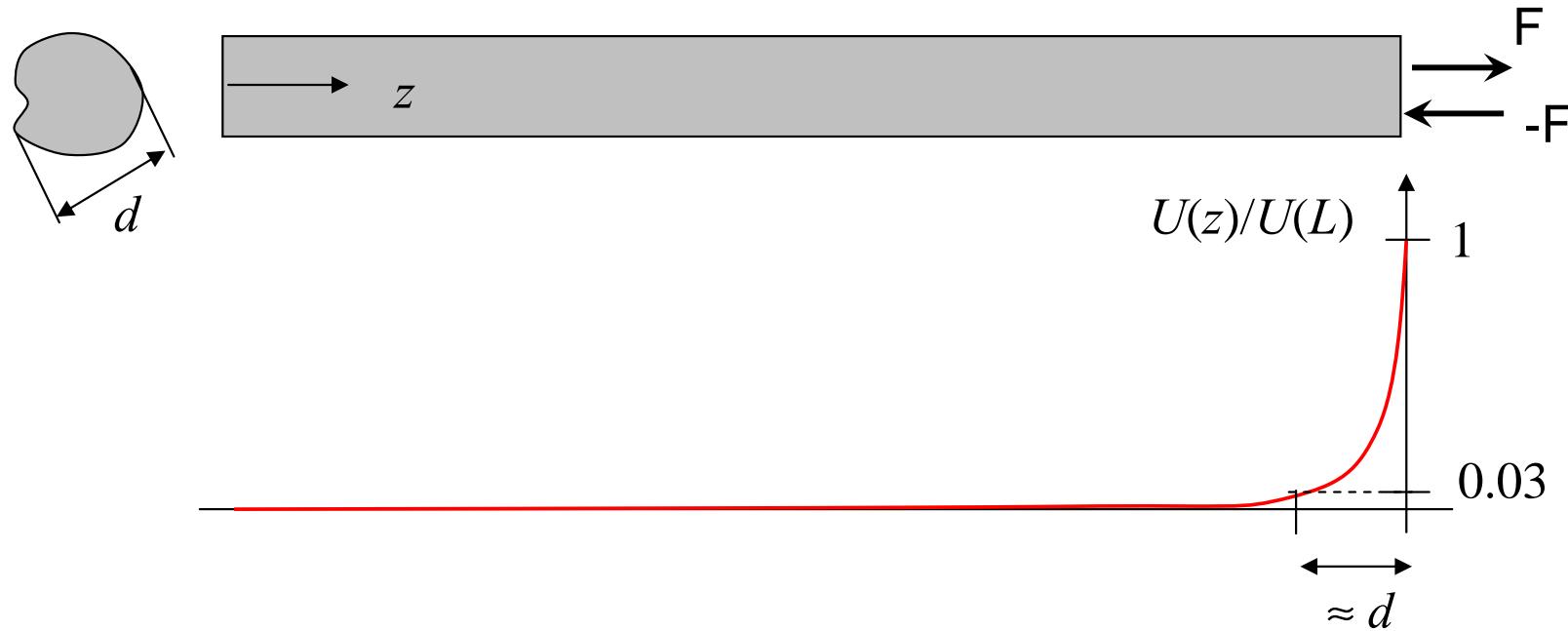
$$U(z) \leq U(L) \exp(-(L - l - z)/C), \quad 0 \leq z \leq L - l$$



In other words: the strain energy for a right cylinder loaded only on one end-sections with a self-equilibrated traction distribution exponentially decays along the beam axis, when the distance from the acted end-section increases.

Toupin R.A.: Saint Venant's principio. Arch. Rat. Mech. Anal. 18 (1965), 83-96.

## de Saint-Venant's principle



for compact cross-sections, an effective decay distance from the loaded end-section has the same of order of magnitude of an equivalent radius for  $\Omega$

# Semi-inverse approach

Hp:

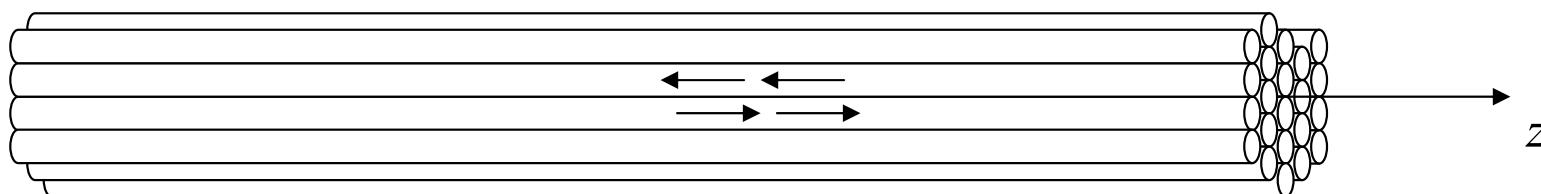
$$\sigma_x = \sigma_y = \tau_{xy} = 0$$



$$[\sigma] = \begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{bmatrix}$$

stress vector on every plane parallel to the beam axis  
is itself parallel to the beam axis

$$(\sigma \cdot \mathbf{n}) \times \hat{\mathbf{k}} = 0, \quad \forall \mathbf{n} : \mathbf{n} \cdot \hat{\mathbf{k}} = 0$$



Clebsch A., Theorie der Elasticitat faster Korper. BG Teubner, Leipzig 1862.

Baldacci R.F., Sull'integrazione diretta del problema di Saint-Venant in termini di tensioni.  
Atti Accad. Scienze Torino 90 (1995-56), 604-610.

## Semi-inverse approach

$$\operatorname{div} \boldsymbol{\sigma} = 0 \quad \text{in } \mathcal{Z}$$



$$\begin{aligned}\boldsymbol{\tau} &= \tau_{zx} \hat{\mathbf{i}} + \tau_{zy} \hat{\mathbf{j}} = \boldsymbol{\tau}(x, y) \quad \text{in } \mathcal{Z} \\ \operatorname{div} \boldsymbol{\tau} + \sigma_{z/z} &= 0\end{aligned}$$

$$\boldsymbol{\sigma} \cdot \mathbf{n}_\Sigma = 0 \quad \text{su } \Sigma$$



$$\boldsymbol{\tau} \cdot \mathbf{n} = \tau_{zx} n_x + \tau_{zy} n_y = 0 \quad \text{su } \partial\Omega$$

$$\operatorname{Rot} \operatorname{Rot} \boldsymbol{\varepsilon} = 0$$

$$\boldsymbol{\varepsilon} = 1/(2G) \boldsymbol{\sigma} - v/E \operatorname{tr}(\boldsymbol{\sigma}) \mathbf{I}$$

$$\begin{aligned}\sigma_{z/xx} &= \sigma_{z/yy} = \sigma_{z/zz} = \sigma_{z/xy} = 0 \\ (\tau_{zy/x} - \tau_{zx/y})_x &= \bar{v} \sigma_{z/yz} \quad \text{in } \mathcal{Z} \\ (\tau_{zy/x} - \tau_{zx/y})_y &= -\bar{v} \sigma_{z/xz}\end{aligned}$$

$$\bar{v} = v/(1+v)$$

## Semi-inverse approach

$$\begin{aligned}\sigma_z &= a + a_1 x + a_2 y - z(b_1 x + b_2 y) = \\ &= \frac{N}{A} - \frac{M_y^- - zT_x}{I_y} x + \frac{M_x^- + zT_y}{I_x} y = \frac{N}{A} - \frac{M_y(z)}{I_y} x + \frac{M_x(z)}{I_x} y\end{aligned}$$

$$\begin{cases} \operatorname{div} \tau = b_1 x + b_2 y & \text{su } \Omega \\ (\operatorname{Rot} \tau) \cdot \hat{\mathbf{k}} = \bar{v}(b_1 y - b_2 x) + c \end{cases}$$

$$\tau \cdot \mathbf{n} = \tau_{zx} n_x + \tau_{zy} n_y = 0 \quad \text{su } \partial\Omega$$

## Semi-inverse approach

$$\begin{cases} \operatorname{div} \boldsymbol{\tau} = b_1x + b_2y \\ (\operatorname{Rot} \boldsymbol{\tau}) \cdot \hat{\mathbf{k}} = \bar{v}(b_1y - b_2x) + c \end{cases} \quad \text{su } \Omega$$

$$\boldsymbol{\tau} \cdot \mathbf{n} = \tau_{zx}n_x + \tau_{zy}n_y = 0 \quad \text{su } \partial\Omega$$



$$\boldsymbol{\tau} = \boldsymbol{\tau}^0 + \bar{\boldsymbol{\tau}}$$

$$\bar{\tau}_{zx} = \frac{1}{2}[b_1(x^2 - \bar{v}y^2) - cy]$$

$$\bar{\tau}_{zy} = \frac{1}{2}[b_2(y^2 - \bar{v}x^2) + cx]$$

$$\boldsymbol{\tau}^0 : \begin{cases} \operatorname{div} \boldsymbol{\tau}^0 = 0 \\ (\operatorname{Rot} \boldsymbol{\tau}^0) \cdot \hat{\mathbf{k}} = 0 \quad \text{su } \Omega \\ \boldsymbol{\tau}^0 \cdot \mathbf{n} = 0 \quad \text{su } \partial\Omega \end{cases}$$

**Baldacci R.F.**, Sull'integrazione diretta del problema di Saint-Venant in termini di tensioni.  
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## Semi-inverse approach

$$\boldsymbol{\tau}^0 : \left\{ \begin{array}{l} \operatorname{div} \boldsymbol{\tau}^0 = 0 \\ (\operatorname{Rot} \boldsymbol{\tau}^0) \cdot \hat{\mathbf{k}} = 0 \\ \boldsymbol{\tau}^0 \cdot \mathbf{n} = 0 \end{array} \right. \begin{array}{l} \text{su } \Omega \\ \text{su } \partial\Omega \end{array}$$

$\exists \Psi(x, y) : \boldsymbol{\tau}^0 = \nabla\Psi$


$$\boxed{\begin{array}{ll} \nabla_2 \Psi = 0 & \text{su } \Omega \\ \nabla\Psi \cdot \mathbf{n} = -\bar{\boldsymbol{\tau}} \cdot \mathbf{n} & \text{su } \partial\Omega \end{array}}$$

well-posed **Neumann-Dini** problem

## Semi-inverse approach

$$\boldsymbol{\tau}^0 : \begin{cases} \operatorname{div} \boldsymbol{\tau}^0 = 0 \\ (\operatorname{Rot} \boldsymbol{\tau}^0) \cdot \hat{\mathbf{k}} = 0 \\ \boldsymbol{\tau}^0 \cdot \mathbf{n} = 0 \end{cases} \quad \begin{matrix} \text{su } \Omega \\ \text{su } \partial\Omega \end{matrix}$$

$$\exists \phi(x, y) : \tau_{zx} = \phi_{/y}, \quad \tau_{zy} = -\phi_{/x}$$

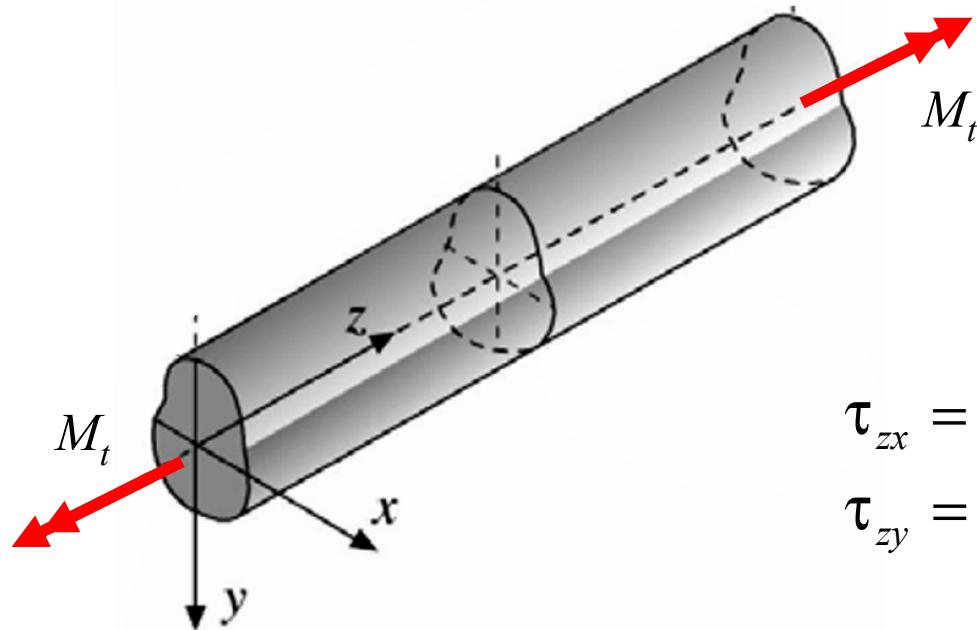


$$\begin{aligned} \nabla_2 \phi &= 0 && \text{su } \Omega \\ \phi &= -\frac{1}{2} b_1 \int_0^s (x^2 - \bar{v} y^2) dy + \frac{1}{2} b_2 \int_0^s (y^2 - \bar{v} x^2) dx + \frac{c}{4} (x^2 + y^2) && \text{su } \partial\Omega \end{aligned}$$

well-posed **Dirichlet** problem

$$\Psi_{/x} = \phi_{/y}, \quad \Psi_{/y} = -\phi_{/x}$$

# An example: pure torsion



$$\sigma_z = 0$$

$$\begin{cases} \operatorname{div} \tau = 0 & \text{su } \Omega \\ (\operatorname{Rot} \tau) \cdot \hat{\mathbf{k}} = c & \\ \tau \cdot \mathbf{n} = 0 & \text{su } \partial\Omega \end{cases}$$

$$\tau_{zx} = G\theta'(\psi_{G/x} - y) = \Psi_{/x} - cy/2$$

$$\tau_{zy} = G\theta'(\psi_{G/y} + x) = \Psi_{/y} + cx/2$$

displacement field

$$s_x = -\theta' zy$$

$$\theta = \theta(z)$$

$$s_y = \theta' zx$$

$$\theta' = \theta_{/z} = \text{cost} \rightarrow \text{unit twisting angle}$$

$$s_z = \theta' \psi_G(x, y)$$

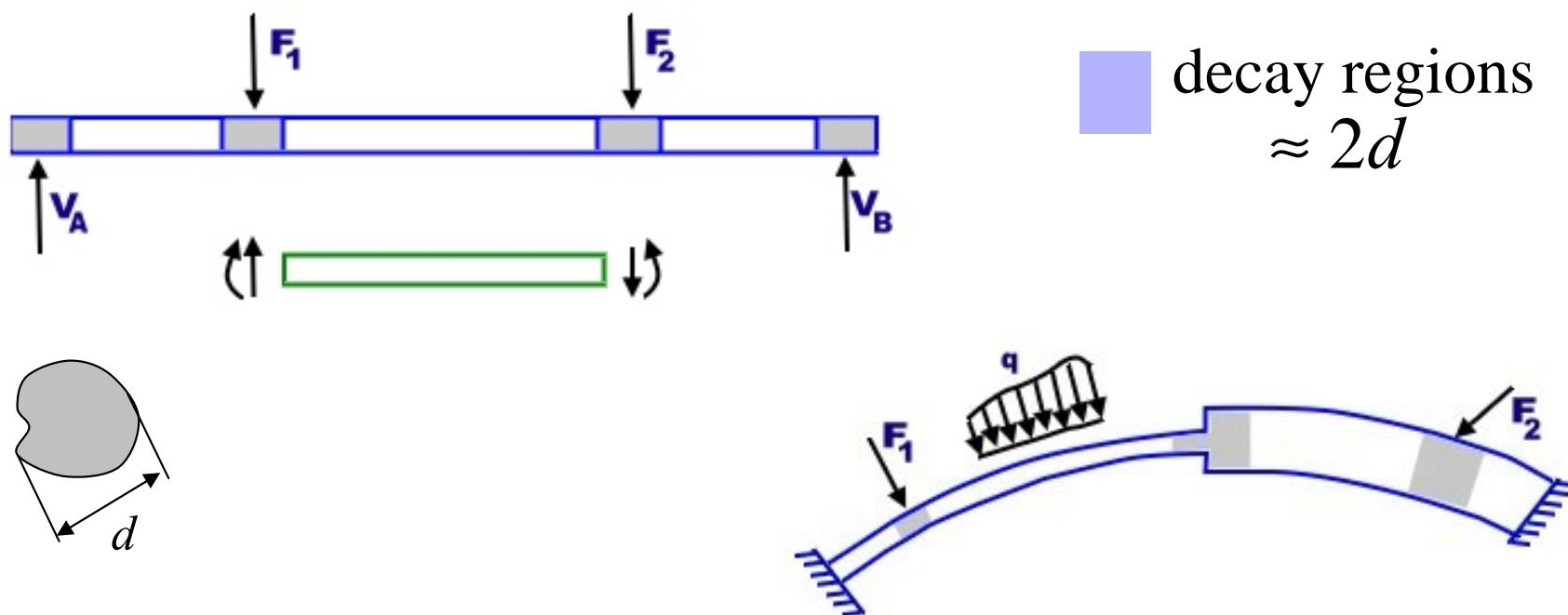


warping function

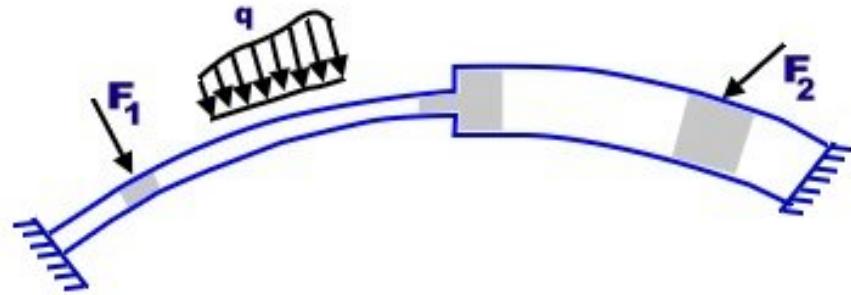
# Limitations and extension to real cases

In real cases:

- constraints
- volume forces
- forces on the mantle and not only on the end-sections (also concentrated)
- the beam cross-section can be not constant along the beam axis
- the beam axis may be curve



## Limitations and extension to real cases

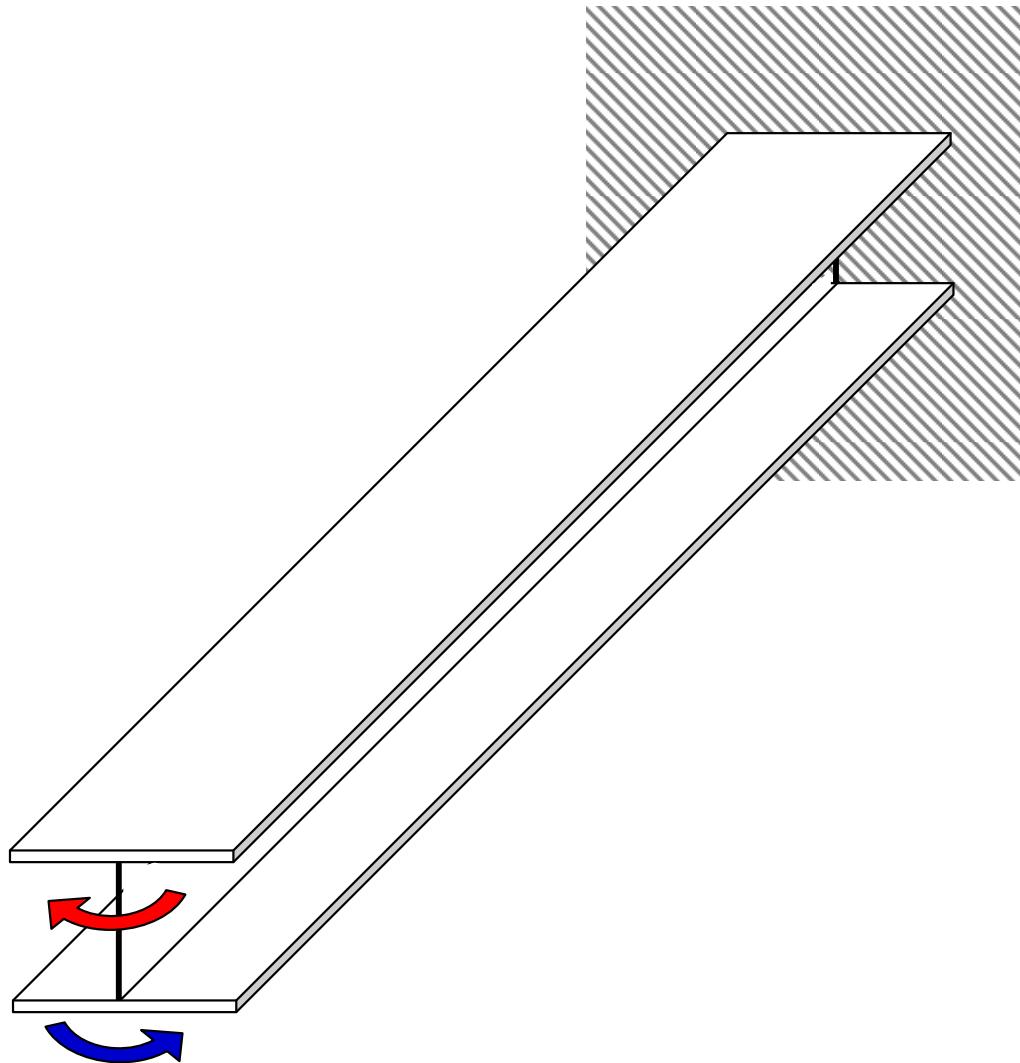


$$\sigma \approx \sigma^{(q)} + \sigma^{(dSV)} \approx \sigma^{(dSV)}$$

Nevertheless, if we consider **thin-walled beams** (especially with open sections), the decay distance may be of the order of  $L$ . That is, the solution of de Saint-Venant is not applicable

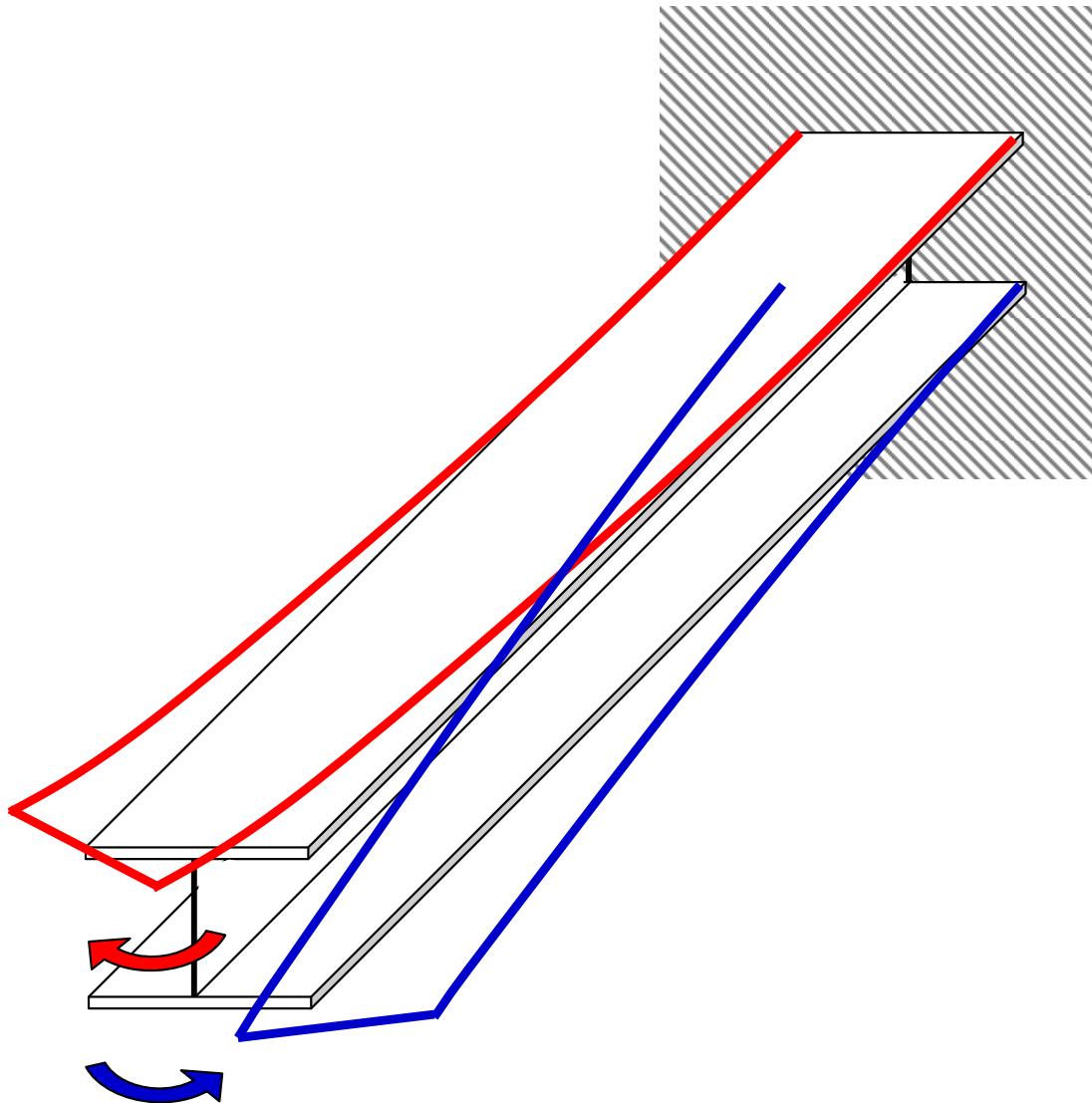
## Limitations and extension to real cases

twin-walled open-section beams



## Limitations and extension to real cases

twin-walled open-section beams



## Limitations and extension to real cases

!

In general, thin-walled beams needs  
treatments different from those  
adopted for classical rods

!



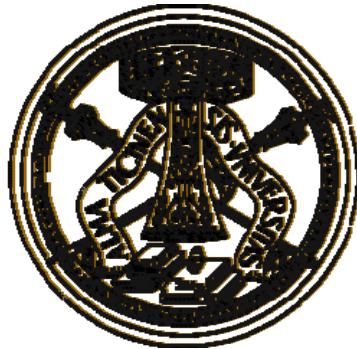
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**7th November, 2008**

**Thank You**



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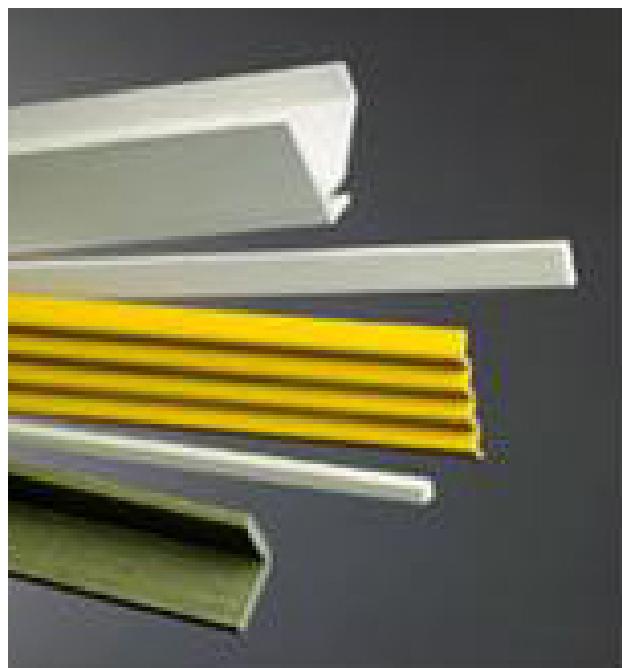
**7th November, 2008**

# **Rational deduction of anisotropic thin-walled beam models from three-dimensional elasticity**

Giuseppe Vairo



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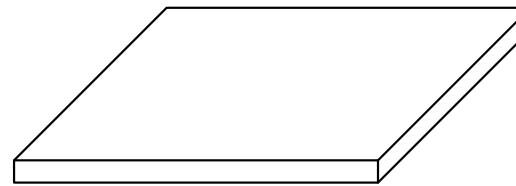
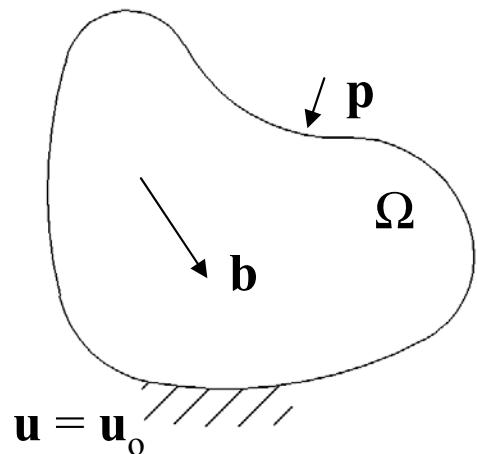
- [1] Kollbrunner CF, Basler K. Torsion in structures: an engineering approach. Springer, Berlin, 1969.
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- [17] Piovan MT, Cortinez VH. Transverse shear deformability in the dynamics of thin-walled composite beams: consistency of different approaches. Journal of Sound and Vibration 285:721-33, 2005.
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The rational deduction and justification of these theories from three-dimensional elasticity and their consistent generalization for anisotropic materials as well as for non-conventional cases (such as laminated beams or unilateral material behaviour) can be truly considered as an open task yet.

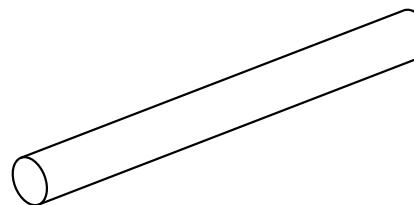
The deduction of thin-walled beam models in a consistent way is not only a speculative issue, but leads to a safer and more complete technical use of these theories.

# Rational deduction of structural theories

3D elastostatic problem



2D model



1D model

- ✓ Asymptotic method
- ✓ Costrained approach

# Asymptotic philosophy

**Main idea:** the **three-dimensional solution of the elasticity equations can be approximated through successive terms of a power series.**

For beams, the slenderness ratio (between diameter of the cross-section and beam length) is taken as a small parameter.

Accordingly, under suitable hypotheses which ensure series convergence, different structural theories can be rationally deduced as **approximate solutions of an exactly-stated problem**, varying series truncation order.

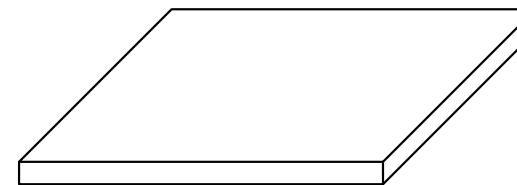
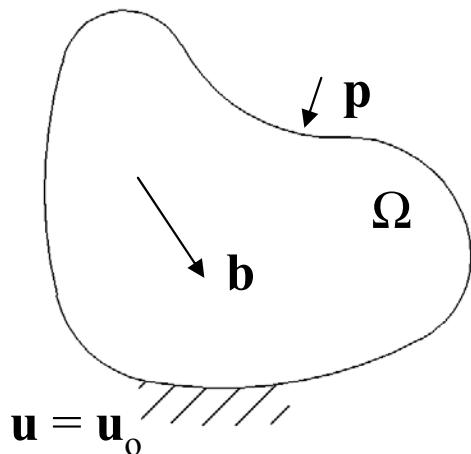
- [19] Trabucho L, Viaño JM. Mathematical modelling of rods. In: Ciarlet PG and Lions JL (eds), Handbook of Numerical Analysis, vol. IV. Elsevier, The Netherlands, pp.487-974, 1996.
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# Constrained philosophy

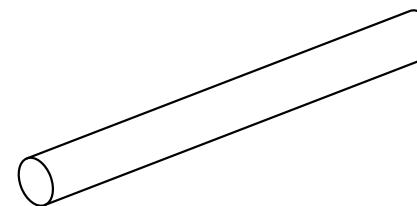
**Main idea:** an exact solution for a simplified constrained problem, i.e. based on approximate representations of the unknown functions, is looked for.

# Constrained elasticity

3D elastostatic problem



2D model



1D model

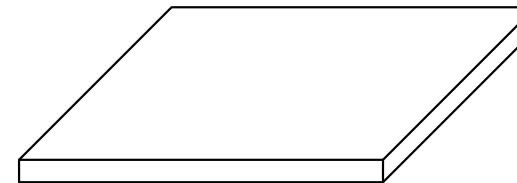
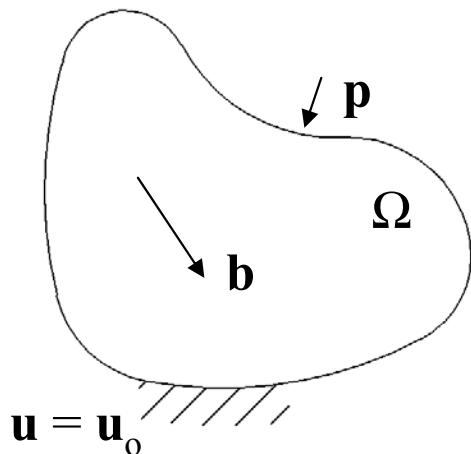
assumptions on strain  
and/or stress fields



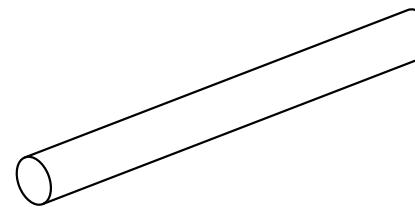
**Internal constraints added to the 3D model**

# Constrained elasticity

3D elastostatic problem



2D model



1D model

assumptions on strain  
and/or stress fields



**Internal constraints** added to the 3D model

Lagrangian multipliers can be employed to formulate the 3D constrained problem:

they give **reactive constraints** belonging to dual spaces  
of those where constrained variables live

## Constrained elasticity

**P Podio-Guidugli.** An exact derivation of thin plates equations. *J Elasticity*, 22:121-33, **1989**.

**M Lembo, P Podio-Guidugli.** Internal constraints, reactive stresses, and the Timoshenko beam theory. *J Elasticity*, 65(1-3):131-48, **2001**.

**Antman et al.** ...

**Internal constraints on strain field and on material behaviour**

# Constrained elasticity

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**Internal constraints on strain field and on material behaviour**

**P Bisegna, E Sacco.** A rational deduction of plate theories from the three-dimensional linear elasticity. *Z Angew Math Mech*, 77:349-66, **1997**.

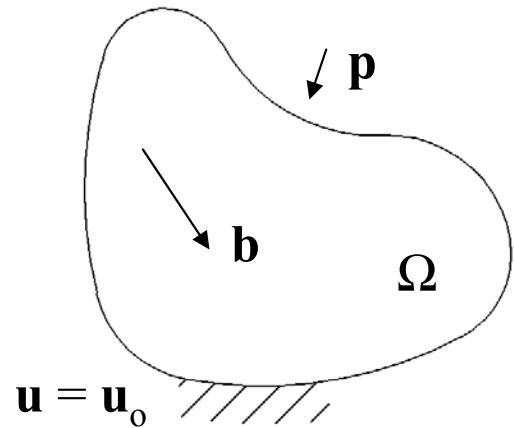
**F Maceri, P Bisegna.** Modellazione strutturale. In: E Giangreco – Ingegneria delle Strutture, II, Utet: 1-90, **2002**.

**F Maceri, G Vairo.** Anisotropic thin-walled beam models: a rational deduction from three-dimensional elasticity. *J of Mech Mater Struct*, to appear, **2008**.

...

**Dual constraints on both stress and strain fields, leaving unchanged the material constitutive law**

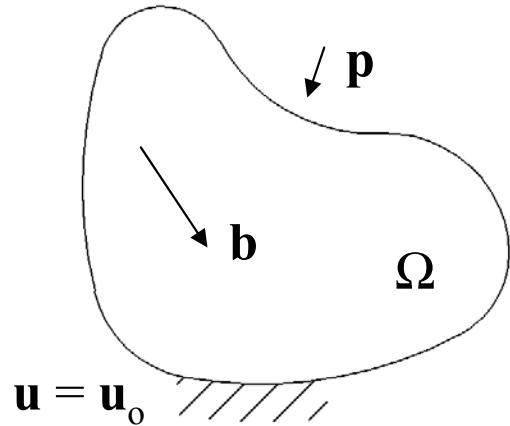
# Dual constraints approach



Hu-Washizu 3D functional

$$\begin{aligned} \mathcal{W}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = & \frac{1}{2} \int_{\Omega} \mathbf{C} \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} dv - \int_{\Omega} \boldsymbol{\sigma} \cdot (\hat{\nabla} \mathbf{u} - \boldsymbol{\varepsilon}) dv \\ & - \int_{\Omega} \mathbf{b} \cdot \mathbf{u} dv - \int_{\partial_f \Omega} \mathbf{p} \cdot \mathbf{u} da - \int_{\partial_u \Omega} \boldsymbol{\sigma} \mathbf{n}_{\partial} \cdot (\mathbf{u} - \mathbf{u}_o) da \end{aligned}$$

# Dual constraints approach



Hu-Washizu 3D functional

$$\mathcal{W}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = \frac{1}{2} \int_{\Omega} \mathbf{C} \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} dv - \int_{\Omega} \boldsymbol{\sigma} \cdot (\hat{\nabla} \mathbf{u} - \boldsymbol{\varepsilon}) dv \\ - \int_{\Omega} \mathbf{b} \cdot \mathbf{u} dv - \int_{\partial_f \Omega} \mathbf{p} \cdot \mathbf{u} da - \int_{\partial_u \Omega} \boldsymbol{\sigma} \mathbf{n}_{\partial} \cdot (\mathbf{u} - \mathbf{u}_o) da$$



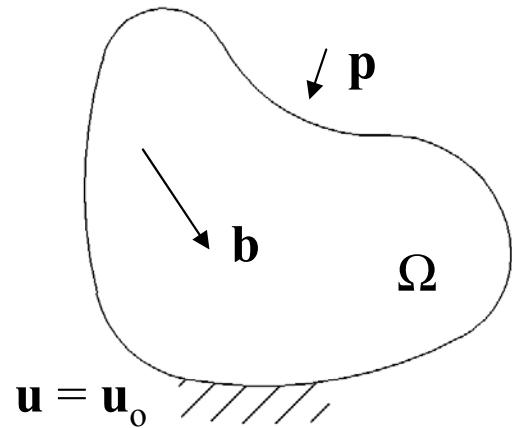
Lagrangian functional

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \chi, \omega) = \mathcal{W}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) - \int_{\Omega} \chi \cdot \mathbf{G} \boldsymbol{\varepsilon} dv - \int_{\Omega} \omega \cdot \mathbf{H} \boldsymbol{\sigma} dv - \int_{\Omega} \mathbf{G}^* \chi \cdot \mathbf{H}^* \omega dv$$

$\mathbf{G}$ ,  $\mathbf{H}$  constraint operators ( $\mathbf{G}^*$ ,  $\mathbf{H}^*$  adjoint operators)

$\omega$ ,  $\chi$  Lagrangian multipliers (reactive part)

# Dual constraints approach



Hu-Washizu 3D functional

$$\mathcal{W}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) = \frac{1}{2} \int_{\Omega} \mathbf{C} \boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} dv - \int_{\Omega} \boldsymbol{\sigma} \cdot (\hat{\nabla} \mathbf{u} - \boldsymbol{\varepsilon}) dv \\ - \int_{\Omega} \mathbf{b} \cdot \mathbf{u} dv - \int_{\partial_f \Omega} \mathbf{p} \cdot \mathbf{u} da - \int_{\partial_u \Omega} \boldsymbol{\sigma} \mathbf{n}_{\partial} \cdot (\mathbf{u} - \mathbf{u}_o) da$$



Lagrangian functional

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \chi, \omega) = \mathcal{W}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) - \int_{\Omega} \chi \cdot \mathbf{G} \boldsymbol{\varepsilon} dv - \int_{\Omega} \omega \cdot \mathbf{H} \boldsymbol{\sigma} dv - \int_{\Omega} \mathbf{G}^* \chi \cdot \mathbf{H}^* \omega dv$$

$\mathbf{G}$ ,  $\mathbf{H}$  constraint operators ( $\mathbf{G}^*$ ,  $\mathbf{H}^*$  adjoint operators)

$\omega$ ,  $\chi$  Lagrangian multipliers (reactive part)

dual constraints

$$\text{total strain} \in \text{Ker}(\mathbf{G}) \\ \text{elastic stress} \in \text{Ker}(\mathbf{H})$$

total stress (strain) =  
elastic part + reactive part

# Dual constraints approach

## stationarity of $\mathcal{L}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}, \boldsymbol{\chi})$

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\chi}, \boldsymbol{\omega}) = \mathcal{W}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) - \int_{\Omega} \boldsymbol{\chi} \cdot \mathbf{G} \boldsymbol{\varepsilon} dV - \int_{\Omega} \boldsymbol{\omega} \cdot \mathbf{H} \boldsymbol{\sigma} dV - \int_{\Omega} \mathbf{G}^* \boldsymbol{\chi} \cdot \mathbf{H}^* \boldsymbol{\omega} dV$$

$\mathbf{u}$ )  $\rightarrow$  
$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{\sigma} \mathbf{n}_\partial = \mathbf{p} & \text{on } \partial_f \Omega \end{cases}$$
 equilibrium

---

$\boldsymbol{\varepsilon}$ )  $\rightarrow$   $\boldsymbol{\sigma} + \mathbf{G}^* \boldsymbol{\chi} = \mathbf{C} \boldsymbol{\varepsilon}$  in  $\Omega$  constitutive law

---

$\boldsymbol{\sigma}$ )  $\rightarrow$  
$$\begin{cases} \boldsymbol{\varepsilon} + \mathbf{H}^* \boldsymbol{\omega} = \hat{\nabla} \mathbf{u} & \text{in } \Omega \\ \mathbf{u} = \mathbf{u}_o & \text{on } \partial_u \Omega \end{cases}$$
 compatibility

---

$\boldsymbol{\chi}, \boldsymbol{\omega}) \rightarrow \mathbf{G}(\boldsymbol{\varepsilon} + \mathbf{H}^* \boldsymbol{\omega}) = \mathbf{0}$   $\mathbf{H}(\boldsymbol{\sigma} + \mathbf{G}^* \boldsymbol{\chi}) = \mathbf{0}$  constraints

---

# Dual constraints approach

stationarity of  $\mathcal{L}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}, \boldsymbol{\chi})$

$$\mathcal{L}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\chi}, \boldsymbol{\omega}) = \mathcal{W}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}) - \int_{\Omega} \boldsymbol{\chi} \cdot \mathbf{G} \boldsymbol{\varepsilon} dV - \int_{\Omega} \boldsymbol{\omega} \cdot \mathbf{H} \boldsymbol{\sigma} dV - \int_{\Omega} \mathbf{G}^* \boldsymbol{\chi} \cdot \mathbf{H}^* \boldsymbol{\omega} dV$$

$\mathbf{u}$ )  $\rightarrow$  
$$\begin{cases} \operatorname{div} \boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} & \text{in } \Omega \\ \boldsymbol{\sigma} \mathbf{n}_{\partial} = \mathbf{p} & \text{on } \partial_f \Omega \end{cases}$$
 equilibrium

$\boldsymbol{\varepsilon}$ )  $\rightarrow$  
$$\boldsymbol{\sigma} + \mathbf{G}^* \boldsymbol{\chi} = \mathbf{C} \boldsymbol{\varepsilon} \quad \text{in } \Omega$$
 constitutive law

$\boldsymbol{\sigma}$ )  $\rightarrow$  
$$\begin{cases} \boldsymbol{\varepsilon} + \mathbf{H}^* \boldsymbol{\omega} = \hat{\nabla} \mathbf{u} & \text{in } \Omega \\ \mathbf{u} = \mathbf{u}_o & \text{on } \partial_u \Omega \end{cases}$$
 compatibility

$\boldsymbol{\chi}, \boldsymbol{\omega}) \rightarrow \mathbf{G}(\boldsymbol{\varepsilon} + \mathbf{H}^* \boldsymbol{\omega}) = \mathbf{0} \quad \mathbf{H}(\boldsymbol{\sigma} + \mathbf{G}^* \boldsymbol{\chi}) = \mathbf{0}$  constraints

total stress  $\underbrace{\boldsymbol{\sigma} + \mathbf{G}^* \boldsymbol{\chi}}_{\text{elastic stress}}$

elastic strain  $\underbrace{\boldsymbol{\varepsilon} + \mathbf{H}^* \boldsymbol{\omega}}_{\text{total strain}}$

## Dual constraints approach

stationary conditions of the Lagrangian functional  $\mathcal{L}(\mathbf{u}, \boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{\omega}, \chi)$  with respect to elastic strain  $\boldsymbol{\varepsilon}$  and total stress  $\boldsymbol{\sigma}$  allow to eliminate  $\boldsymbol{\varepsilon}$  and  $\boldsymbol{\sigma}$  themselves and to obtain the energy-type functional

$$\mathcal{E}(\mathbf{u}, \chi, \boldsymbol{\omega}) = \frac{1}{2} \int_{\Omega} \mathbf{C}(\hat{\nabla} \mathbf{u} - \mathbf{H}^* \boldsymbol{\omega}) \cdot (\hat{\nabla} \mathbf{u} - \mathbf{H}^* \boldsymbol{\omega}) dv - \int_{\Omega} \chi \cdot \mathbf{G} \hat{\nabla} \mathbf{u} dv - \int_{\Omega} \mathbf{b} \cdot \mathbf{u} dv - \int_{\partial_f \Omega} \mathbf{p} \cdot \mathbf{u} da$$

defined on the manifold  $\mathbf{u} = \mathbf{u}_o$  on  $\partial_u \Omega$

$$\boldsymbol{\varepsilon}^{tot} = \hat{\nabla} \mathbf{u}$$

$$\boldsymbol{\sigma}^{tot} = \mathbf{C}(\hat{\nabla} \mathbf{u} - \mathbf{H}^* \boldsymbol{\omega}) - \mathbf{G}^* \chi$$

$$\boldsymbol{\varepsilon}^{el} = (\hat{\nabla} \mathbf{u} - \mathbf{H}^* \boldsymbol{\omega})$$

$$\boldsymbol{\sigma}^{el} = \mathbf{C}(\hat{\nabla} \mathbf{u} - \mathbf{H}^* \boldsymbol{\omega})$$

$$\boldsymbol{\varepsilon}^{react} = \mathbf{H}^* \boldsymbol{\omega}$$

$$\boldsymbol{\sigma}^{react} = -\mathbf{G}^* \chi$$

# Dual constraints approach

**Maceri F, Bisegna P.** Modellazione strutturale (in Italian). In: Elio Giangreco - Ingegneria delle Strutture, vol. II. Utet, Torino, pp. 1-90, **2002**.



Rational deduction of classical plates and planar beam theories

$$\mathbf{G}\mathbf{A} = \mathbf{H}\mathbf{A} \quad \forall \mathbf{A} \in Sym$$

# Dual constraints approach

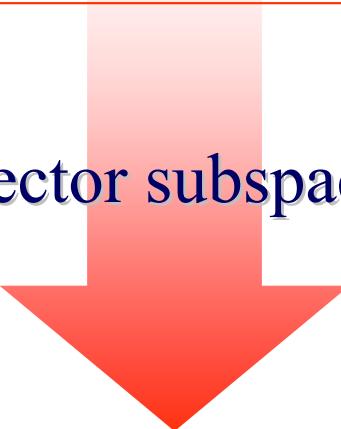
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Rational deduction of classical plates and planar beam theories

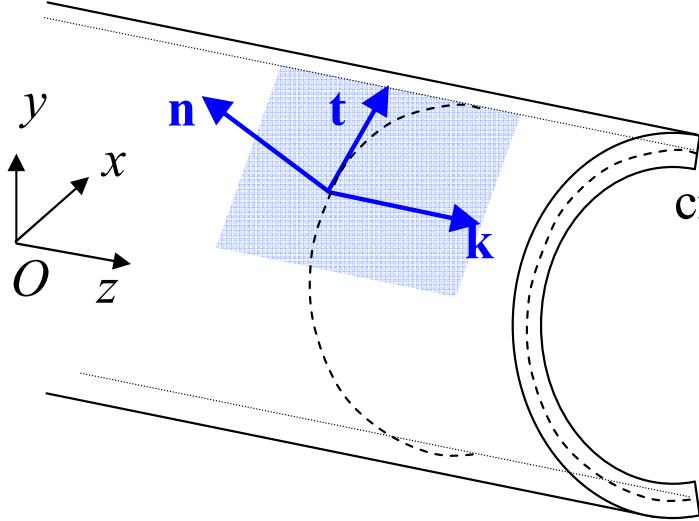
$$\mathbf{G}\mathbf{A} = \mathbf{H}\mathbf{A} \quad \forall \mathbf{A} \in Sym$$

$\chi$  and  $\omega$  belong to dual vector subspaces with same dimensions

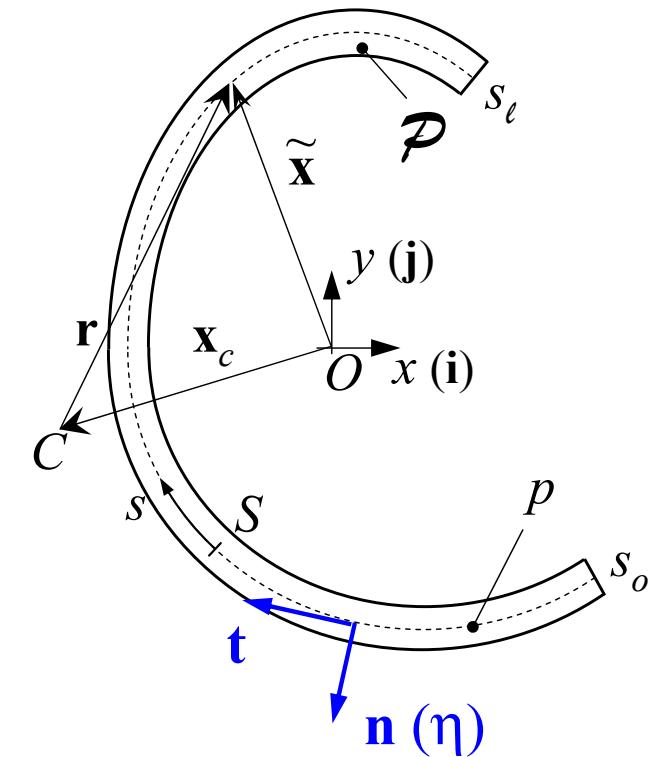


Once, kinematical constraints are chosen,  
consistent stress assumptions directly arise

# Thin-walled beam model



centerline  $p$   
 thickness  $2\delta$   
 cross-section  $\mathcal{P} = p \times ]-\delta, \delta[$   
 mantle  $\Sigma = \partial \mathcal{P} \times ]-L, L[$   
 beam  $\Omega = \mathcal{P} \times ]-L, L[$   
 ends  $\mathcal{P}|_{\pm L} = \mathcal{P} \times \{\pm L\}$



$$\mathbf{x}(s, \eta, z) = \tilde{\mathbf{x}}(s) + \eta \mathbf{n}(s) + z \mathbf{k}$$

$$s \in [s_o, s_e]$$

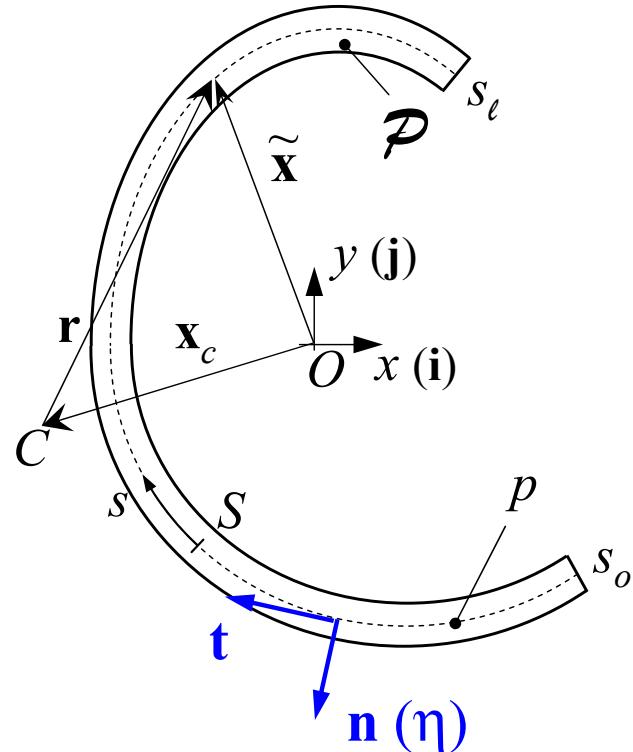
$$\mathbf{t}(s) = \tilde{\mathbf{x}}_s, \quad \mathbf{n}(s) = -\tilde{y}_s \mathbf{i} + \tilde{x}_s \mathbf{j}$$

$$\mathbf{r} = \tilde{\mathbf{x}} - \mathbf{x}_c$$

$$\mathbf{t}_s = \kappa(s) \mathbf{n} \quad \mathbf{n}_s = -\kappa(s) \mathbf{t}$$

$\kappa(s)$ : curvature (with sign) of  $p$

# Thin-walled beam model



$$\int_{\mathcal{P}} (\cdot) da = \int_{p-\delta}^{\delta} \int_{-\delta}^{\delta} (\cdot) j(s, \eta) d\eta ds \quad j(s, \eta) = [1 - \eta \kappa(s)]$$

average over the thickness domain

$$\bar{f}(s, z) = \frac{1}{2\delta} \int_{-\delta}^{\delta} f(s, \eta, z) j(s, \eta) d\eta$$

$$\nabla f = \frac{f_{/s}}{j} \mathbf{t} + f_{/\eta} \mathbf{n} + f' \mathbf{k} \quad f' = \frac{\partial f}{\partial z}$$

## Thin-walled beam model

## material symmetry

The beam is assumed to be homogeneous and comprising a linearly elastic material, having at least a monoclinic symmetry, with symmetry plane orthogonal to the beam axis.

$$C_{\alpha\beta\gamma 3} = C_{\alpha 333} = 0$$



This symmetry includes the case of fiber-reinforced composite profiles commonly used in civil engineering and produced by pultrusion technology

### index rules

- Greek indices imply values in {1,2} and denote components in the plane of  $\mathcal{P}$  referred to the local tangent frame;
- index '3' denotes components along  $z$ -axis

strain and stress dual constraints

## No-shear beam model

Vlasov / Euler-Bernoulli

## Total strains

1. in-plane (dilatation and shear) total strain components vanish everywhere on  $\mathcal{P}$ ;
2. shear total strain between  $z$ -axis and direction  $\mathbf{n}(s)$  vanishes everywhere on  $\mathcal{P}$ ;
3. flux through the thickness of the in-plane shear total strain vector

$\gamma = 2\epsilon_{13} \mathbf{t} + 2\epsilon_{23} \mathbf{n} = 2\epsilon_{x3} \mathbf{i} + 2\epsilon_{y3} \mathbf{j}$  is equal to zero;

$$\mathbf{G}\boldsymbol{\epsilon} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \bar{\epsilon}_{13}, \epsilon_{23}\}^T$$

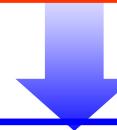
1.            3.            2.

### Total strains

1. in-plane (dilatation and shear) total strain components vanish everywhere on  $\mathcal{P}$ ;
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$$\gamma = 2\epsilon_{13} \mathbf{t} + 2\epsilon_{23} \mathbf{n} = 2\epsilon_{x3} \mathbf{i} + 2\epsilon_{y3} \mathbf{j} \text{ is equal to zero;}$$

$$\mathbf{G}\boldsymbol{\epsilon} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, \bar{\epsilon}_{13}, \epsilon_{23}\}^T$$



$$\mathbf{H}\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}, \bar{\sigma}_{13}, \sigma_{23}\}^T$$

### Elastic stresses

1. elastic stress vector on every plane parallel to the  $z$ -axis is parallel to  $\mathbf{k}$ ;
2. shear elastic stress between  $z$ -axis and direction  $\mathbf{n}(s)$  vanishes everywhere on  $\mathcal{P}$ ;
3. flux through the thickness of the in-plane shear elastic stress vector

$$\tau = \tau_{13} \mathbf{t} + \tau_{23} \mathbf{n} = \tau_{x3} \mathbf{i} + \tau_{y3} \mathbf{j} \text{ is equal to zero;}$$

**Remark:**  $\bar{\epsilon}_{13}^{tot} = 0, \bar{\sigma}_{13}^{el} = 0 \Rightarrow \omega_{13} = \omega_{13}(s, z), \chi_{13} = \chi_{13}(s, z)$

$$\begin{aligned} \mathcal{E}(\mathbf{u}, \chi, \omega) = & \frac{1}{2} \int_{\Omega} \{ C_{\alpha\beta\gamma\delta} (\Lambda_{\alpha\beta} - \omega_{\alpha\beta}) (\Lambda_{\gamma\delta} - \omega_{\gamma\delta}) + 2C_{\alpha\beta 33} (\Lambda_{\alpha\beta} - \omega_{\alpha\beta}) u'_3 + C_{3333} (u'_3)^2 + \\ & + 4C_{\alpha 3 \beta 3} (\Lambda_{\alpha 3} - \omega_{\alpha 3}) (\Lambda_{\beta 3} - \omega_{\beta 3}) \} dv - \int_{\Omega} (\chi_{\alpha\beta} \Lambda_{\alpha\beta} + 2\chi_{\alpha 3} \Lambda_{\alpha 3}) dv - \Pi_{ext} \end{aligned}$$

where

$$\Lambda = \hat{\nabla} \mathbf{u}$$

$$\Pi_{ext} = \int_{\Omega} (b_{\alpha} u_{\alpha} + b_3 u_3) dv + \int_{\Sigma} (\hat{p}_{\alpha} u_{\alpha} + \hat{p}_3 u_3) da + \int_{\mathcal{P}|_{\pm L}} (p_{\alpha}^{\pm} u_{\alpha}|_{\pm L} + p_3^{\pm} u_3|_{\pm L}) da$$

stationarity of  $\mathcal{E}(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\chi})$  with respect to  $\boldsymbol{\chi}$  gives the equations

$$\Lambda_{11} = [u_{1/1} - \kappa u_2]/j = 0$$

$$\Lambda_{22} = u_{2/2} = 0$$

$$2\Lambda_{12} = [u_{2/1} + \kappa u_1]/j + u_{1/2} = 0$$

$$4\delta\bar{\Lambda}_{13} = \int_{-\delta}^{\delta} (u_{3/1} + ju'_1) d\eta = 0$$

$$2\Lambda_{23} = u'_2 + u_{3/2} = 0$$

where  $u_1 = \mathbf{u} \cdot \mathbf{t}$ ,  $u_2 = \mathbf{u} \cdot \mathbf{n}$  and  $u_3 = \mathbf{u} \cdot \mathbf{k}$ .

stationarity of  $\mathcal{E}(\mathbf{u}, \omega, \chi)$  with respect to  $\chi$  gives the equations

$$\begin{aligned}\Lambda_{11} &= [u_{1/1} - \kappa u_2]/j = 0 \\ \Lambda_{22} &= u_{2/2} = 0 \\ 2\Lambda_{12} &= [u_{2/1} + \kappa u_1]/j + u_{1/2} = 0\end{aligned}$$

$$\begin{aligned}4\delta\bar{\Lambda}_{13} &= \int_{-\delta}^{\delta} (u_{3/1} + ju'_1) d\eta = 0 \\ 2\Lambda_{23} &= u'_2 + u_{3/2} = 0\end{aligned}$$

where  $u_1 = \mathbf{u} \cdot \mathbf{t}$ ,  $u_2 = \mathbf{u} \cdot \mathbf{n}$  and  $u_3 = \mathbf{u} \cdot \mathbf{k}$ .

from which, by integration, the displacement field is obtained and represented in Cartesian form

$$\begin{aligned}u_x(s, \eta, z) &= \mathbf{u} \cdot \mathbf{i} = u_c(z) - \theta(z)[y(s, \eta) - y_c] \\ u_y(s, \eta, z) &= \mathbf{u} \cdot \mathbf{j} = v_c(z) + \theta(z)[x(s, \eta) - x_c] \\ u_3(s, \eta, z) &= \mathbf{u} \cdot \mathbf{k} = w_c(z) - u'_c(z)x(s, \eta) - v'_c(z)y(s, \eta) + \theta'(z)\psi_c(s, \eta)\end{aligned}$$

warping function

$$\psi_c(s, \eta) = \psi_o + \int_0^s \mathbf{r} \cdot \mathbf{n} ds - \eta \mathbf{r} \cdot \mathbf{t} - \frac{1}{3} \int_0^s \delta^2 \kappa ds$$

$$\mathbf{r} = \widetilde{\mathbf{x}} - \mathbf{x}_c \quad \int_{\mathcal{P}} \psi_c da = \int_{\mathcal{P}} x \psi_c da = \int_{\mathcal{P}} y \psi_c da = 0$$

stationarity of  $\mathcal{E}(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\chi})$  with respect to  $\boldsymbol{\omega}$  gives the equations

$$\omega_{\alpha\beta}) \quad \mathcal{C}_{\alpha\beta\gamma\delta}[\Lambda_{\gamma\delta} - \omega_{\gamma\delta}] + \mathcal{C}_{\alpha\beta 33}u'_3 = 0$$

$$\omega_{13}) \quad \int_{-\delta}^{\delta} \mathcal{C}_{\alpha 313}[\Lambda_{\alpha 3} - \omega_{\alpha 3}] j(s, \eta) d\eta = 0$$

$$\omega_{23}) \quad \mathcal{C}_{\alpha 323}[\Lambda_{\alpha 3} - \omega_{\alpha 3}] = 0$$

from which  $\omega_{ij}$  are obtained:

$$\omega_{\alpha\beta} = (\mathcal{C}_{\alpha\beta\gamma\delta})^{-1} \mathcal{C}_{\gamma\delta 33} u'_3$$

$$\omega_{13} = 0$$

$$\omega_{23} = \mathcal{C}_{1323}\Lambda_{13}/\mathcal{C}_{2323}$$

By substituting  $\boldsymbol{\omega}$  and  $\mathbf{u}$  into the functional  $\mathcal{E}(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\chi})$  and by integrating over  $\mathcal{P}$ , the energy functional  $\hat{\mathcal{E}}$  is obtained

$$\hat{\mathcal{E}}(u_c, v_c, w_c, \theta) = \frac{1}{2} \int_{-L}^L \mathbf{D}\mathbf{e} \cdot \mathbf{e} dz - \int_{-L}^L \mathbf{q} \cdot \hat{\mathbf{s}} dz - \mathbf{Q}^\pm \cdot \hat{\mathbf{s}} \Big|_{\pm L}$$

$$\hat{\mathcal{E}}(u_c, v_c, w_c, \theta) = \frac{1}{2} \int_{-L}^L \mathbf{D}\mathbf{e} \cdot \mathbf{e} dz - \int_{-L}^L \mathbf{q} \cdot \hat{\mathbf{s}} dz - \mathbf{Q}^\pm \cdot \hat{\mathbf{s}} \Big|_{\pm L}$$

$\hat{\mathbf{s}}$ : generalized displacements

$\mathbf{e}$ : generalized total strains

$\mathbf{q}$ : generalized distributed forces

$\mathbf{Q}^\pm$ : generalized end-located forces

$\mathbf{D}$ : generalized elasticity matrix of the beam

$$\hat{\mathbf{s}} = \{u_c, v_c, w_c, -v'_c, u'_c, \theta, \theta'\}^T$$

$$\mathbf{q} = \{q_x, q_y, q_z, m_x, m_y, m_z, m_\psi\}^T$$

$$\mathbf{e} = \{w'_c, -v''_c, u''_c, \theta'', \theta'\}^T$$

$$\mathbf{Q}^\pm = \{Q_x^\pm, Q_y^\pm, Q_z^\pm, M_x^\pm, M_y^\pm, M_z^\pm, M_\psi^\pm\}^T$$

$$\{q_x, q_y, q_z\} = \int_{\mathcal{P}} \{b_x, b_y, b_z\} da + \int_{\partial\mathcal{P}} \{\hat{p}_x, \hat{p}_y, \hat{p}_z\} d\varrho$$

$$\{Q_x^\pm, Q_y^\pm, Q_z^\pm\} = \int_{\mathcal{P}} \{p_x^\pm, p_y^\pm, p_z^\pm\} da$$

$$\{m_x, m_y, m_\psi\} = \int_{\mathcal{P}} \{y, -x, \psi_c\} b_z da + \int_{\partial\mathcal{P}} \{y, -x, \psi_c\} \hat{p}_z d\varrho$$

$$\{M_x^\pm, M_y^\pm, M_\psi^\pm\} = \int_{\mathcal{P}} \{y, -x, \psi_c\} p_z^\pm da$$

$$m_z = \int_{\mathcal{P}} [b_y(x - x_c) - b_x(y - y_c)] da + \int_{\partial\mathcal{P}} [\hat{p}_y(x - x_c) - \hat{p}_x(y - y_c)] d\varrho$$

$$M_z^\pm = \int_{\mathcal{P}} [p_y^\pm(x - x_c) - p_x^\pm(y - y_c)] da$$

$$\hat{\mathcal{E}}(u_c, v_c, w_c, \theta) = \frac{1}{2} \int_{-L}^L \mathbf{D}\mathbf{e} \cdot \mathbf{e} dz - \int_{-L}^L \mathbf{q} \cdot \hat{\mathbf{s}} dz - \mathbf{Q}^\pm \cdot \hat{\mathbf{s}} \Big|_{\pm L}$$

$$\mathbf{D} = \begin{bmatrix} \hat{\mathcal{C}}_{3333}A & \hat{\mathcal{C}}_{3333}S_x & \hat{\mathcal{C}}_{3333}S_y & \hat{\mathcal{C}}_{3333}S_\psi & 0 \\ \hat{\mathcal{C}}_{3333}I_x & \hat{\mathcal{C}}_{3333}I_{xy} & \hat{\mathcal{C}}_{3333}I_{x\psi} & 0 \\ & \hat{\mathcal{C}}_{3333}I_y & \hat{\mathcal{C}}_{3333}I_{y\psi} & 0 \\ sym & & \hat{\mathcal{C}}_{3333}I_\psi & 0 \\ & & & \hat{\mathcal{C}}_{1313}J_\theta & \end{bmatrix}$$

$$\{A, S_x, S_y, S_\psi\} = \int_{\mathcal{P}} \{1, y, -x, \psi_c\} da$$

$$\{I_x, I_{xy}, I_y, I_{x\psi}, I_{y\psi}, I_\psi\} = \int_{\mathcal{P}} \{y^2, -xy, x^2, y\psi_c, -x\psi_c, \psi_c^2\} da$$

$$J_\theta = \int_{\mathcal{P}} [\kappa(\eta^2 - \delta^2/3) - 2\eta]^2/j^2 da$$

## Reduced elastic moduli

$$\hat{\mathcal{C}}_{1313} = \mathcal{C}_{1313} - \frac{\mathcal{C}_{1323}^2}{\mathcal{C}_{2323}}$$

$$\hat{\mathcal{C}}_{3333} = \mathcal{C}_{3333} - \mathcal{C}_{\alpha\beta 33} (\mathcal{C}_{\alpha\beta\gamma\delta})^{-1} \mathcal{C}_{\gamma\delta 33}$$

reduced constitutive law comes out from the procedure adopted and is not a-priori enforced by a constrained constitutive law

$O$ : centroid for  $\Omega$ ,  
 $x$  and  $y$  principal for  $\mathcal{P}$ ,

$C$ : twinsting center for  $\mathcal{P}$ ,  
 $S$ : sectorial centroid for  $p$



**D**: diagonal

$$\hat{\mathcal{E}}(u_c, v_c, w_c, \theta) = \mathcal{A}(w_c) + \mathcal{F}_x(v_c) + \mathcal{F}_y(u_c) + \mathcal{T}(\theta),$$

where

$$\mathcal{A}(w_c) = \frac{1}{2} \hat{\mathcal{C}}_{3333} A \int_{-L}^L (w'_c)^2 dz - \int_{-L}^L q_z w_c dz - Q_z^\pm w_c|_{\pm L},$$

$$\mathcal{F}_x(v_c) = \frac{1}{2} \hat{\mathcal{C}}_{3333} I_x \int_{-L}^L (v''_c)^2 dz - \int_{-L}^L (q_y v_c - m_x v'_c) dz - (Q_y^\pm v_c|_{\pm L} - M_x^\pm v'_c|_{\pm L}),$$

$$\mathcal{F}_y(u_c) = \frac{1}{2} \hat{\mathcal{C}}_{3333} I_y \int_{-L}^L (u''_c)^2 dz - \int_{-L}^L (q_x u_c + m_y u'_c) dz - (Q_x^\pm u_c|_{\pm L} + M_y^\pm u'_c|_{\pm L})$$

$$\begin{aligned} \mathcal{T}(\theta) = & \frac{1}{2} \hat{\mathcal{C}}_{3333} I_\psi \int_{-L}^L (\theta'')^2 dz + \frac{1}{2} \hat{\mathcal{C}}_{1313} J_\theta \int_{-L}^L (\theta')^2 dz + \\ & - \int_{-L}^L (m_z \theta + m_\psi \theta') dz - (M_z^\pm \theta|_{\pm L} + M_\psi^\pm \theta'|_{\pm L}). \end{aligned}$$

stationarity of  $\hat{\mathcal{E}}(u_c, v_c, w_c, \theta)$  with respect to the unknown displacement functions gives equilibrium field equations and natural boundary conditions

$$\mathcal{A}(w_c) \quad \hat{\mathcal{C}}_{3333} A w_c'' + q_z = 0, \quad \hat{\mathcal{C}}_{3333} A w_c'|_{-L}^L = Q_z^\pm.$$

$$\mathcal{F}_x(v_c) \quad \hat{\mathcal{C}}_{3333} I_x v_c^{IV} - q_y - m_x' = 0, \quad \begin{cases} \hat{\mathcal{C}}_{3333} I_x v_c'''|_{-L}^L - m_x|_{-L}^L = -Q_y^\pm \\ \hat{\mathcal{C}}_{3333} I_x v_c''|_{-L}^L = -M_x^\pm \end{cases}$$

$$\mathcal{F}_y(u_c) \quad \hat{\mathcal{C}}_{3333} I_y u_c^{IV} - q_x + m_y' = 0, \quad \begin{cases} \hat{\mathcal{C}}_{3333} I_y u_c'''|_{-L}^L + m_y|_{-L}^L = -Q_x^\pm \\ \hat{\mathcal{C}}_{3333} I_y u_c''|_{-L}^L = M_y^\pm. \end{cases}$$

$$\mathcal{T}(\theta)$$

$$\hat{\mathcal{C}}_{3333} I_\psi \theta^{IV} - \hat{\mathcal{C}}_{1313} J_\theta \theta'' - m_z + m_\psi' = 0, \quad \begin{cases} [\hat{\mathcal{C}}_{3333} I_\psi \theta''' - \hat{\mathcal{C}}_{1313} J_\theta \theta' + m_\psi]|_{-L}^L = -M_z^\pm \\ \hat{\mathcal{C}}_{3333} I_\psi \theta''|_{-L}^L = M_\psi^\pm. \end{cases}$$

$$\mathbf{e} = \{w'_c, -v''_c, u''_c, \theta'', \theta'\}^T$$



$$\mathbf{S} = \{N \ C_x \ C_y \ C_\psi \ C_z\}^T = \mathbf{D} \ \mathbf{e}$$

$\mathbf{S}$  satisfies the global equilibrium equations

$$\{N, C_x, C_y, C_\psi\} = \int_{\mathcal{P}} \{1, y, -x, \psi_c\} \sigma_{33} \, da, \quad C_z = \int_{\mathcal{P}} (\sigma_{y3}x - \sigma_{x3}y) \, da$$

$C_z$ : primary twisting moment

$C_\psi$ : bimoment (warping torque)

$$\mathbf{e} = \{w'_c, -v''_c, u''_c, \theta'', \theta'\}^T$$



$$\begin{aligned}\sigma_{\alpha\beta}^{(el)} &= \sigma_{23}^{(el)} = 0 \\ \sigma_{13}^{(el)} &= \hat{\mathcal{C}}_{1313} \theta' [\kappa(\eta^2 - \delta^2/3) - 2\eta]/j \\ \sigma_{33}^{(el)} &= \hat{\mathcal{C}}_{3333} (w'_c - u''_c x - v''_c y + \theta'' \psi_c)\end{aligned}$$



$\boldsymbol{\sigma}^{(el)}$  does not satisfy local equilibrium equations

$$\boldsymbol{\sigma}^{tot} = \mathbf{C}(\hat{\nabla} \mathbf{u} - \mathbf{H}^* \boldsymbol{\omega}) - \mathbf{G}^* \boldsymbol{\chi}$$

$$(\mathbf{G}^* \boldsymbol{\chi})_{33} = 0 \quad \rightarrow \quad \sigma_{33}^{el} = \sigma_{33}^{tot}$$

disregarding high-order effects  
(related to curvature and thickness)



Bauld NR, Tzeng L. A Vlasov theory for fiber-reinforced beams with thin-walled open cross-sections. International Journal of Solids and Structures 20:277-97, 1984.

also, isotropic material

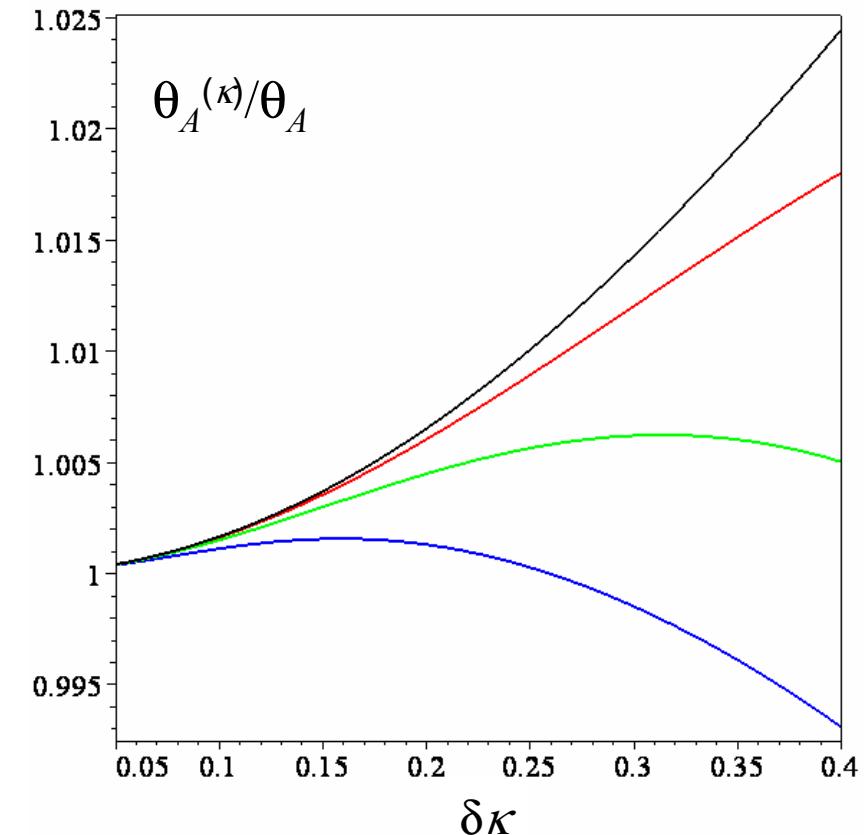
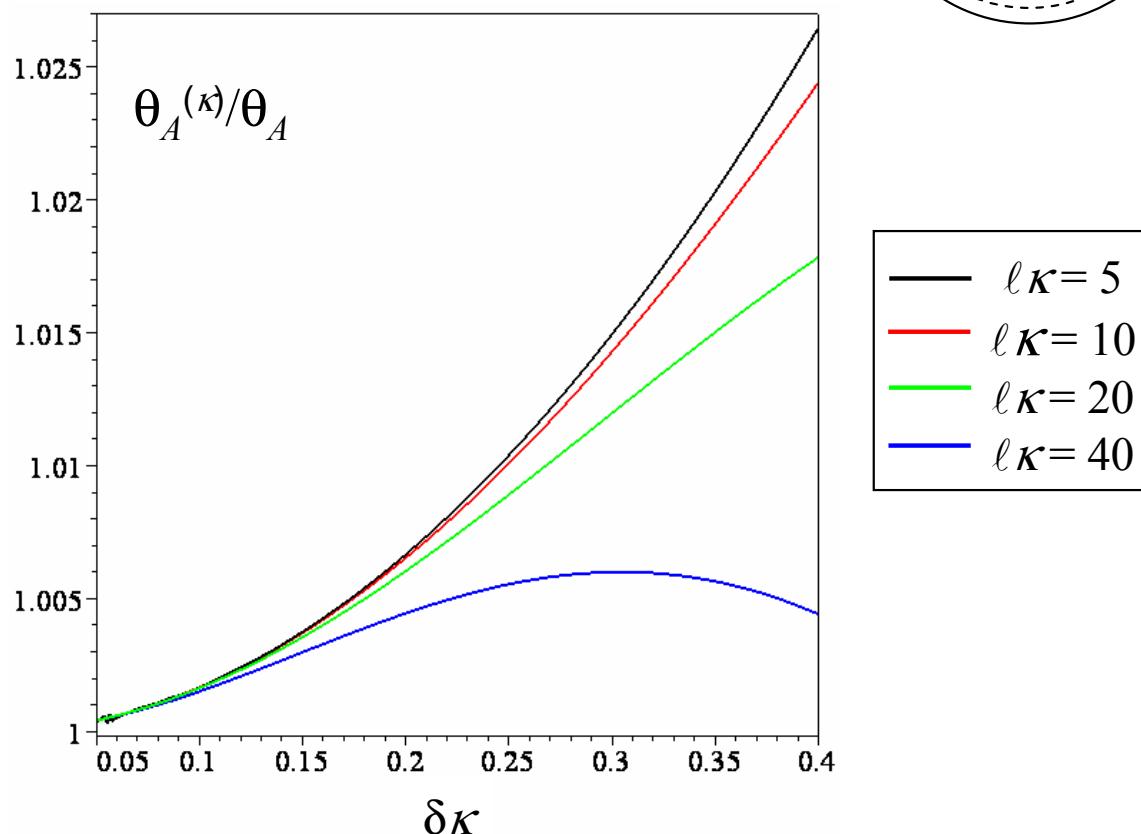
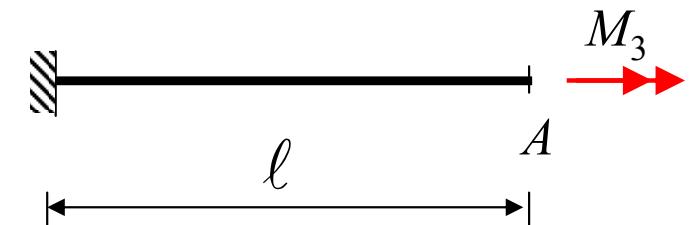
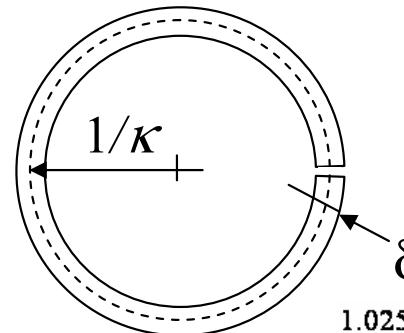
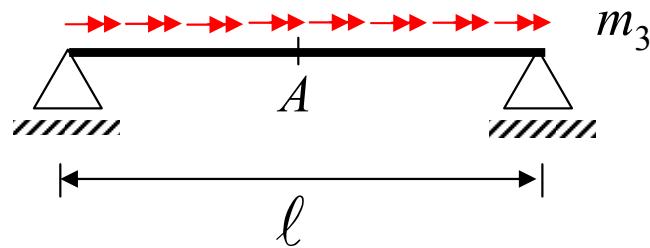


Classical Vlasov theory

# Curvature's influence

Homogeneous beam, comprised by material with a constitutive symmetry at least monoclinic and symmetry plane orthogonal to  $x_3$

$$\frac{\hat{C}_{3333}}{\hat{C}_{1313}} = 2.5$$



**Hp:**

$$\frac{2\delta}{L} \ll 1, \quad \left| \frac{d\delta}{ds} \right| \ll 1, \quad |\delta\kappa| \ll 1, \quad \delta / \int_p ds \ll 1 \quad \rightarrow \quad j(s, \eta) \equiv 1$$

strain and stress dual constraints

# First-order shear-deformable beam model

Vlasov / Timoshenko

### Total strains

1. in-plane (dilatation and shear) total strain components vanish everywhere on  $\mathcal{P}$ ;
2. shear total strain between  $z$ -axis and direction  $\mathbf{n}(s)$  vanishes everywhere on  $\mathcal{P}$ ;
3. flux through the thickness of the in-plane shear total strain vector
$$\gamma = 2\epsilon_{13} \mathbf{t} + 2\epsilon_{23} \mathbf{n} = 2\epsilon_{x3} \mathbf{i} + 2\epsilon_{y3} \mathbf{j}$$
is equal to zero;

### Elastic stresses

1. elastic stress vector on every plane parallel to the  $z$ -axis is parallel to  $\mathbf{k}$ ;
2. shear elastic stress between  $z$ -axis and direction  $\mathbf{n}(s)$  vanishes everywhere on  $\mathcal{P}$ ;
3. flux through the thickness of the in-plane shear elastic stress vector
$$\tau = \tau_{13} \mathbf{t} + \tau_{23} \mathbf{n} = \tau_{x3} \mathbf{i} + \tau_{y3} \mathbf{j}$$
is equal to zero;

## Total strains

1. in-plane (dilatation and shear) total strain components vanish everywhere on  $\mathcal{P}$ ;
  - 2'. shear total strain between  $z$ -axis and direction  $\mathbf{n}(s)$  vanishes everywhere on  $\mathcal{P}$ ;
  - 3'. flux through the thickness of the in-plane shear total strain vector  
 $\gamma = 2\epsilon_{13} \mathbf{t} + 2\epsilon_{23} \mathbf{n} = 2\epsilon_{x3} \mathbf{i} + 2\epsilon_{y3} \mathbf{j}$  is equal to zero;
- is constant over  $\mathcal{P}$*

## Elastic stresses

1. elastic stress vector on every plane parallel to the  $z$ -axis is parallel to  $\mathbf{k}$ ;
  - 2'. shear elastic stress between  $z$ -axis and direction  $\mathbf{n}(s)$  vanishes everywhere on  $\mathcal{P}$ ;
  - 3'. flux through the thickness of the in-plane shear elastic stress vector  
 $\tau = \tau_{13} \mathbf{t} + \tau_{23} \mathbf{n} = \tau_{x3} \mathbf{i} + \tau_{y3} \mathbf{j}$  is equal to zero;
- is constant over  $\mathcal{P}$*

$$\mathbf{G}\boldsymbol{\varepsilon} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, (2\delta\bar{\epsilon}_{13})_1, \epsilon_{23/1}, \epsilon_{23/2}\}^T$$

$$\mathbf{H}\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}, (2\delta\bar{\sigma}_{13})_1, \sigma_{23/1}, \sigma_{23/2}\}^T$$

Constraints 2' e 3' can be referred to  $\varepsilon_{x3}^{(\text{tot})}$  and  $\varepsilon_{y3}^{(\text{tot})}$

$$(2\delta\bar{\Lambda}_{x3})/1\mathbf{i} + (2\delta\bar{\Lambda}_{y3})/1\mathbf{j} \cong (2\delta\bar{\Lambda}_{13})/1\mathbf{t} + (2\delta\bar{\Lambda}_{23})/1\mathbf{n} = \mathbf{0}$$

$$2\bar{\Lambda}_{x3} = \gamma_x(z), \quad 2\bar{\Lambda}_{y3} = \gamma_y(z)$$

Analogously, constraints 2' e 3' can be referred to  $\sigma_{x3}^{(\text{el})}$  and  $\sigma_{y3}^{(\text{el})}$

$$\begin{aligned}
\mathcal{E}(\mathbf{u}, \boldsymbol{\chi}, \boldsymbol{\omega}) = & \frac{1}{2} \int_{\Omega} \{ \mathcal{C}_{\alpha\beta\gamma\delta} (\Lambda_{\alpha\beta} - \omega_{\alpha\beta})(\Lambda_{\gamma\delta} - \omega_{\gamma\delta}) + 2\mathcal{C}_{\alpha\beta 33} (\Lambda_{\alpha\beta} - \omega_{\alpha\beta}) u'_3 + \mathcal{C}_{3333} (u'_3)^2 + \\
& + 4\mathcal{C}_{1313} [\Lambda_{13} + (2\delta\omega_{13})_1]^2 + 8\mathcal{C}_{1323} [\Lambda_{13} + (2\delta\omega_{13})_1] [\Lambda_{23} + \tilde{\omega}_{\alpha 3/\alpha}] + \\
& + 4\mathcal{C}_{2323} [\Lambda_{23} + \tilde{\omega}_{\alpha 3/\alpha}]^2 \} dv - \int_{\Omega} [\chi_{\alpha\beta} \Lambda_{\alpha\beta} + 2\chi_{13} (2\delta\bar{\Lambda}_{13})_1 + \\
& + 2\tilde{\chi}_{\alpha 3} \Lambda_{23/\alpha}] dv - \Pi_{ext},
\end{aligned}$$

where

$$\boldsymbol{\Lambda} = \hat{\nabla} \mathbf{u}$$

$$\Pi_{ext} = \int_{\Omega} (b_{\alpha} u_{\alpha} + b_3 u_3) dv + \int_{\Sigma} (\hat{p}_{\alpha} u_{\alpha} + \hat{p}_3 u_3) da + \int_{\mathcal{P}|_{\pm L}} (p_{\alpha}^{\pm} u_{\alpha}|_{\pm L} + p_3^{\pm} u_3|_{\pm L}) da$$

strain and stress dual constraints

## First-order shear-deformable beam model accounting for warping shear effects

### Total strains

1. in-plane (dilatation and shear) total strain components vanish everywhere on  $\mathcal{P}$ ;
- 2''. shear total strain between  $z$ -axis and direction  $\mathbf{n}(s)$  at every position  $(s, \eta)$  in  $\mathcal{P}$  does not depend upon the thickness coordinate  $\eta$  and varies linearly along  $p$ ;
- 3'. flux through the thickness of the in-plane shear total strain vector  $\gamma = 2\epsilon_{13} \mathbf{t} + 2\epsilon_{23} \mathbf{n} = 2\epsilon_{x3} \mathbf{i} + 2\epsilon_{y3} \mathbf{j}$  is constant over  $\mathcal{P}$ ;

$$\mathbf{G}\boldsymbol{\epsilon} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, (2\delta\bar{\epsilon}_{13})_{/1}, \epsilon_{23/11}, \epsilon_{23/2}\}^T$$

1.      3.      2.

### Total strains

1. in-plane (dilatation and shear) total strain components vanish everywhere on  $\mathcal{P}$ ;
- 2''. shear total strain between  $z$ -axis and direction  $\mathbf{n}(s)$  at every position  $(s, \eta)$  in  $\mathcal{P}$  does not depend upon the thickness coordinate  $\eta$  and varies linearly along  $p$ ;
- 3'. flux through the thickness of the in-plane shear total strain vector  $\gamma = 2\epsilon_{13} \mathbf{t} + 2\epsilon_{23} \mathbf{n} = 2\epsilon_{x3} \mathbf{i} + 2\epsilon_{y3} \mathbf{j}$  is constant over  $\mathcal{P}$ ;

$$\mathbf{G}\boldsymbol{\epsilon} = \{\epsilon_{11}, \epsilon_{22}, \epsilon_{12}, (2\delta\bar{\epsilon}_{13})_1, \epsilon_{23/11}, \epsilon_{23/2}\}^T$$



$$\mathbf{H}\boldsymbol{\sigma} = \{\sigma_{11}, \sigma_{22}, \sigma_{12}, (2\delta\bar{\sigma}_{13})_1, \sigma_{23/11}, \sigma_{23/2}\}^T$$

### Elastic stresses

1. elastic stress vector on every plane parallel to the  $z$ -axis is parallel to  $\mathbf{k}$ ;
- 2''. shear elastic stress between  $z$ -axis and direction  $\mathbf{n}(s)$  at every position  $(s, \eta)$  in  $\mathcal{P}$  does not depend upon the thickness coordinate  $\eta$  and varies linearly along  $p$ ;
- 3'. flux through the thickness of the in-plane shear elastic stress vector  $\tau = \tau_{13} \mathbf{t} + \tau_{23} \mathbf{n} = \tau_{x3} \mathbf{i} + \tau_{y3} \mathbf{j}$  is constant over  $\mathcal{P}$  ;

$$\begin{aligned}
\mathcal{E}(\mathbf{u}, \chi, \omega) = & \frac{1}{2} \int_{\Omega} \left\{ C_{\alpha\beta\gamma\delta} (\Lambda_{\alpha\beta} - \omega_{\alpha\beta})(\Lambda_{\gamma\delta} - \omega_{\gamma\delta}) + 2C_{\alpha\beta 33} (\Lambda_{\alpha\beta} - \omega_{\alpha\beta}) u'_3 + \right. \\
& + C_{3333} (u'_3)^2 + 8C_{1323} [\Lambda_{13} + (2\delta\omega_{13})_{/1}] [\Lambda_{23} - \tilde{\omega}_{13/11} + \tilde{\omega}_{23/2}] + \\
& + 4C_{1313} [\Lambda_{13} + (2\delta\omega_{13})_{/1}]^2 + 4C_{2323} [\Lambda_{23} - \tilde{\omega}_{13/11} + \tilde{\omega}_{23/2}]^2 \} dv + \\
& \left. - \int_{\Omega} [\chi_{\alpha\beta} \Lambda_{\alpha\beta} + 2\chi_{13} (2\delta\Lambda_{13})_{/1} + 2(\tilde{\chi}_{13} \Lambda_{23/11} + \tilde{\chi}_{23} \Lambda_{23/2})] dv - \Pi_{ext} \right.
\end{aligned}$$

defined on the manifold:  $\tilde{\chi}_{\alpha 3} = \tilde{\chi}_{13/1} = 0$  on  $\partial \mathcal{P} \times ]-L, L[$ ;  $\tilde{\omega}_{\alpha 3} = \tilde{\omega}_{13/1} = 0$  on  $\partial \mathcal{P} \times ]-L, L[$   
 $\chi_{13} = 0$  on  $\partial p \times ]-L, L[$ ;  $\omega_{13} = 0$  on  $\partial p \times ]-L, L[$

where

$$\boldsymbol{\Lambda} = \hat{\nabla} \mathbf{u}$$

$$\Pi_{ext} = \int_{\Omega} (b_{\alpha} u_{\alpha} + b_3 u_3) dv + \int_{\Sigma} (\hat{p}_{\alpha} u_{\alpha} + \hat{p}_3 u_3) da + \int_{\mathcal{P}|_{\pm L}} (p_{\alpha}^{\pm} u_{\alpha} \Big|_{\pm L} + p_3^{\pm} u_3 \Big|_{\pm L}) da$$

stationarity of  $\mathcal{E}(\mathbf{u}, \omega, \chi)$  with respect to  $\chi$  gives the equations

$$u_{1/1} - \kappa u_2 = 0$$

$$u_{2/2} = 0$$

$$u_{2/1} + \kappa u_1 + u_{1/2} = 0$$

$$\int_{-\delta}^{\delta} [u_{3/11} + u'_{1/1}] dn = 0$$

$$u'_{2/11} + u_{3/211} = 0$$

$$u'_{2/2} + u_{3/22} = 0$$

stationarity of  $\mathcal{E}(\mathbf{u}, \omega, \chi)$  with respect to  $\chi$  gives the equations

$$\begin{aligned} u_{1/1} - \kappa u_2 &= 0 \\ u_{2/2} &= 0 \\ u_{2/1} + \kappa u_1 + u_{1/2} &= 0 \end{aligned}$$

$$\begin{aligned} \int_{-\delta}^{\delta} [u_{3/11} + u'_{1/1}] dn &= 0 \\ u'_{2/11} + u_{3/211} &= 0 \\ u'_{2/2} + u_{3/22} &= 0 \end{aligned}$$

from which, by integration, the displacement field is obtained and represented in Cartesian form

$$u_x(s, \eta, z) = u_c(z) - \theta(z)[y(s, \eta) - y_c]$$

$$u_y(s, \eta, z) = v_c(z) + \theta(z)[x(s, \eta) - x_c]$$

$$u_3(s, \eta, z) = w_c(z) - \phi_y(z)x(s, \eta) + \phi_x(z)y(s, \eta) + \phi_z(z)\psi_c(s, \eta)$$

$$\phi_x(z) = \gamma_y(z) - v'_c(z)$$

$$\phi_y(z) = u'_c(z) - \gamma_x(z)$$

$$\phi_z(z) = \theta'(z) - \gamma_\psi(z)$$

warping function

$$\psi_c(s, \eta) = \psi_o + \int_0^s \mathbf{r} \cdot \mathbf{n} ds - \eta \mathbf{r} \cdot \mathbf{t}$$

$$2\bar{\Lambda}_{xz} = \bar{\gamma}_{xz} = \gamma_x(z) - \gamma_\psi(z)(\tilde{y} - y_c)$$

$$2\bar{\Lambda}_{yz} = \bar{\gamma}_{yz} = \gamma_y(z) + \gamma_\psi(z)(\tilde{x} - x_c)$$

$$\mathbf{r} = \tilde{\mathbf{x}} - \mathbf{x}_c$$

$$\int_{\mathcal{P}} \psi_c da = \int_{\mathcal{P}} x \psi_c da = \int_{\mathcal{P}} y \psi_c da = 0$$

stationarity of  $\mathcal{E}(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\chi})$  with respect to  $\boldsymbol{\omega}$  gives the equations

$$\omega_{\alpha\beta}) \quad C_{\alpha\beta\gamma\delta}[\Lambda_{\gamma\delta} - \omega_{\gamma\delta}] + C_{\beta\beta 33}u'_3 = 0$$

$$\omega_{13}) \quad \left\{ \int_{-\delta}^{\delta} C_{1313}[\Lambda_{13} + (2\delta\omega_{13})_{/1}] + C_{1323}[\Lambda_{23} - \tilde{\omega}_{13/11} + \tilde{\omega}_{23/2}] d\eta \right\}_{/1} = 0$$

$$\tilde{\omega}_{13}) \quad \{C_{1323}[\Lambda_{13} + (2\delta\omega_{13})_{/1}] + C_{2323}[\Lambda_{23} - \tilde{\omega}_{13/11} + \tilde{\omega}_{23/2}]\}_{/11} = 0$$

$$\tilde{\omega}_{23}) \quad \{C_{1323}[\Lambda_{13} + (2\delta\omega_{13})_{/1}] + C_{2323}[\Lambda_{23} - \tilde{\omega}_{13/11} + \tilde{\omega}_{23/2}]\}_{/2} = 0$$

from which  $\omega_{ij}$  are obtained.

By substituting  $\boldsymbol{\omega}$  and  $\mathbf{u}$  into the functional  $\mathcal{E}(\mathbf{u}, \boldsymbol{\omega}, \boldsymbol{\chi})$  and by integrating over  $\mathcal{P}$ , the energy functional  $\hat{\mathcal{E}}$  is obtained

$$\hat{\mathcal{E}}(u_c, v_c, w_c, \phi_x, \phi_y, \phi_z, \theta) = \frac{1}{2} \int_{-L}^L \mathbf{D}\mathbf{e} \cdot \mathbf{e} dz - \int_{-L}^L \mathbf{q} \cdot \hat{\mathbf{s}} dz - \mathbf{Q}^\pm \cdot \hat{\mathbf{s}}|_{\pm L}$$

$$\hat{\mathcal{E}}(u_c, v_c, w_c, \phi_x, \phi_y, \phi_z, \theta) = \frac{1}{2} \int_{-L}^L \mathbf{D}\mathbf{e} \cdot \mathbf{e} dz - \int_{-L}^L \mathbf{q} \cdot \hat{\mathbf{s}} dz - \mathbf{Q}^\pm \cdot \hat{\mathbf{s}} \Big|_{\pm L}$$

$\hat{\mathbf{s}}$ : generalized displacements

$\mathbf{e}$ : generalized total strains

$\mathbf{q}$ : generalized distributed forces

$\mathbf{Q}^\pm$ : generalized end-located forces

$\mathbf{D}$ : generalized elasticity matrix of the beam

$$\hat{\mathbf{s}} = \{u_c, v_c, w_c, \phi_x, \phi_y, \theta, \phi_z\}^T$$

$$\mathbf{q} = \{q_x, q_y, q_z, m_x, m_y, m_z, m_\psi\}^T$$

$$\mathbf{e} = \{w'_c, \phi'_x, \phi'_y, \phi'_z, \theta', \gamma_x, \gamma_y, \gamma_\psi\}^T$$

$$\mathbf{Q}^\pm = \{Q_x^\pm, Q_y^\pm, Q_z^\pm, M_x^\pm, M_y^\pm, M_z^\pm, M_\psi^\pm\}^T$$

$$\{q_x, q_y, q_z\} = \int_{\mathcal{P}} \{b_x, b_y, b_z\} da + \int_{\partial\mathcal{P}} \{\hat{p}_x, \hat{p}_y, \hat{p}_z\} d\varrho$$

$$\{Q_x^\pm, Q_y^\pm, Q_z^\pm\} = \int_{\mathcal{P}} \{p_x^\pm, p_y^\pm, p_z^\pm\} da$$

$$\{m_x, m_y, m_\psi\} = \int_{\mathcal{P}} \{y, -x, \psi_c\} b_z da + \int_{\partial\mathcal{P}} \{y, -x, \psi_c\} \hat{p}_z d\varrho$$

$$\{M_x^\pm, M_y^\pm, M_\psi^\pm\} = \int_{\mathcal{P}} \{y, -x, \psi_c\} p_z^\pm da$$

$$m_z = \int_{\mathcal{P}} [b_y(x - x_c) - b_x(y - y_c)] da + \int_{\partial\mathcal{P}} [\hat{p}_y(x - x_c) - \hat{p}_x(y - y_c)] d\varrho$$

$$M_z^\pm = \int_{\mathcal{P}} [p_y^\pm(x - x_c) - p_x^\pm(y - y_c)] da$$

## generalized elasticity matrix

$\mathbf{D}^{(0)}$ : Bernoulli/Vlasov terms

$\mathbf{D}^{(1)}$ : Timoshenko/warping shear terms

## FSDT beam model with warping shear

$$\mathbf{D}_{8 \times 8} = \begin{bmatrix} \mathbf{D}_{5 \times 5}^{(0)} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{3 \times 3}^{(1)} \end{bmatrix}$$

$$D_{11} = \int_{\mathcal{P}} \hat{C}_{3333} da$$

$$D_{12} = \int_{\mathcal{P}} \hat{C}_{3333} y(s, \eta) da$$

$$D_{13} = - \int_{\mathcal{P}} \hat{C}_{3333} x(s, \eta) da$$

$$D_{14} = \int_{\mathcal{P}} \hat{C}_{3333} \psi_c(s, \eta) da$$

$$D_{22} = \int_{\mathcal{P}} \hat{C}_{3333} y^2 da$$

$$D_{23} = - \int_{\mathcal{P}} \hat{C}_{3333} xy da$$

$$D_{24} = \int_{\mathcal{P}} \hat{C}_{3333} y \psi_c da$$

$$D_{33} = \int_{\mathcal{P}} \hat{C}_{3333} x^2 da$$

$$D_{34} = - \int_{\mathcal{P}} \hat{C}_{3333} x \psi_c da$$

$$\mathbf{D} D_{44} = \int_{\mathcal{P}} \hat{C}_{3333} \psi_c^2 da$$

$$D_{55} = 4 \int_{\mathcal{P}} \hat{C}_{1313} \eta^2 da$$

$$D_{66} = \int_{\mathcal{P}} [C_{1313} (\tilde{x}_{/1})^2 - 2C_{1323} \tilde{x}_{/1} \tilde{y}_{/1} + C_{2323} (\tilde{y}_{/1})^2] da$$

$$D_{67} = \int_{\mathcal{P}} \{(C_{1313} - C_{2323}) \tilde{x}_{/1} \tilde{y}_{/1} + C_{1323} [(\tilde{x}_{/1})^2 - (\tilde{y}_{/1})^2]\} da$$

$$D_{68} = \int_{\mathcal{P}} [C_{1313} \tilde{x}_{/1} \mathbf{r} \cdot \mathbf{n} + C_{2323} \tilde{y}_{/1} \mathbf{r} \cdot \mathbf{t} - C_{1323} (\tilde{x}_{/1} \mathbf{r} \cdot \mathbf{t} + \tilde{y}_{/1} \mathbf{r} \cdot \mathbf{n})] da \quad D_{77} = \int_{\mathcal{P}} [C_{1313} (\tilde{y}_{/1})^2 + 2C_{1323} \tilde{x}_{/1} \tilde{y}_{/1} + C_{2323} (\tilde{x}_{/1})^2] da$$

$$D_{78} = \int_{\mathcal{P}} [C_{1313} \tilde{y}_{/1} \mathbf{r} \cdot \mathbf{n} - C_{2323} \tilde{x}_{/1} \mathbf{r} \cdot \mathbf{t} + C_{1323} (\tilde{x}_{/1} \mathbf{r} \cdot \mathbf{n} - \tilde{y}_{/1} \mathbf{r} \cdot \mathbf{t})] da \quad D_{88} = \int_{\mathcal{P}} [C_{1313} (\mathbf{r} \cdot \mathbf{n})^2 + C_{2323} (\mathbf{r} \cdot \mathbf{t})^2 - 2C_{1323} (\mathbf{r} \cdot \mathbf{n})(\mathbf{r} \cdot \mathbf{t})] da$$

## Reduced elastic moduli

$$\hat{C}_{3333} = C_{3333} - C_{\alpha\beta 33} (C_{\alpha\beta\gamma\delta})^{-1} C_{\gamma\delta 33}$$

$$\hat{C}_{1313} = C_{1313} - \frac{(C_{1323})^2}{C_{2323}}$$

stationarity of  $\hat{\mathcal{E}}(u_c, v_c, w_c, \phi_x, \phi_y, \phi_z, \theta)$  with respect to the unknown displacement functions gives equilibrium field equations and boundary conditions



$$u_c, v_c, w_c, \phi_x, \phi_y, \phi_z, \theta$$

$$\mathbf{e} = \{w'_c, \phi'_x, \phi'_y, \phi'_z, \theta', \gamma_x, \gamma_y, \gamma_\psi\}^T$$



$$\mathbf{S} = \{N \ C_x \ C_y \ C_\psi \ C_z \ T_x \ T_y \ C_z^\psi\}^T = \mathbf{D} \ \mathbf{e}$$

$\mathbf{S}$  satisfies the global equilibrium equations

$$\{N, C_x, C_y, C_\psi\} = \int_{\mathcal{P}} \{1, y, -x, \psi_c\} \sigma_{33} da, \quad C_z = \int_{\mathcal{P}} (\sigma_{y3}x - \sigma_{x3}y) da$$

$$C_z^\psi = \int_{\mathcal{P}} (\sigma_{13} \mathbf{r} \cdot \mathbf{t} - \sigma_{23} \mathbf{r} \cdot \mathbf{n}) da$$

$$\{T_x, T_y\} = \int_{\mathcal{P}} \boldsymbol{\tau} \cdot \{\mathbf{i}, \mathbf{j}\} da = \int_{\mathcal{P}} \sigma_{13} \{\tilde{x}_{/1}, \tilde{y}_{/1}\} da + \int_{\mathcal{P}} \sigma_{23} \{-\tilde{y}_{/1}, \tilde{x}_{/1}\} da$$

$C_z$ : primary twisting moment

$C_\psi$ : bimoment (warping torque)

$C_z^\psi$ : secondary twisting moment

$$\mathbf{e} = \{w'_c, \phi'_x, \phi'_y, \phi'_z, \theta', \gamma_x, \gamma_y, \gamma_\psi\}^T$$



$$\sigma_{\alpha\beta}^{(el)} = 0$$

$$\sigma_{13}^{(el)} = -2\hat{C}_{1313}\theta' \eta + (C_{1313}\tilde{x}_{/1} - C_{1323}\tilde{y}_{/1})\gamma_x + (C_{1313}\tilde{y}_{/1} + C_{1323}\tilde{x}_{/1})\gamma_y - (C_{1313}\mathbf{r} \cdot \mathbf{n} - C_{1323}\mathbf{r} \cdot \mathbf{t})\gamma_\psi$$

$$\sigma_{23}^{(el)} = (C_{1323}\tilde{x}_{/1} - C_{2323}\tilde{y}_{/1})\gamma_x + (C_{1323}\tilde{y}_{/1} + C_{2323}\tilde{x}_{/1})\gamma_y - (C_{1323}\mathbf{r} \cdot \mathbf{n} - C_{2323}\mathbf{r} \cdot \mathbf{t})\gamma_\psi$$

$$\sigma_{33}^{(el)} = \hat{C}_{3333}(w'_c - \phi'_y x + \phi'_x y + \phi'_z \psi_c)$$



$\sigma^{(el)}$  does not satisfy local equilibrium equations

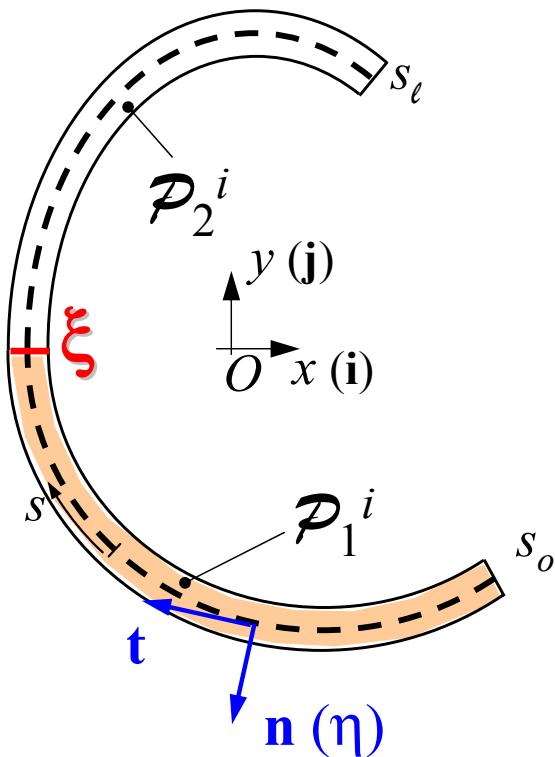
$$\boldsymbol{\sigma}^{tot} = \mathbf{C}(\hat{\nabla} \mathbf{u} - \mathbf{H}^* \boldsymbol{\omega}) - \mathbf{G}^* \boldsymbol{\chi}$$

$$(\mathbf{G}^* \boldsymbol{\chi})_{33} = 0 \quad \rightarrow \quad \sigma_{33}^{el} = \sigma_{33}^{tot}$$

# Total shear stresses recovering

$$\mathcal{P}^i = \mathcal{P} \times \{z^i\}$$

internal cross-section



$$\mathcal{P}^i = \mathcal{P}_1^i \cup \mathcal{P}_2^i \text{ and } \mathcal{P}_1^i \cap \mathcal{P}_2^i = \xi$$

along- $z$  field and boundary equilibrium eqs.

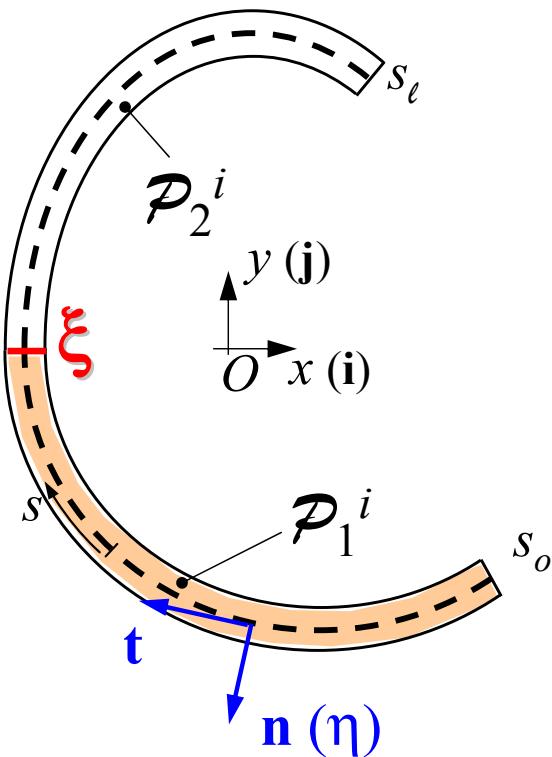
$$\begin{aligned} \operatorname{Div} \boldsymbol{\tau} + \sigma_{33/3} + b_3 &= 0 & \text{in } \Omega \\ \boldsymbol{\tau} \cdot \mathbf{n}_\partial &= \hat{p}_3 & \text{on } \Sigma \end{aligned}$$

$$\mathbf{n}_\partial = \begin{cases} \pm \mathbf{n} & \text{on } \partial \mathcal{P}|_{\eta=\pm\delta} \\ -\mathbf{t} & \text{on } \partial \mathcal{P}|_{s=s_o} \\ \mathbf{t} & \text{on } \partial \mathcal{P}|_{s=s_\ell}. \end{cases}$$

# Total shear stresses recovering

$$\mathcal{P}^i = \mathcal{P} \times \{z^i\}$$

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$$\mathbf{n}_\partial = \begin{cases} \pm \mathbf{n} & \text{on } \partial \mathcal{P}|_{\eta=\pm\delta} \\ -\mathbf{t} & \text{on } \partial \mathcal{P}|_{s=s_o} \\ \mathbf{t} & \text{on } \partial \mathcal{P}|_{s=s_\ell}. \end{cases}$$

along- $z$  field and boundary equilibrium eqs.

$$\begin{aligned} \operatorname{Div} \boldsymbol{\tau} + \sigma_{33/3} + b_3 &= 0 & \text{in } \Omega \\ \boldsymbol{\tau} \cdot \mathbf{n}_\partial &= \hat{p}_3 & \text{on } \Sigma \end{aligned}$$

$$\sigma_{33}^{el} = \sigma_{33}^{tot}$$

$$\bar{\sigma}_{13}(s^i, z) = -\frac{1}{2\delta} \left[ \int_{\mathcal{P}_1^i} (\sigma_{33/3} + b_3) da + \int_{\partial \mathcal{P}_1^i \setminus \xi} \hat{p}_3 d\varrho \right]$$

$$\sigma_{13}(s^i, \eta, z) \cong \sigma_{13}^{(el)} + (\bar{\sigma}_{13} - \bar{\sigma}_{13}^{(el)}) = -2\hat{\mathcal{C}}_{1313}\theta'_\eta + \bar{\sigma}_{13}$$

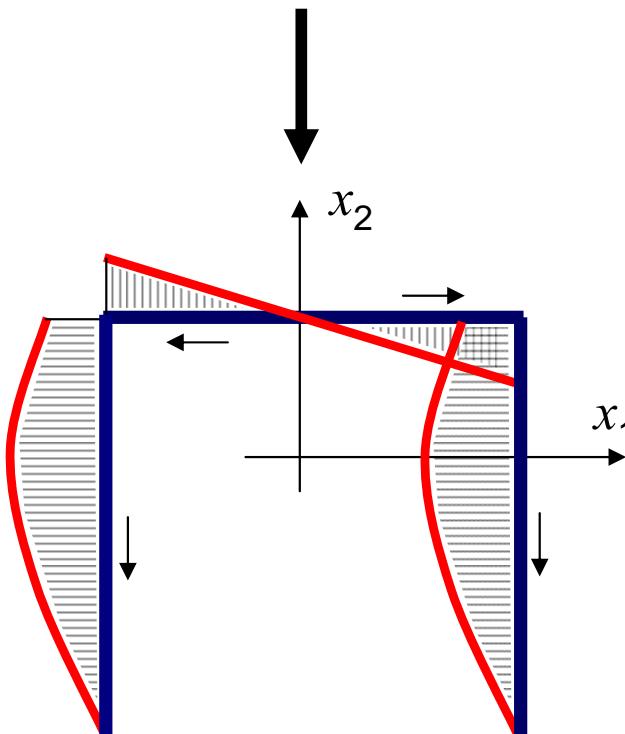
$$\sigma_{23/2} + (\sigma_{33/3} - \bar{\sigma}_{33/3}) + (b_3 - \bar{b}_3) - \frac{\hat{p}_3^+ + \hat{p}_3^-}{2\delta} = 0$$

↷  $\sigma_{23}^{tot}$

# **Other refinements and generalizations**

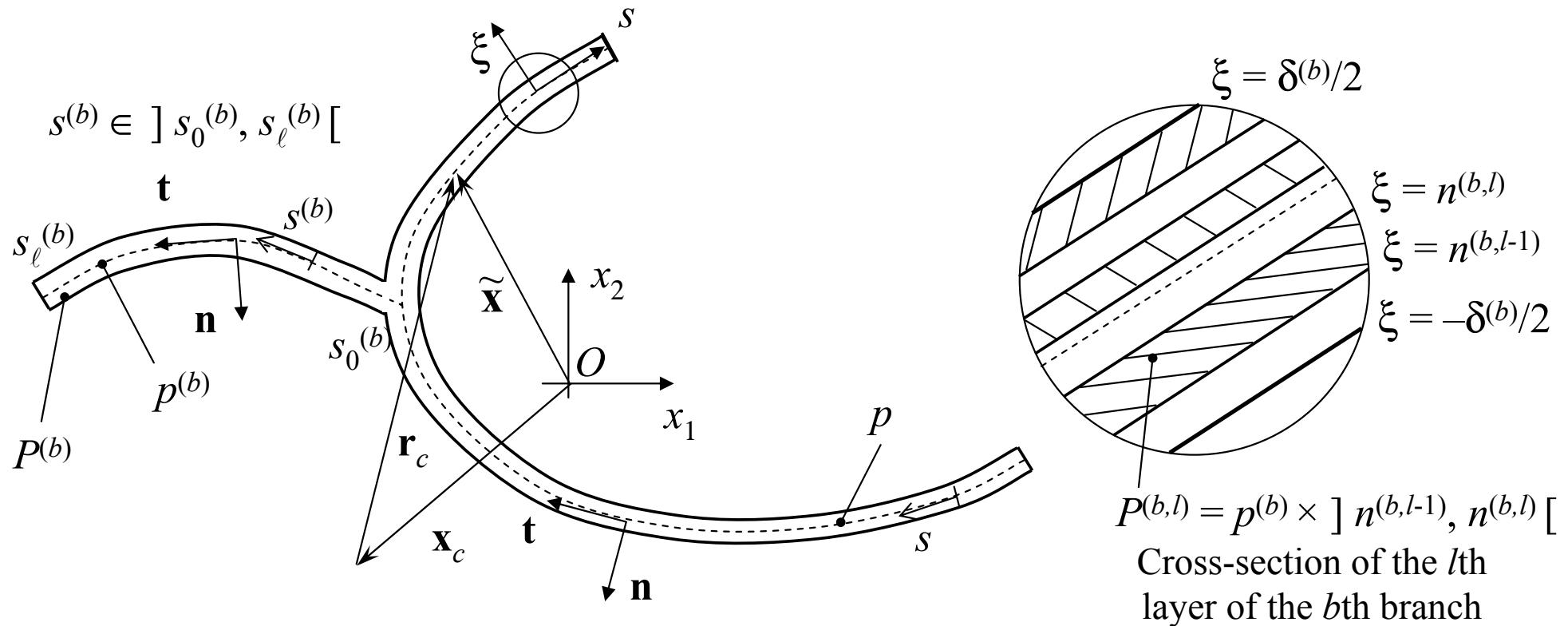
## Multi-branch refined assumptions

Average over the thickness of total shear strains and elastic shear stresses are piecewise constant functions of the curvilinear abscissa  $s$ , i.e. they are assumed to be constant on each branch.



**F. Maceri, G. Vairo.** Anisotropic thin-walled beam models: a rational deduction from three-dimensional elasticity. *to appear on J. of Mechanics of Materials and Structures* (2008).

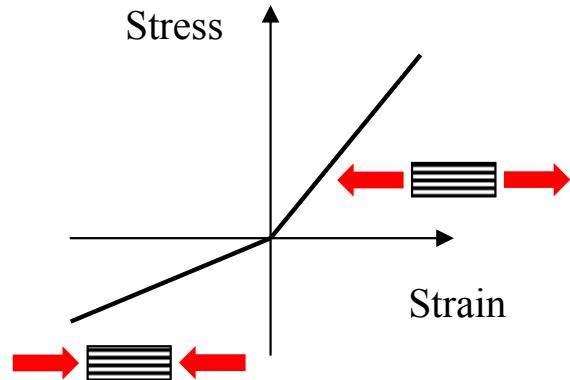
# Laminated thin-walled beams



**F. Maceri, G. Vairo** (2007) Rational derivation of a single-layer model for laminated thin-walled beams, Proc. 6th Int. Conf. on Intelligent Processing and Manufacturing of Materials, 2007.

**F. Maceri, G. Vairo.** Anisotropic thin-walled beam models: a rational deduction from three-dimensional elasticity. *to appear on J. of Mechanics of Materials and Structures* (2008).

## Bimodular thin-walled beams



$$\Phi(\boldsymbol{\varepsilon}) = \frac{1}{2} \{h\mathbf{C}^+ + (1-h)\mathbf{C}^-\}[\boldsymbol{\varepsilon}] \cdot \boldsymbol{\varepsilon}$$

unilateral anisotropic behaviour of fiber-reinforced layers

**F. Maceri, G. Vairo.** Bimodular thin-walled beams: a variational model based on a dual constrained approach. WCCM8-ECCOMAS08, 2008.

**F. Maceri, G. Vairo.** Anisotropic thin-walled beam models: a rational deduction from three-dimensional elasticity. *to appear on J. of Mechanics of Materials and Structures* (2008).



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**Thank You**



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