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DISPLACEMENT/MIXED FINITE ELEMENT FORMULATION FOR BEAM AND FRAME PROBLEMS

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ABSTRACT

In this work two parallel approaches - displacement and mixed finite element based methods are employed for seeking the solutions of small strain/large displacement of in-plane beam and frame problems, and further its consistent numerical implementation in a finite element program is achieved. The adopted kinematics hypothesis is based on the geometrically exact theory for beams with straight beam element under static loading conditions. An updated lagrangian approach is used for the structure geometry at any stage. Shear deformation is considered in both approaches. Shear locking is observed in displacement based formulation which is removed by reduced integration rule whereas no locking is observed in force based formulation.

Displacement based method is the immediate work from literature- a two-dimensional geometrically exact theory of beams while an independent attempt is made in mixed finite element method to include the geometric nonlinearity. This uses the approach of Hu-Washizu variational formulation at element level: with in the general framework of the displacement method for the solution of the global structural problem. Equilibrium and compatibility are always satisfied along the element in force based formulation. Some planar problems are studied to validate the results from both models.

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CONTENTS

1. INTRODUCTION

Nonlinear structural analysis has been the subject of very extensive research. More specifically, several studies on the nonlinear behavior of frames have been conducted over the last four decades. As the number of these studies is vast, only a few of the works that include nonlinear geometric effects are listed herein. Most of the past literature are based on the displacement formulation, with only a few based on force or mixed formulations. Mixed finite element formulation is the recent approach, still in developing stage for analysis in which variable field -displacement, stresses and strain are interpolated independently. Some of the past literature on the both element formulation is presented in this chapter and in the end objective of the work is mentioned.

1.1 DISPLACEMENT BASED FINITE ELEMENT METHOD

Non-linear analysis of beam like structural system has been extensively studied by the displacement based approach in the last four decades of finite element solution method. The approach can model to three-dimensional deformations of curved beam elements considering both issues of geometric and material non-linearity under dynamic loading. The state of the art employs the usual finite element principles means non-linear strain displacement relation are considered, and polynomial interpolation functions are assumed for the displacement fields. Further the linearization of the equilibrium equations is done for calculating the element parameters in which displacement field is the primary unknowns. This involves the formation of stiffness matrix and residual vectors for the global structure and solve by any non-linear solution algorithm methods(i.e, full or modified Newton Raphson type solution). The development of elements for elastic nonlinear analysis of frames started in the sixties. Some of the earliest papers on elastic nonlinear analysis are, for instance, Argyris et al. (1964)(2) and Connor et al. (1968)(3). In the development of a geometrically nonlinear beam element, basically an updated Lagrangian or a total Lagrangian formulation can be employed. One important early study on large displacement analysis of frame structures is the paper by Bathe and Bolourchi (1979)(4), which presented an updated Lagrangian and a total Lagrangian formulation for three-dimensional straight beam elements derived from the principles of continuum mechanics.

A two-dimensional geometrically exact beam theory of straight element beams(rods)was developed by Reissner(1972)(1) and was later extended to curved three-dimensional beam element theory by Reissner(1981)(5), but the exactness in the theory is lost due to the simplifications in the rotation matrix. The theory later revisited by Simo (1985)(6) and Simo et al. 1986)(7) mentioning dynamic form with geometrical aspects for three dimensional beams. Straight beam axis is the special case of the curved reference axis. The extension of two-dimensional formulations to three dimensions is by no means trivial for geometric non-linear problems. This is due to the non-vectorial nature of large rotations in space. In geometrically linear problems, rotations are considered infinitesimal, and therefore can be treated as vectors. However, in spatial problems with large displacements, rotations are not vector entities, as can be easily confirmed by verifying that the commutative property of vectors does not hold for large rotations in space. This can be treated by imposing a sequence of rotations to a body, around two or three orthogonal axes, and concluding that the final position of the body depends on the sequence of the imposed rotations. Other publications on curved beam element is mentioned by Stolarski and Belytschko(1982)(8), Saje (1991)(9), and Ibrahimbegovic and Frey (1992)(10).

Further some other publications on large displacement inelastic frame analysis, not specifically for steel frames, are Cichon (1984)(11), Simo et al. (1984)(12), Tuomala and Mikkola (1984)(13), Nedergaard and Pedersen (1985)(14), Chan (1988)(15), Gendy and Saleeb (1993)(16), Ovunc and Ren (1996)(17), Park and Lee (1996)(18), and Waszczyszyn and Janus-Michalska (1998)(19).

1.2 FORCE BASED FORMULATION

Displacement based models encounter serious numerical problems in the description of the nonlinear response of structures under severe ground motions. Often, very small subdivision of the structural members is necessary to obtain reasonable results. Even, so numerical instabilities are not uncommon specially under cyclic loading.

Recent studies show that elements which are based on the flexibility method, and thus on force interpolation functions inside the element, are better suited to describe the nonlinear behavior of structural members, particularly under conditions of strain softening or load reversal and the lower number of model degrees of freedom for comparable accuracy in global and local response. Only a few elements based on the force approach have been proposed for the nonlinear analysis of frames. A brief description of these elements is given below.

Backlund (1976)(20) proposed a hybrid-type beam element for analysis of elasto- plastic plane frames with large displacements. In this work, the flexibility matrix is computed based on an assumed distribution of forces along the element. However, the method also uses displacement interpolation functions that assume linearly varying curvature and a constant axial strain to compute the section deformations from the end displacements. Section forces are obtained from these section deformations using the constitutive relation, but the section forces calculated in this way are not in equilibrium with the applied loads. These deviations only decrease as the number of elements is increased in the member discretization. Large displacement effects are taken into account by updating the structure geometry.

Kondoh and Atluri (1987)(21) employed an assumed-stress approach to derive the tangent stiffness of a plane frame element, subject to conservative or non-conservative loads. The element is assumed to undergo arbitrarily large rigid rotations but small axial stretch and relative (non-rigid) point-wise rotations. It is shown that the tangent stiffness can be derived explicitly, if a plastic-hinge method is employed. Shi and Atluri (1988)(22) extended these ideas to three-dimensional frames, claiming that the proposed element could undergo arbitrarily large rigid rotations in space. However, as also noticed by Abbasnia and Kassimali (1995)(23), the rotations of the joints are treated by Shi and Atluri as vectorial quantities. This limits the application of the element to problems with small rotations, leading to inaccurate results when the proposed element is used in structures subject to large rotations.

Carol and Murcia (1989)(24) presented a hybrid-type formulation valid for nonlinear material and second order plane frame analysis. The authors refer to the method as being exact in the sense that the equilibrium equations are satisfied strictly. However, second order effects are considered using a linear strain-displacement relation, which restricts the formulation to relatively small deformations. Besides, the second order effect is not correctly accounted for in the stiffness matrix expression, leading to an inconsistent tangent stiffness, and consequently causing low convergence rate.

Neuenhofer and Filippou (1998)(25) presented a force-based element for geometrically nonlinear analysis of plane frame structures, assuming linear elastic material response, and moderately large rotations. The basic idea of the formulation consists in using a force interpolation function for the bending moment field that depends on the transverse displacements, such that the equilibrium equations are satisfied in the deformed configuration. Consistently, the adopted strain displacement relation is nonlinear. The weak form of this kinematic equation leads to a relation between nodal displacements and section deformations. In this work, a new method, called Curvature-Based Displacement Interpolation (CBDI), was proposed in order to derive the transverse displacements from the curvatures using Lagrangian polynomial interpolation by integrating twice. The motivation for this work was the extension of the force-based element proposed in Neuenhofer and Filippou (1997)(26) to include geometrically nonlinear behavior. This latter work was, in turn, based on the force formulation that was initially proposed by Ciampi and Carlesimo (1986)(27), and was continually developed in several other works, including Spacone (1994)(28), Spacone et al. (1996a)(29), Spacone et al. (1996b)(30), and Petrangeli and Ciampi (1997)(31).

Further work, Ranzo and Petrangeli (1998)(32) and Petrangeli et al. (1999)(33) introduced shear effects in the analysis of reinforced concrete structures, following the idea of the force-based formulation presented in Petrangeli and Ciampi (1997)(31). Another new extension, accounting for the bond-slip effect in reinforced concrete sections, is presented by Monti and Spacone (2000)(34).

Souza (2000)(35) proposed a force-based formulation for inelastic large displacement analysis of planar and spatial frames under the Hellinger-Reissner variational principle. In that study full geometric-nonlinearity for large displacement is included with the co-rotational formulation. The formulation was geometrically approximate, as opposed to the geometrically exact theories. Further material non-linearity is included by fiber-discretization method which means integration at the material point over the cross-section by appropriate stress-strain relationship.

More recent work by R.L.Taylor (2003)(36) proposed the elasto-plastic mixed beam element model but with in the regime of small deformation for plane frame structures. The approach uses the three field (displacement, strain, stress) formulation based on the Hu-Washizu principle where forces, displacement and strain are interpolated independently using shape functions. The advantage of using this variational principle is that the shear can be included without giving concerned about the locking in elements.

It is important to emphasize that exact force distributions are easily determined for one-dimensional elements only. In case of continuum elements, exact force interpolations functions are not available. Therefore, force based formulations seem especially suited for the nonlinear analysis of frames.

1.3 OBJECTIVE AND SCOPE

The main objective of this dissertation is to present the exact geometric non-linearity beam theory under the displacement and mixed finite beam element formulation and their consistent numerical implementation in finite element program. For the brevity of the work, elastic constitutive laws are considered and further the target for analysis is planar beam and frame structures. Shear effect is included in both models. The work is divided into the chapters with the following brief overview:

- chapter 2 describe the kinematic hypothesis of the element that is considered for both models. It also speaks about the compatibility equations and constitutive laws that is considered in the model.
- chapter 3 describes the model for the displacement based formulation. A brief description of the variational form that is considered and further the solution algorithm is described in the end.
- chapter 4 describes the model for the mixed finite beam element. The variational form that is considered in the model with the solution algorithm in the end is mentioned.
- chapter 5 shows some examples to validate the model. In all the examples their accuracy is being checked by existing standalone finite element programs and with the results available in literature.
- The conclusions and possible future work drawn from this study are presented in chapter 6

2. GEOMETRICALLY NON-LINEAR PROBLEMS

The geometrically non-linear analysis of elastic in-plane oriented bodies e.g beams, frames is presented in updated Lagrangian approach. Displacement and Rotations are unrestricted in magnitude. The vectorial nature of large rotations is preserved in the plane problems and therefore commutative property of vectors hold good in this regime. Ressiner (1972)(1) approach for exact geometrical beam theory for plane frame is implemented in this chapter.

2.1 KINEMATICS AND DEFORMATION

This section introduces the brief summary of the basic equations for finite deformation solid mechanics problems, which for detailed description should be referred to R.L. Taylor et al.(2005)(37). Though the equations is presented in three dimensional plane - the two dimensional plane being a special case of these. Further the model is also framed in two dimensional plane in the present work.

In cartesian coordinates the position vector of the material points in a fixed reference configuration, also called un-deformed configuration, Ω_0, is described in terms of its components as :

$$\mathbf{X} = X_I \mathbf{E_I}; \qquad I = 1, 2, 3 \tag{2.1}$$

where $\mathbf{E_I}$ are unit orthogonal base vectors and summation convention is used for repeated indices.

After the body is loaded each material point is described by its position vector, $\mathbf{x}$, in the current

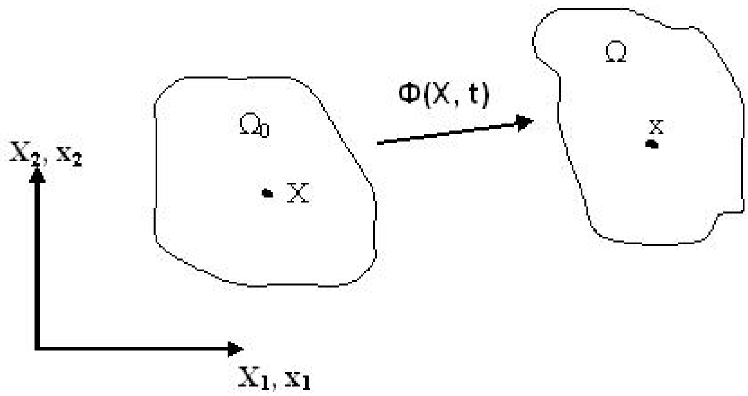


Figure 2.1: Reference and current deformed configuration for finite deformation problems

deformed configuration, Ω , by the component as:

$$\mathbf{x} = x_i \mathbf{e_i}; \qquad i = 1, 2, 3 \tag{2.2}$$

where $\mathbf{e_i}$ are unit base vectors for the current time t, further common origins and directions of the reference and current coordinates are used for brevity. The position vector at the current time is related to the reference configuration position vector through the mapping

$$x_i = \phi_i(X_I, t) \tag{2.3}$$

The mapping function ϕ_i is required as part of any solution and is analogous to the displacement vector, which is mentioned next. Since the common origins and directions for the coordinate are used, a displacement vector may be introduced as the change between the two frames. Hence accordingly,

$$x_i = \delta_{iI}(X_I + U_I) \tag{2.4}$$

where δ_{iI} is a rank-two shifter tensor between the two coordinate frames, and is defined by a quantity Kronnecker delta in mathematical literature. Using the shifter, a displacement component may be written with respect to either the reference configuration or the current configuration and related through

$$u_i = \delta_{iI} U_I \;\; and \;\; U_I = \delta_{iI} u_i \tag{2.5}$$

Thus either u_i or U_I can be used equally to develop finite element parameters. A fundamental measure of deformation is described by the deformation gradient relative to X_I given by

$$F_{iI} = \frac{\partial \phi_i}{\partial X_I} \tag{2.6}$$

with subject to the constraint

$$J = \det F_{iI} \; > \; 0 \tag{2.7}$$

to ensure that material volume elements remain positive. The deformation gradient helps in mapping a differential line element and volume element in the reference configuration into one in the current configuration as

$$dx_i = \frac{\partial \phi_i}{\partial X_I} dX_I \;\; and \;\; dv = J dV \tag{2.8}$$

The deformation gradient may be expressed in terms of the displacement as

$$F_{iI} = \delta_{iI} + \frac{\partial u_i}{\partial X_I} = \delta_{iI} + u_{i,I} \tag{2.9}$$

Since the deformation gradient is a two-point tensor which is referred to both the reference and the current configurations. It is common to introduce deformation measures which are completely related to either the reference or the current configuration. For the reference configuration, the right Cauchy-Green deformation tensor, C_{IJ}, is introduced as

$$C_{IJ} = F_{iI} F_{iJ} \tag{2.10}$$

Alternatively the Green strain tensor, E_{IJ}, is given as

$$E_{IJ} = \frac{1}{2}(C_{IJ} - \delta_{IJ}) \tag{2.11}$$

may be used. The Green strain may be expressed in terms of the reference displacements as

$$E_{IJ} = \frac{1}{2}[\frac{\partial U_I}{\partial X_J} + \frac{\partial U_J}{\partial X_I} + \frac{\partial U_K}{\partial X_I}\frac{\partial U_K}{\partial X_J}] \tag{2.12}$$

Similarly, we can express stresses with respect to reference and current states. Cauchy(true) stress, σ_{ij}, and the Kirchhoff stress, τ_{ij},are symmetric measure of stress defined in the current configuration which are related through the determinant of the deformation gradient as

$$\tau_{ij} = J\sigma_{ij} \tag{2.13}$$

The second Piola-Kirchhoff stress, S_{IJ} , is a symmetric stress measure with respect to the reference configuration and is related to the Kirchhoff stress through the deformation gradient as

$$\tau_{ij} = F_{iI}S_{IJ}F_{jJ} \tag{2.14}$$

2.2 LARGE DISPLACEMENT THEORY OF BEAMS

The behavior for the bending of a beam for the small strain/large deformation is developed in this section. Displacement and rotations along the element can be arbitrarily large(Geometrically exact theory). Such formulation is realistic for most practical slender structures such as beams, frames and shells. In these developments the normal to the cross-section is followed, as contrasted to following the tangent to the beam axis, by an orthogonal frame. For keeping the model simple straight beam element is assumed. The motion for the beam for which the orthogonal triad, $\mathbf{a_i}$ of the beam cross-section can be written as

$$\phi_i \equiv x_i = x_i^0 + \Lambda_{iI}Z_I \tag{2.15}$$

where the orthogonal matrix is related to the$\mathbf{a_i}$vectors as

$$\mathbf{\Lambda} = [\mathbf{a_1} \;\; \mathbf{a_2} \;\; \mathbf{a_3}\,] \tag{2.16}$$

The response in torsion is assumed uncoupled from the axial and flexural response. Consequently, the displacements and forces associated with torsion are omitted in the following discussion for

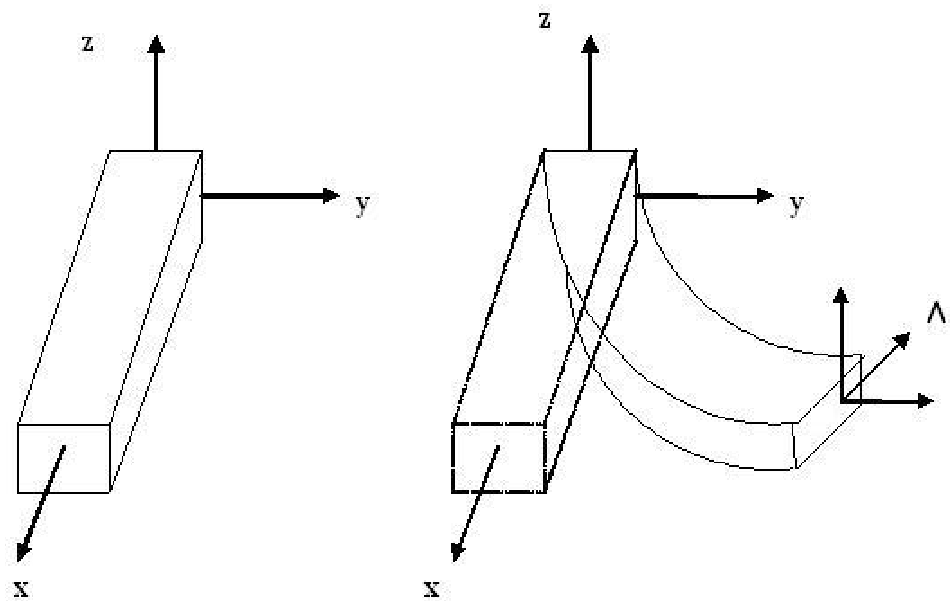


Figure 2.2: Displacement field of the beam

simplicity. Therefore the matrix form for the above motion of the beam can be represented as

$$\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} X \\ 0 \\ 0 \end{Bmatrix} + \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} + \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} \end{bmatrix} \begin{Bmatrix} 0 \\ Y \\ Z \end{Bmatrix} \tag{2.17}$$

where the reference coordinate $X_1(X)$ is the beam axis $X_2(Y), X_3(Z)$ are cross-section axes, $u(X), v(Y), and\, w(Z)$ are displacements of the beam reference axis. Here the rotation matrix $\Lambda(X)$ of the beam cross-section does not necessarily remain normal to the beam axis and thus admits the possibility of transverse shearing deformations. It is also assumed that the cross-section do not distort in their own planes. We consider the two-dimensional case where the motion is restricted to the e.g $X - Z$ plane. The orthogonal matrix may then be represented as

$$\mathbf{\Lambda} = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \tag{2.18}$$

Inserting this in eq. (2.2.3) and expanding, the deformed position then is described compactly as

$$x = X + u(X) + Z\sin\beta(X)$$
$$y = Y \tag{2.19}$$
$$z = w(X) - Z\cos\beta(X)$$

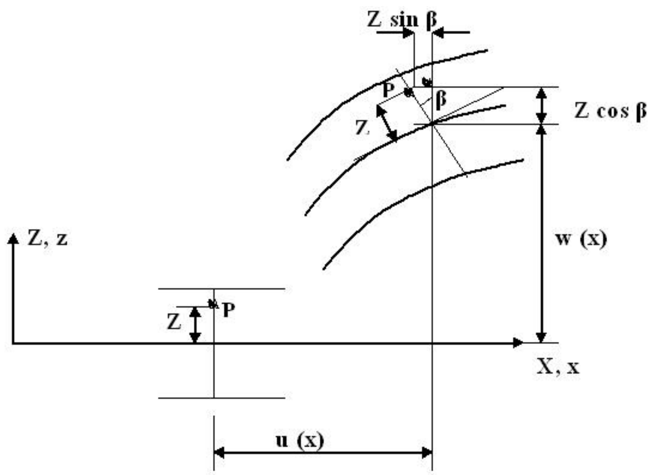


Figure 2.3: Displacement field of the beam

The deformation gradient for this displacement is given by the relation

$$\mathbf{F_{iI}} = \begin{bmatrix} [1 + u_{,X} + Z\beta_{,,X}\cos\beta] & 0 & \sin\beta \\ 0 & 1 & 0 \\ [w_{,X} - Z\beta_{,,X}\sin\beta] & 0 & \cos\beta \end{bmatrix} \tag{2.20}$$

The two non-zero Green strain are obtained using eq. (2.1.12) while, ignoring the quadratic term in Z, are expressed by

$$\begin{aligned} E_{XX} &= u_{,X} + \frac{1}{2}(u_{,X}^2 + w_{,X}^2) + Z\Lambda\beta_{,X} = E^0 + ZK^b \\ 2E_{XZ} &= (1 + u_{,X})\sin\beta + w_{,X}\cos\beta = \Gamma \end{aligned} \tag{2.21}$$

where E^0 and Γ are strains which are constant on the cross-section and K^b measures change in rotation (curvature) of the cross-sections and where

$$\Lambda = (1 + u_{,X})\cos\beta - w_{,X}\sin\beta \tag{2.22}$$

2.2.1 Constitutive laws

For simplicity, the strains are small and the constitution may be expressed by a linear elastic relation between the Green-Lagrange strains and the second Piola-Kirchhoff stresses. It can be expressed as,

$$S_{XX} = EE_{XX} \quad and \quad S_{XZ} = 2GE_{XZ} \tag{2.23}$$

where E is the Young's modulus and G a shear modulus.

3. DISPLACEMENT BASED FINITE ELEMENT METHOD

3.1 INTRODUCTION

This chapter describes the small strain/large displacement theory of the plane frame beam column element based on the displacement method, employing updated lagrangian approach. In the present work the beam theory developed by Reissner(1972) for two-dimensional beams is considered here which is able to accommodate large displacement and large rotations. This theory stand contrast with the corotational approach, means that all the element arrays are handled directly in the global structure coordinate system, rather than in the local element(rotating) coordinate system. Therefore, no final local-global transformations for element arrays need to be done. The material properties, Kinematics hypothesis and deformation assumptions is same as described in chapter 1. Static unidirectional loading is considered with no inertial term in the formulation. Shear deformation is considered with reduced quadrature rule to avoid 'shear locking'. Solution strategy is also mentioned in the end of the chapter. The model is supported by some examples and also crosschecked by existing standalone programs in the chapters 6.

3.2 VARIATIONAL FORMULATION

Variational description for finite deformation can be derived from Galerkin (weak) or variational form. The formulation can be done in either the reference configuration or in the current configuration, which is the standard practice in the finite element approximations. The suggested process requires the minimization of the total potential energy of the system in terms of a prescribed displacement field. For the beam column element displacement field is given by eq.

(2.2.5) and the compatibility relation is given by eq (2.2.7).

The following assumptions are necessary for the given variational principle:

- Conservation of external loads (body forces and boundary tractions).
- Hyperelastic material behavior.

The external loads are conservative if the external work done is equal to the sum of the work done by imposed forces which mathematically expressed as.

$$\Pi_{ext}(u) = -\int_{\Gamma_t} \bar{t}^T u \, d\Gamma \tag{3.1}$$

where $\Pi_{ext}(u)$ denotes the terms from end forces and loading along the length, $\bar{t}$ are the imposed tractions on the part Γ_t of the element boundary Γ. This functional is referred to as the potential energy of the external loading. A common example of conservative loads are gravity loading (dead loads) with constant direction.

A material model is hyper-elastic (or Green elastic) if there exists a stored energy functions $W(\varepsilon)$, such that the stress (or second Piola -Kirchhoff stress if we are in reference configuration) can be expressed as a function of strain ε as

$$\sigma = \frac{\partial W(\varepsilon)}{\partial \varepsilon} \tag{3.2}$$

If this constitutive relation has a unique inverse, i.e, if $W(\varepsilon)$ is strictly convex, a unique strain ε can be found for a given stress, using the complementary energy density

$$\chi(\sigma) = \sigma\varepsilon(\sigma) - W(\varepsilon(\sigma)) \tag{3.3}$$

Taking the derivative of eq. (3.1.3) with respect to σ gives

$$\begin{aligned} \frac{\partial \chi(\sigma)}{\partial \sigma} &= \varepsilon(\sigma) + \sigma \frac{\partial \varepsilon(\sigma)}{\partial \sigma} - \frac{\partial W(\varepsilon(\sigma))}{\partial \varepsilon} \frac{\partial \varepsilon(\sigma)}{\partial \sigma} \\ &= \varepsilon(\sigma) + \sigma \frac{\partial \varepsilon(\sigma)}{\partial \sigma} - \sigma \frac{\partial \varepsilon(\sigma)}{\partial \sigma} \\ &= \varepsilon(\sigma) \end{aligned} \tag{3.4}$$

The inverse form is possible for most elastic material models in the range of small strain, but this is not always the case for large elastic strains.

A variational equation for finite elasticity for the beam may be written in the reference configuration as,

$$\delta\Pi = \int_{\Omega} (\delta E_{XX} S_{XX} + 2\delta E_{XZ} S_{XZ})\, dV - \delta\Pi_{ext} \tag{3.5}$$

By resolving the volume integral into one along the length times an integral over the beam cross-section area A and define force resultants as

$$T^p = \int_A S_{XX}\, dA, \quad S^p = \int_A S_{XZ}\, dA, \quad M^b = \int_A S_{XX} Z\, dA \tag{3.6}$$

Integrating the above equation using eq. (2.2.9) and eq. (2.2.7) the elastic behaviour of the beam resultant becomes

$$T^p = EAE^0 \quad S^p = \kappa GA\Gamma, \quad M^b = EIK^b \tag{3.7}$$

the variational equation may be written compactly with the help eq. (3.1.6) as

$$\delta\Pi = \int_{Ł} (\delta E^0 T^p + \delta\Gamma S^p + \delta K^b M^b)\, dX - \delta\Pi_{ext} \tag{3.8}$$

where virtual strains with the help of eq. (2.2.7) can be written as

$$\begin{aligned} \delta E^0 &= (1 + u_{,X})\, \delta u_{,X} + w_{,X}\, \delta w_{,X} \\ \delta\Gamma &= sin\beta\, \delta u_{,X} + cos\beta\, \delta w_{,X} + \Lambda\, \delta\beta \\ \delta K^b &= \Lambda\, \delta\beta_{,X} + \beta_{,X}(cos\beta\, \delta u_{,X} - sin\beta\, \delta w_{,X} - \Gamma\, \delta\beta) \end{aligned} \tag{3.9}$$

3.3 FINITE ELEMENT APPROXIMATION

A finite element approximation for the displacement field may be introduced as

$$\begin{Bmatrix} u \\ w \\ \beta \end{Bmatrix} = {N_\alpha}^d(X) \begin{Bmatrix} \widetilde{u}_\alpha \\ \widetilde{w}_\alpha \\ \widetilde{\beta}_\alpha \end{Bmatrix} \quad \alpha = 1, 2 \tag{3.10}$$

where the shape functions ${N_\alpha}^d(X)$ for each displacement field are the same. The linear shape functions is used for each displacement field.

$$[N_1^d \;\; N_2^d] = [(1-\frac{X}{L}) \;\; \frac{X}{L}] \tag{3.11}$$

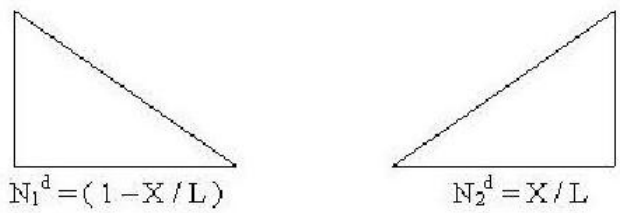


Figure 3.1: Displacement field shape functions.

Using this approximation eq. (3.1.8) can be manipulated as

$$\delta\Pi = [\delta\widetilde{u}_\alpha \;\; \delta\widetilde{w}_\alpha \;\; \delta\widetilde{\beta}_\alpha\,] \int_L \mathbf{B}_\alpha^{\mathbf{T}} \begin{Bmatrix} T^p \\ S^p \\ M^b \end{Bmatrix} dX - \delta\Pi_{ext} \tag{3.12}$$

where

$$\mathbf{B}_\alpha = \begin{bmatrix} (1+u_{,X})\,N_{\alpha,X}^d & w_{,X}\,N_{\alpha,X}^d & 0 \\ sin\,\beta\,N_{\alpha,X}^d & cos\,\beta\,N_{\alpha,X}^d & \Lambda N_\alpha^d \\ \beta_{,X} cos\,\beta\,N_{\alpha,X}^d & -\beta_{,X} sin\,\beta\,N_{\alpha,X}^d & (\Lambda\,N_{\alpha,X}^d - \Gamma\,\beta_{,X}\,N_\alpha^d) \end{bmatrix} \tag{3.13}$$

or

$$\mathbf{B} = [\,\mathbf{B}_1 \;\; \mathbf{B}_2\,] \tag{3.14}$$

The non- linear equilibrium equation for a quasi-static problem that is solved at each load step is given by

$$\mathbf{\Psi_{n+1}} = \underbrace{f_{n+1}}_{external force} - \underbrace{\int_L \mathbf{B}_\alpha^{\mathbf{T}} \begin{Bmatrix} T_{n+1}^p \\ S_{n+1}^p \\ M_{n+1}^b \end{Bmatrix} dX}_{internal force(F_{int})} = 0 \tag{3.15}$$

The behavior of axial and bending deformations occur at a cross-section is assumed uncoupled. Further the elastic material behavior is assumed given by constitutive laws in eq. (2.2.9), therefore stress resultants can be expressed as

$$T^p = EAE^0, \;\; S^p = \kappa GA\Gamma, \;\;\; M^b = EIK^b \tag{3.16}$$

or in matrix form can be expressed as

$$\mathbf{F} = \mathbf{D}_T\mathbf{E} \tag{3.17}$$

where

$$\mathbf{D_T} = \begin{bmatrix} EA & & \\ & \kappa GA & \\ & & EI \end{bmatrix} \tag{3.18}$$

where F is the stress vector. Further $\mathbf{E}$ may be expressed as

$$\mathbf{E} = \mathbf{B}^T \, \widetilde{u} \tag{3.19}$$

For a Newton-Raphson type solution the tangent stiffness matrix is deduced by a linearization of eq. (3.1.15) as,

$$\mathbf{K}_T = \int_L \mathbf{B}^T \frac{\partial F}{\partial \widetilde{u}} \, dX + \int_L \frac{(\partial \mathbf{B}^T \, F)}{\partial \widetilde{u}} \, dX \tag{3.20}$$

where $\widetilde{u} = [\, \widetilde{u}_1 \; \widetilde{w}_1 \; \widetilde{\beta}_1 \; \widetilde{u}_2 \; \widetilde{w}_2 \; \widetilde{\beta}_2 \,]$ and with the assumption that $\frac{\partial f_{n+1}}{\partial \widetilde{u}} = 0$ means that no load changes with deformation

Using eq. (3.1.17), eq. (3.1.18) and eq. (3.1.19), the tangent matrix eq. (3.1.20) can be expressed as

$$[\mathbf{K}_T]_{6\times 6} = \underbrace{\int_L \mathbf{B}^T \mathbf{D_T} \mathbf{B} \, dX}_{(K_m)_(6\times 6)} + [\mathbf{K_g}]_{6\times 6} \tag{3.21}$$

where

$$\mathbf{K_g} = \begin{bmatrix} (\mathbf{K_g})_{11} & (\mathbf{K_g})_{12} \\ (\mathbf{K_g})_{21} & (\mathbf{K_g})_{22} \end{bmatrix} \tag{3.22}$$

and also

$$(\mathbf{K}_g)_{\alpha\beta} = \int_L (N_{\alpha,X} \begin{bmatrix} T^p & 0 & M^b \cos\beta \\ 0 & T^p & -M^b \sin\beta \\ M^b \cos\beta & -M^b \sin\beta & 0 \end{bmatrix} N_{\beta,X} + N_\alpha \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G_3 \end{bmatrix} N_\beta$$

$$+ N_{\alpha,X} \begin{bmatrix} 0 & 0 & G_1 \\ 0 & 0 & G_2 \\ 0 & 0 & -M^b \Gamma \end{bmatrix} N_\beta + N_\alpha \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ G_1 & G_2 & -M^b \Gamma \end{bmatrix} N_{\beta,X})\, dX \qquad (3.23)$$

where $\alpha,\ \beta = 1, 2$ and

$$\begin{aligned} G_1 &= S^p \cos\beta - M^b\, \beta_{,X} \sin\beta \\ G_2 &= -S^p \sin\beta - M^b\, \beta_{,X} \cos\beta \\ G_3 &= -S^p\, \Gamma - M^b\, \beta_{,X}\, \Lambda \end{aligned} \qquad (3.24)$$

More comprehensive derivation of $\mathbf{K}_g$ is expressed in Appendix A

3.4 SUMMARY OF SOLUTION ALGORITHM

Description of the solution algorithm for theory developed in previous sections is presented in this section. Graphical overview of the entire process is presented in the flow chart of the structure state determination. Full Newton-Raphson method is used for the solution algorithm which means in particular the tangent stiffness matrix, $\mathbf{K}_T$, is reformed at each iterations. The above interpolation functions will lead to 'shear locking' and it is necessary to compute integrals for shear stress by using a 'reduced quadrature'. This implies that for a two-noded beam element, use one quadrature point exactly in the middle for integrating each element. The solution strategy can be breakdown in the following steps.

1. The Full Newton Raphson iteration (i) starts with the given load step on global structure
2. For the first iteration step set $j = 0$
   $\widetilde{a}_{i+1}^{j=0} = \widetilde{a}_i$
   For solution at step $(i + 1)$ assume the state at the previous step (i) is known.

3. Start the element state determination. Loop over all the elements in the structure for state at iteration j.

4. Compute $\mathbf{K}_m$, $\mathbf{K}_g$, $\mathbf{F}_{int}$ using eq. (3.1.21) and eq. (3.1.15) .Assemble the global stiffness matrix and internal force vector.

5. Check convergence on global residual force vector R. If converge move to next load step(set $i = i + 1$) and start from step 2, else

6. Solve $d\widetilde{a} = \mathbf{K}_T^{-1}\,\mathbf{R}$ and update the geometry $\widetilde{a}^{j+1} = \widetilde{a}^{j} + d\widetilde{a}$ Go to step 3

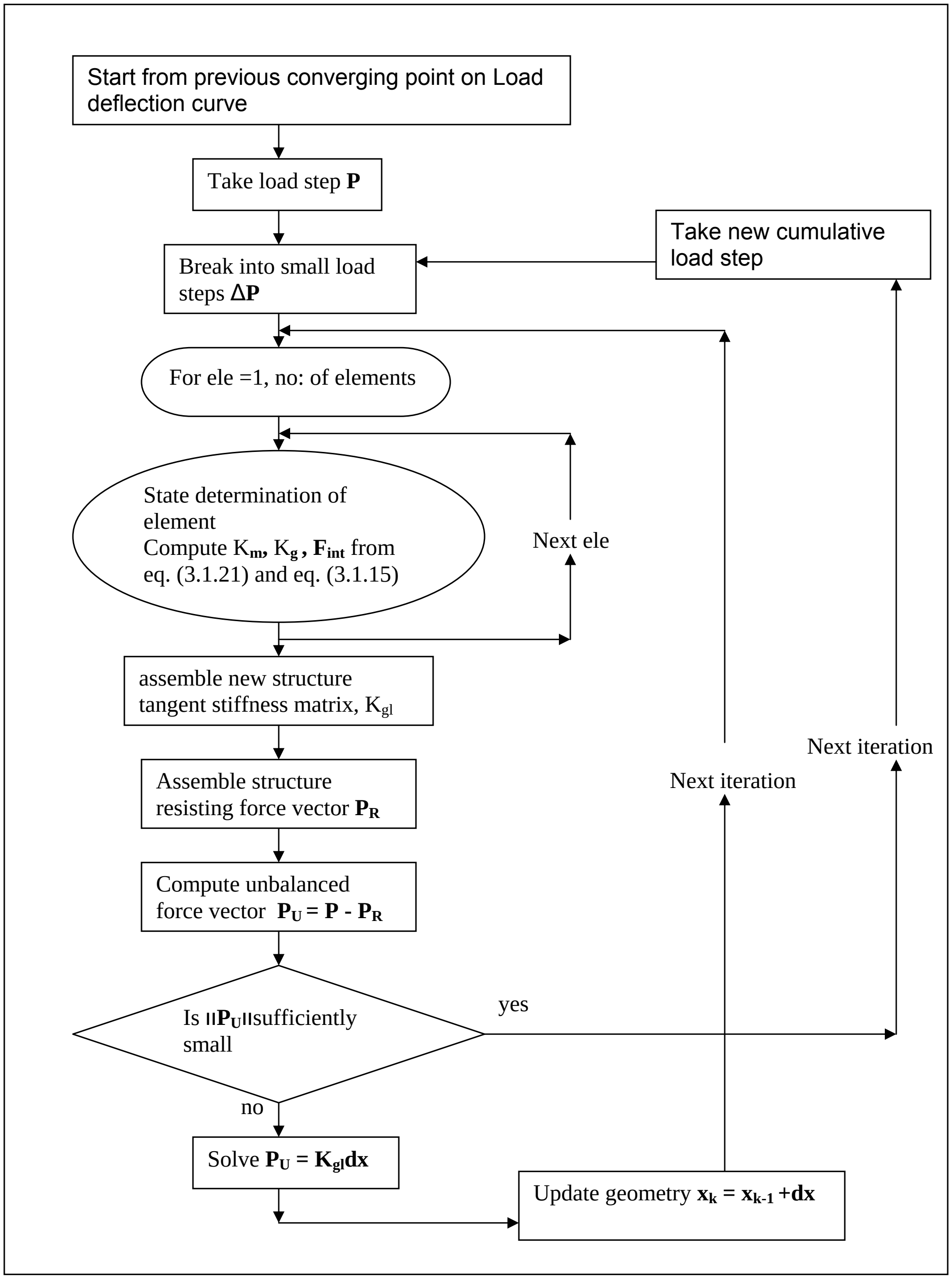


Figure 3.2: Flow chart of structure state determination.

4. MIXED FINITE ELEMENT METHOD

4.1 INTRODUCTION

This chapter describes the proposed formulation of the plane frame beam column element with mixed approach in the variational form. The material properties, kinematic hypothesis and deformation is same as described in chapter 2 in the present discussion. Formulation follows the usual step as in displacement based formulation of finding the minimization or stationarity of the potential function. The approach we now present is based on the use of a three-field (displacement, strain, stress) formulation based on the Hu-Washizu variational principle. The state determination procedure of the structure iteratively determines the element resisting forces and stiffness matrix while strictly satisfying element equilibrium and compatibility in each iteration. This procedure is considerably more involved than for displacement based element. Shear deformation can be readily included without inducing the shear locking in the element and , thus the behavior is independent of the number of integration points along the element.

4.2 VARIATIONAL FORMULATION

For the case at hand, the displacement field is given by eq. (2.2.5), the compatibility relation with two non zero strains in the corresponding directions given by eq. (2.2.7). For an elastic material with stress (T, M, S) and strain (E^0, K^b, Γ) the Hu- Washizu principle may be written as

$$\begin{aligned}\Pi_{hw}(\sigma,\varepsilon,u) &= \int_{\Omega} W(\varepsilon)\, d\Omega + \int_{\Omega} S(\,\Gamma^{c} - \Gamma\,)\, d\Omega \\ &+ \int_{\Omega} \sigma(\,T^{p}, M,)(\,E^{c}(\,E^{0}, K^{b}) - E\,)\, d\Omega - \Pi_{ext}\end{aligned} \tag{4.1}$$

where $W(\varepsilon)$ is the stored energy function introduced also in eq. (3.1.2) and Π_{ext} is the potential for the body and boundary loading. Here the Γ^{c}, E^{c} comes from the appropriate strain compatibility equations mentioned in eq. (2.21)

The stationarity of the Hu-Washizu principle is imposed by taking its first variation with respect to the independent fields ($u,\ \varepsilon,\ \sigma$) and setting it equal to zero

$$\begin{aligned}\delta\Pi_{hw} &= \frac{\partial\Pi_{hw}}{\partial u}\delta u + \frac{\Pi_{hw}}{\partial S}\delta S + \frac{\Pi_{hw}}{\partial E}\delta E \\ &= \delta_{u}\Pi_{hw} + \delta_{S}\Pi_{hw} + \delta_{E}\Pi_{hw} - \delta\Pi_{ext} = 0\end{aligned} \tag{4.2}$$

which can be re-written as after putting the corresponding terms in the above equation

$$\begin{aligned}\delta\Pi_{hw} = &\int_{L} \{\delta E_{0}\,[\,T^{c} - T\,] + \delta T\,[\,u_{,X} + \frac{1}{2}(u_{,X}^{2} + w_{,X}^{2}) - E_{0}\,] \\ &\quad + \delta u_{,X}[(1+u_{,X})\,T + sin\,\beta\, S + \cos\,\beta_{,X}\, M\,]\}\, dX \\ &+ \int_{L} \{\delta K_{b}\,[\,M^{c} - M\,] + \delta M\,[\,\Lambda\,\beta_{,X} - K^{b}\,] \\ &\quad + \delta w_{,X}[\,T w_{,X} - \beta_{,X}\, sin\,\beta\, M + cos\beta\, S\,]\}\, dX \\ &+ \int_{L} \{\delta\Gamma\,[\,S^{c} - S\,] + \delta S\,[(1+u_{,X})\sin\,\beta + w_{,X}\cos\,\beta - \Gamma\,] \\ &\quad + \delta\beta_{,X}\,[\,M\,\Lambda\,] + \delta\beta\,[\,S\Lambda - \Gamma\beta_{,X}M\,]\}\, dX - \delta\Pi_{ext} = 0\end{aligned} \tag{4.3}$$

The model assumes elastic material behaviour, however the inelastic constitutive forms can be introduced from any constitutive model in terms of specified strains, strain rates (plasticity), or functional of strain (viscoelasticity). The equilibrium equations, the compatibility equation and the consistent force field shape functions with the constitutive laws at section can be deduce from the above variational form of potential.

4.2.1 Equilibrium Equation

Since the displacement variations field can be arbitrarily chosen in this derivation (i.e., a displacement interpolation function was not adopted), the equilibrium equations are satisfied pointwise (strong form).

$$\begin{aligned}\delta_u\Pi_{hw} = \int_L \{&\delta u_{,X}[(1+u_{,X})\,T + sin\,\beta\,S + \cos\,\beta_{,X}\,M] \\ &+ \delta w_{,X}[\,Tw_{,X} - \beta_{,X} sin\,\beta\,M + cos\beta\,S\,] + \delta\beta_{,X}[\,M\Lambda\,] \\ &+ \delta\beta[\,S\Lambda - \Gamma\beta_{,X}\,M\,]\}\,dX - \delta\Pi_{ext} = 0\end{aligned} \tag{4.4}$$

4.2.2 Compatibility Equation

The compatibility equations are imposed weakly as from the eq. (4.1.3)

$$\begin{aligned}\delta_S\Pi_{hw} = \int_L \{&\delta T\,[\,u_{,X} + \frac{1}{2}(u_{,X}^2 + w_{,X}^2) - E_0\,] + \delta M\,[\,\Lambda\beta_{,X} - K^b] \\ &+ \delta S\,[(1+u_{,X})\sin\,\beta + w_{,X}\cos\,\beta - \Gamma\,]\}\,dX = 0\end{aligned} \tag{4.5}$$

If this equation could be satisfied for all statically admissible variations δT, δM, and δS (i.e., all virtual force systems in equilibrium), it would imply the strong form of the compatibility relation eq. (2.2.7)

4.2.3 Constitutive laws - section force consistency Equation

If this equation could be satisfied for all statically admissible variations δE_0, δK_b, and $\delta\Gamma$ consistent approach for section forces equilibrium with the constitutive laws can be deduced

$$\begin{aligned}\delta_E\Pi_{hw} = \int_L \{&\delta E_0\,[\,T^c - T\,] + \delta K_b\,[\,M^c - M\,] \\ &+ \delta\Gamma\,[\,S^c - S\,]\,\}\,dX = 0\end{aligned} \tag{4.6}$$

which means

$$\begin{aligned}T^c &= T \\ M^c &= M \\ S^c &= S\end{aligned} \tag{4.7}$$

in order to make the stationarity of the variational form.

There is an advantage in deriving the present formulation form a variational principle: it allows the concentration of all intrinsic characteristics- equilibrium equations, compatibility equations and constitutive terms of the problem in a single expression. Based on this, it should be clear that the proposed element formulation can be used to solve more general problems such as, for instance, elasto-plastic analysis.

We also note that the shear in each element may be computed from moment equilibrium as

$$S = -\frac{\partial M}{\partial X} \tag{4.8}$$

and, thus it is not necessary to add additional force parameter to the element.

4.3 FINITE ELEMENT APPROXIMATION

The finite element approximations for the displacement, stress, and strain field can be introduced respectively as,

$$\begin{Bmatrix} u \\ w \\ \beta \end{Bmatrix} = {N_\alpha}^d(X) \begin{Bmatrix} \widetilde{u}_\alpha \\ \widetilde{w}_\alpha \\ \widetilde{\beta}_\alpha \end{Bmatrix} \quad \alpha = 1, 2 \tag{4.9}$$

$$\begin{Bmatrix} E_0 \\ K_b \\ \Gamma \end{Bmatrix} = \begin{Bmatrix} {N_\alpha}^E(X)\widetilde{E}_0 \\ {N_\alpha}^{K_b}(X)\widetilde{K}_b \\ {N_\alpha}^\Gamma(X)\widetilde{\Gamma} \end{Bmatrix} \quad and \quad \begin{Bmatrix} T \\ S \\ M \end{Bmatrix} = \begin{Bmatrix} {N_\alpha}^T(X)\widetilde{T} \\ {N_\alpha}^S(X)\widetilde{S} \\ {N_\alpha}^M(X)\widetilde{M}_\alpha \end{Bmatrix} \quad \alpha = 1, 2 \tag{4.10}$$

where the shape functions for each field is assumed as,

$$[N_1^d \;\; N_2^d] = [N_1^M \;\; N_2^M] = [(1 - \frac{X}{L}) \;\; \frac{X}{L}] \tag{4.11}$$

$${N_\alpha}^E(X) = {N_\alpha}^{K_b}(X) = {N_\alpha}^\Gamma(X) = {N_\alpha}^T(X) = {N_\alpha}^S(X) = 1 \tag{4.12}$$

Using this approximation the shear force given by eq. (4.1.8) can be re-written as

$$S = -\frac{\partial M}{\partial X} = \frac{1}{L}(\widetilde{M_1} - \widetilde{M_2}) \quad or \quad \delta S = \frac{1}{L}(\delta\widetilde{M_1} - \delta\widetilde{M_2}) \tag{4.13}$$

Introducing the above approximation eq. (4.1.9), eq. (4.1.10) and eq. (4.1.13) , into eq. (4.1.3) we obtain the variational functional form as

$$
\begin{aligned}
\delta\Pi_{hw} = \int_L & \{\delta\widetilde{E}_0\,[\,T^c - T\,] + \delta\widetilde{T}\,[\,u_{,X} + \frac{1}{2}(u_{,X}^2 + w_{,X}^2) - E_0\,] \\
& + \delta\widetilde{u}_\alpha\,[(1+u_{,X})\,N^d_{\alpha,X}\,T + sin\,\beta\,N^d_{\alpha,X}\,S + \cos\,\beta_{,X}\,N^d_{\alpha,X}\,M\,]\}\,dX \\
+ \int_L & \{\delta\widetilde{K}_b\,[\,M^c - M\,] + \delta\widetilde{M}_\alpha\,N^M_\alpha\,[\,\Lambda\,\beta_{,X} - K^b\,] \\
& + \delta\widetilde{w}_\alpha\,[\,w_{,X}\,N^d_{\alpha,X}\,T - \beta_{,X}\,N^d_{\alpha,X}\,sin\,\beta\,M + cos\beta\,N^d_{\alpha,X}\,S\,]\}\,dX \\
+ \int_L & \{\delta\widetilde{\Gamma}\,[\,S^c - S\,] - \delta\widetilde{M}_\alpha\,N^M_{\alpha,X}\,[(1+u_{,X})\sin\,\beta + w_{,X}\cos\,\beta - \Gamma\,] \\
& + \delta\widetilde{\beta}_\alpha\,[\,M\,\Lambda\,N^d_{\alpha,X} + S\Lambda\,N^d_\alpha - \Gamma\beta_{,X}\,N^d_\alpha\,M\,]\}\,dX - \delta\Pi_{ext} = 0 \qquad \alpha = 1,2
\end{aligned}
\tag{4.14}
$$

which can be re-written in more compact form as

$$
\begin{aligned}
\delta\Pi = [\delta\widetilde{u}_\alpha \;\; \delta\widetilde{w}_\alpha \;\; \delta\widetilde{\beta}_\alpha\,] \int_L \mathbf{B}_\alpha^{\mathbf{T}} \begin{Bmatrix} T^p \\ S^p \\ M^b \end{Bmatrix} dX + [\delta\widetilde{N} \;\; \delta\widetilde{M}_\alpha\,] \int_L \mathbf{A_1}\,dX \\
+ [\,\delta\widetilde{E}_0 \;\; \delta\widetilde{K}_b \;\; \delta\widetilde{\Gamma}\,] \int_L \mathbf{A_2}\,dX - \delta\Pi_{ext} = 0
\end{aligned}
\tag{4.15}
$$

where the matrix $\mathbf{B}_\alpha$ is same as eq. (3.1.13) , vector $\mathbf{A_1}$, $\mathbf{A_2}$ can be given as

$$
\mathbf{A_1} = \begin{Bmatrix} u_{,X} + \frac{1}{2}(u_{,X}^2 + w_{,X}^2) - E_0 \\ {N_1}^M\,(\,\Lambda\beta_{,X} - K_b)\; + \;[\,\frac{(1+u_{,X})\sin\beta + w_{,X}\cos\beta - \Gamma}{L}\,] \\ {N_2}^M\,(\,\Lambda\beta_{,X} - K_b)\; - \;[\,\frac{(1+u_{,X})\sin\beta + w_{,X}\cos\beta - \Gamma}{L}\,] \end{Bmatrix}
\tag{4.16}
$$

$$
\mathbf{A_2} = \begin{Bmatrix} T^c - T \\ M^c - M \\ S^c - S \end{Bmatrix}
\tag{4.17}
$$

Applying a linearization to eq . (4.1.15) give the incremental form for a Newton Raphson solution process as

$$
\begin{Bmatrix} \delta\widetilde{a} \\ \delta\widetilde{f} \\ \delta\widetilde{e} \end{Bmatrix}^T \left\{ \begin{bmatrix} K_g & H^T & 0 \\ H & 0 & -b^T \\ 0 & -b & k \end{bmatrix} \begin{Bmatrix} d\widetilde{a} \\ d\widetilde{f} \\ d\widetilde{e} \end{Bmatrix} = \begin{Bmatrix} R_a \\ R_e \\ R_f \end{Bmatrix} \right\}
\tag{4.18}
$$

where " d " is an increment, Here K_g is the geometric stiffness matrix which is same as eq. (3.1.22).the residual expression is given below. More comprehensive derivation can be found in Appendix A

$$R_f = \int_L \begin{Bmatrix} T^c - T \\ M^c - M \\ S^c - S \end{Bmatrix} dX \tag{4.19}$$

$$R_e = \int_L \begin{Bmatrix} u_{,X} + \frac{1}{2}(u_{,X}^2 + w_{,X}^2) - E_0 \\ {N_1}^M \,(\,\Lambda\beta_{,X} - K_b)\; + \;[\,\frac{(1+u_{,X})\sin\beta + w_{,X}\cos\beta - \Gamma}{L}\,] \\ {N_2}^M \,(\,\Lambda\beta_{,X} - K_b)\; - \;[\,\frac{(1+u_{,X})\sin\beta + w_{,X}\cos\beta - \Gamma}{L}\,] \end{Bmatrix} dX \tag{4.20}$$

$$R_a = \int_L \mathbf{B^T} \begin{Bmatrix} T \\ S \\ M \end{Bmatrix} dX - F_{ext} \tag{4.21}$$

$$\mathbf{k} = \begin{bmatrix} EAL & & \\ & EIL & \\ & & \kappa GAL \end{bmatrix} \tag{4.22}$$

$$\mathbf{b} = \Sigma_q \mathbf{b_q} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 - \frac{X_q}{L} & \frac{X_q}{L} \\ 0 & \frac{1}{L} & -\frac{1}{L} \end{bmatrix} \tag{4.23}$$

where q is the quadrature point of the element H matrix is the following :

$$H = \int_L \mathbf{H_\alpha^1}\, dX + \int_L \mathbf{H_\alpha^2}\, dX \tag{4.24}$$

where

$$\mathbf{H_\alpha^1} = \begin{bmatrix} (1+u_{,X})\, N_{\alpha,X}^d & w_{,X}\, N_{\alpha,X}^d & 0 \\ \beta_{,X} cos\,\beta\, N_{\alpha,X}^d\, N_1^M & -\beta_{,X} sin\,\beta\, N_{\alpha,X}^d\, N_1^M & (\Lambda\, N_{\alpha,X}^d - \Gamma\,\beta_{,X}\, N_\alpha^d)\, N_1^M \\ \beta_{,X} cos\,\beta\, N_{\alpha,X}^d\, N_2^M & -\beta_{,X} sin\,\beta\, N_{\alpha,X}^d\, N_2^M & (\Lambda\, N_{\alpha,X}^d - \Gamma\,\beta_{,X}\, N_\alpha^d)\, N_2^M \end{bmatrix} \tag{4.25}$$

and

$$\mathbf{H_\alpha^2} = \begin{bmatrix} 0 & 0 & 0 \\ \frac{sin\,\beta\, N_{\alpha,X}^d}{L} & \frac{cos\,\beta\, N_{\alpha,X}^d}{L} & \frac{\Lambda N_\alpha^d}{L} \\ -\frac{sin\,\beta\, N_{\alpha,X}^d}{L} & -\frac{cos\,\beta\, N_{\alpha,X}^d}{L} & -\frac{\Lambda N_\alpha^d}{L} \end{bmatrix} \tag{4.26}$$

4.4 SUMMARY OF SOLUTION ALGORITHM

The continuity between elements is enforced only for displacement degrees of freedom between the elements through the compatibility conditions in eq. (4.1.5) weak form. Forces and strain may be discontinuous between elements. Therefore, the parameters for forces and strains may be eliminated at the element level resulting in a stiffness matrix for displacement parameter determination. The elimination may be performed by static condensation in the following way:

Re-writing matrix eq. (4.1.18) in individual equations as

$$\begin{aligned} K_g\, d\widetilde{a} + H^T\, d\widetilde{f} &= R_a \\ H\, d\widetilde{a} - b^T\, d\widetilde{e} &= R_e \\ -b\, d\widetilde{f} + k\, d\widetilde{e} &= R_f \end{aligned} \tag{4.27}$$

Eliminating section strain components using third equation of eq. (4.1.27) as

$$d\widetilde{e} = k^{-1}\,(R_f + b\, d\widetilde{f}) \tag{4.28}$$

and substituting back into second equation of eq. (4.1.27) and rearranging terms will generate stress increment of the element as

$$d\widetilde{f} = F^{-1}(\,H\, d\widetilde{a} - \overline{R_e}) \tag{4.29}$$

where

$$\begin{aligned} F &= b^T k^{-1} b \\ \overline{R_e} &= R_e + b^T k^{-1} R_f \end{aligned} \tag{4.30}$$

are the element flexibility matrix and modified residual vector respectively.

Substituting eq. (4.1.30) back in the first equation of the eq. (4.1.27) and re-arranging the terms gives

$$\overline{K}\, d\widetilde{a} = \overline{R}_a \tag{4.31}$$

where

$$\begin{aligned} \overline{K} &= K_g + H^T\, F^{-1}\, H \\ \overline{R}_a &= R_a + H^T\, F^{-1}\, \overline{R_e} \end{aligned} \tag{4.32}$$

are the element stiffness matrix and modified element residual vector respectively. The resulting stiffness and residual vector of the element now may be assembled into the global equations in an identical manner to the usual displacement based formulation. The solution strategy can be breakdown in the following steps.

1. The Full Newton Raphson iteration (i) starts with the given load step on global structure
2. For the first iteration step $j = 0$
   $\widetilde{a}_{i+1}^{j=0} = \widetilde{a}_i, \; \widetilde{f}_{i+1}^{j=0} = \widetilde{f}_i, \; \widetilde{e}_{i+1}^{j=0} = \widetilde{e}_i,$
   For solution at step $(i + 1)$ assume the state at the previous step (i) is known.
3. Start the element state determination. Loop over all the elements in the structure for state at iteration j.
4. Compute R_e and R_f at each element and check its convergence. If converge go to step 8, else make a loop for the convergence of R_e and R_f in each element (set $k = 0$), and follow as
5. Compute element stress parameter using eq. (4.1.29) and update the stresses inside the element
   $\widetilde{f}_{k+1}^{j} = \widetilde{f}_k^{j} + d\widetilde{f}.$
6. Compute R_f again from the updated element stresses and solve for element strain parameters using eq. (4.1.28) . Update the strain parameters
   $\widetilde{e}_{k+1}^{j} = \widetilde{e}_k^{j} + d\widetilde{e}$
7. Check convergence of R_e and R_f from the updated stress and strain parameters. If not, go back to step 5
8. Condenses arrays and residual for each element as described in the above section. Assemble the global stiffness matrix.
9. Check convergence on global residual vector for the displacement $\overline{R}_a$. If converge move to next load step(set $i = i + 1$) and start from step 2, else follow as
10. Solve $d\widetilde{a} = \overline{K}^{-1}\,\overline{R}_a$ and update the geometry of the structure
    $\widetilde{a}^{j+1} = \widetilde{a}^{j} + d\widetilde{a}$
    Go to step 3 and set $(j = j + 1)$,and follow the steps subsequently

It is necessary to save the parameters for stress, $\widetilde{f}$, and strain $\widetilde{e}$ for each element in the load steps.

5. EXAMPLES

5.1 EXAMPLES

The objective in this analysis was to investigate the performance of the beam column element in large displacement and rotation problems by both displacement and forced based elements models by some examples. For some cases, it should be emphasized that, as most papers only provide the results in graphical form, only a reasonable accurate comparison can be done, due to the inaccuracy in obtaining numerical values from the presented plots. Since the minimum number of load step required to converge strictly depends on the magnitude of the external load. It is emphasized that no special effort is made to optimize the total number of loading steps for a given calculation. Unless stated in the following examples, Full Newton Raphson stategy is employed for non-linear global equations. Through out all the examples discussed below, the constitutive model defined by eq. (2.2.9) is considered. The model results is cross-checked by the open source standalone software Opensees(38). Opensees is freely available software based on the forced based formulation for Earthquake Engineering Simulation and handles both geometric and material non-linearity. Geometric non-linearity is handle by co-rotational approach.

5.1.1 Clamped cantilever with concentrated transversal end load

The cantilever problem represented in Figure 5.1 has been analyzed elastically (with different input parameters)in several works, such as Oran and Kassimali(1976)(39), White(1985)(40), Chan(1988)(41), and among many others. Analytical solutions of this problem were determined by different authors, including Frish-Fay(1962)(42). The example discuss the issue of large deflection with moderate rotation analysis relative to the cantilever length. It should be noted that the vertical displacement will be around 80 % of the original length.

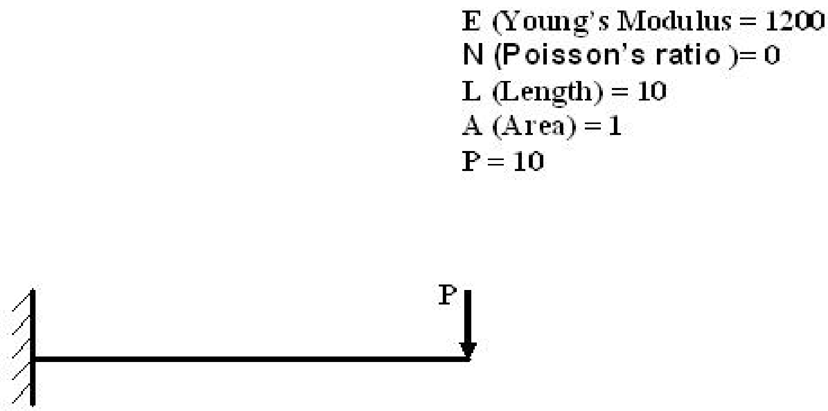


Figure 5.1: Cantilever Beam with Transverse Loading.

The geometrical and material properties chosen are: Young'S modulus $E = 1200$, Poisson's ration $\nu = 0$ and the square cross-section of unit area $A = 1$, the length of the cantilever is $L = 10$. Further transverse load of $P = 10$ is applied at the tip of the beam.

Table 5.1: Cantilever under transverse load free end displacement components.

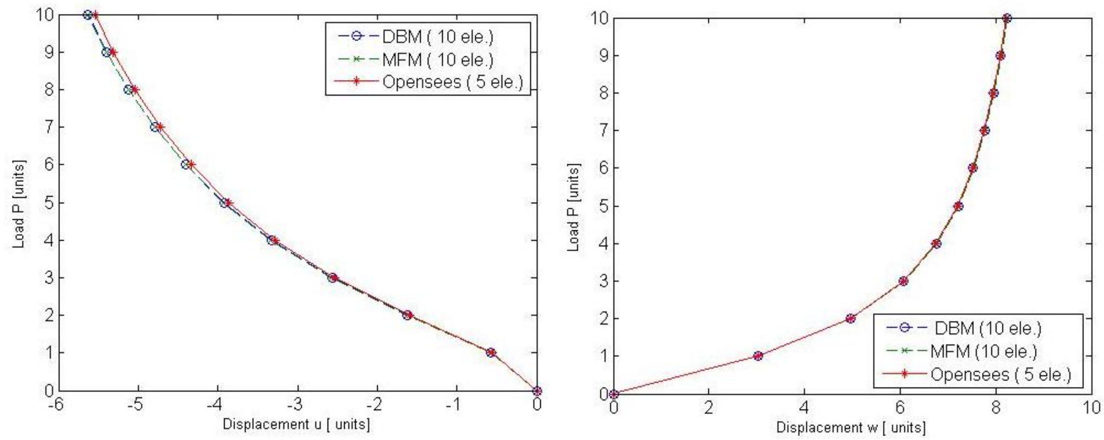
Model	No. of elem.	Hori. displ	verti displ	Rot
displacement based element	10	-5.6291	8.2350	1.4382
Mixed finite element	10	-5.6351	8.2448	1.4313
FEAPpv element	5	-5.5262	8.1957	1.4410
Opensees element	5	-5.5418	8.2226	1.4382

It is being observed that the numerical values presented here is very close to the analytical solution mentioned by Frish-Fay(1962)(42). The exact values is not presented due to imprecision of the presented plots but could be referred to Souza (2000)(35) for the suitable plots. The author Souza (2000)(35) also claims in his work that the above problem can be solved by just one element per member. The results of the model is approximated with different number of elements is presented in the Table 5.2 and Table 5.3.

Table 5.2: Displacement based formulation.

No. of elem.	Hori. displ	verti displ	Rot
5	-5.7075	8.2874	1.4478
10	-5.6291	8.2350	1.4382
20	-5.6093	8.2208	1.4283

The convergence rate is also shown in the Table 5.4 for the both models.



(a) Horizontal displacement component. (b) Vertical displacement component.

Figure 5.2: Equilibrium path for the cantilever under transverse load.

Table 5.3: Mixed element formulation.

No. of elem.	Hori. displ	verti displ	Rot
5	-5.7173	8.2988	1.4471
10	-5.6351	8.2448	1.4313
20	-5.6154	8.2310	1.4274

Table 5.4: Convergence rates for cantilever beam.

No. of iterations	Residual Norm (DBM)	Residual Norm (MFM)
0	1.66667	1.66667
1	625.97970	610.60902
2	118.15220	124.26377
3	10.69949	11.99674
4	0.18409	0.30111
5	0.05465	0.52958
6	0.000561	0.00261
7	2.95×10^{-5}	2.70×10^{-5}
8	2.05×10^{-7}	9.656×10^{-12}
9	1.29×10^{-8}	0

5.1.2 Clamped cantilever subjected to concentrated end moment

This problem has been analyzed by a number of researchers including Bathe and Bolourch(1979)(4), Simo and Vu-Quoc(1986)(7), Crisfield(1990)(43) considering the straight beam element. Clearly for prismatic beam, the exact solution for the deformed shape of this problem is a perfect circle, since the bending moment, and hence the curvature, is constant along the beam. However, we make the approximation in the variable field and the subsequent result is tabulated below for checking the accuracy of the element formulation. The geometrical and material properties chosen are: Young'S modulus $E = 1200$, Poisson's ration $\nu = 0$ and the square cross-section of unit area $A = 1$, the length of the cantilever is $L = 10$. Beam is discretized in ten elements. Further Bending moment of $M = 10\pi$ can rolled the beam in half circle theoretically. The free-end displacement components are presented in Table 5.5, along with the result obtained by the corresponding model of ten 3-node straight-beam elements as discussed by Simo and Vu-Quoc (1986)(7) and also by open sees results.

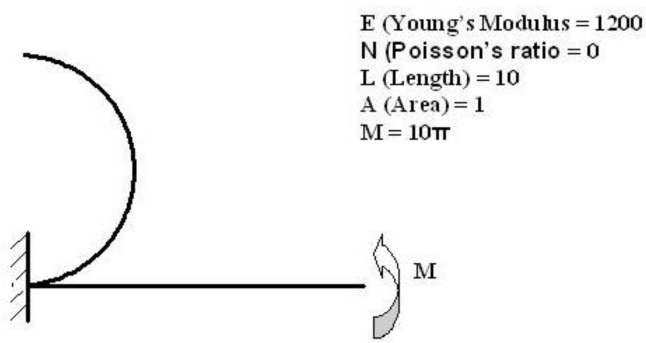


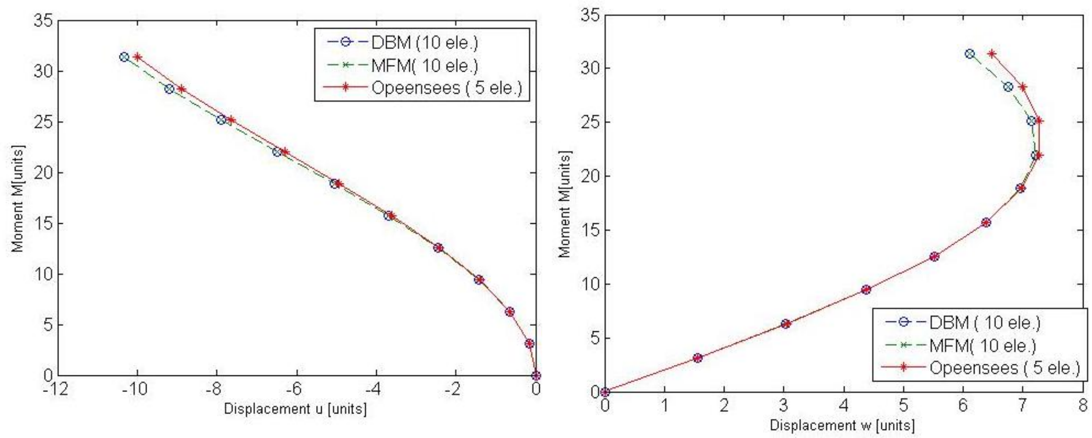
Figure 5.3: Deformed shape of the cantilever beam under free-end moment.

Table 5.5: Cantilever under pure moment load free end displacement components.

Model	No. of elem.	Hori. displ	verti displ	Rot
displacement based element	10	-10.3386	6.1034	3.252
Mixed finite element	10	-10.3398	6.1025	-3.252
Simo-Vu model	10	-10.1580	6.1590	-3.090
Opensees model	5	-10.0000	6.4721	-3.141
Analytical Solution	–	10.0000	6.3660	-3.141

Figure 5.4 shows the results by both models and also the open sees results is mentioned for cross-

check. The results of the model is approximated with different number of elements is presented in the Table 5.6 and Table 5.7.



(a) Horizontal displacement component. (b) Vertical displacement component.

Figure 5.4: Equilibrium path for the cantilever under pure moment load.

Table 5.6: Displacement based formulation.

No. of elem.	Hori. displ	verti displ	Rot
5	-10.9189	5.6890	3.4629
10	-10.3386	6.1034	-3.2522
20	-10.2065	6.1798	-3.2084

Table 5.7: Mixed finite element.

No. of elem.	Hori. displ	verti displ	Rot
5	-10.9461	5.6614	-3.4727
10	-10.3398	6.1025	-3.252
20	-10.2066	6.1797	-3.2084

The convergence rate is also shown in the Table 5.8 for the both models.

5.1.3 One story Portal Frame with transverse loading

The geometrical and material properties chosen are: Young'S modulus $E = 1200$, Poisson's ration $\nu = 0$ and the square cross-section of unit area $A = 1$, the length of the each section is $L = 10$. Each section is discretized into elements and analyzed by both methods and cross checked by OpenSees results.

Table 5.8: Convergence rates for cantilever beam.

No. of iterations	Residual Norm (DBM)	Residual Norm (MFM)
0	5.23333	5.23333
1	182.02560	180.64850
2	22.05138	22.49339
3	0.62304	0.61768
4	0.29973	0.14301
5	0.00157	0.00073
6	6.66×10^{-6}	7.36×10^{-7}
7	1.01×10^{-12}	3.35×10^{-13}

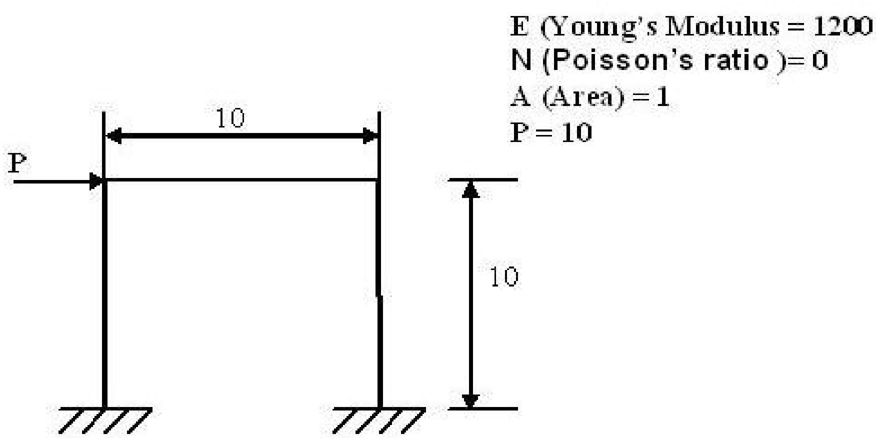
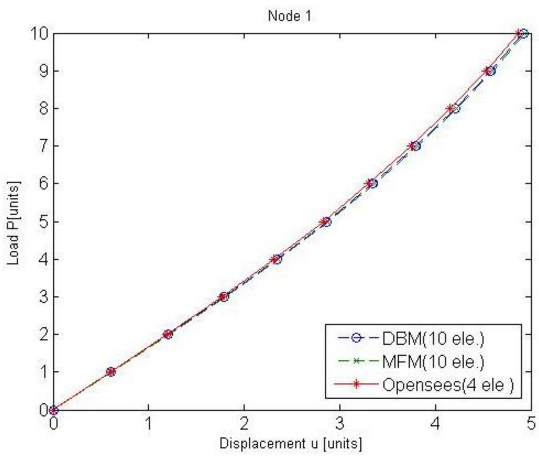


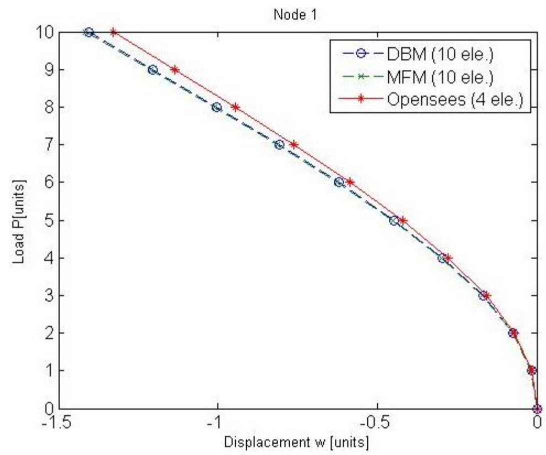
Figure 5.5: One story Portal Frame.

Table 5.9: One story Portal Frame Node 1 displacement components

Model	No. of elem.	Hori. displ	verti displ	Rot
displacement based element	5	4.8711	-1.3625	0.3279
displacement based element	10	4.9210	-1.4045	0.3344
Forced based element	5	4.8908	-1.3728	0.3289
Forced based element	10	4.9408	-1.4148	0.3353
Opensees	4	4.9973	-1.27295	-0.3159

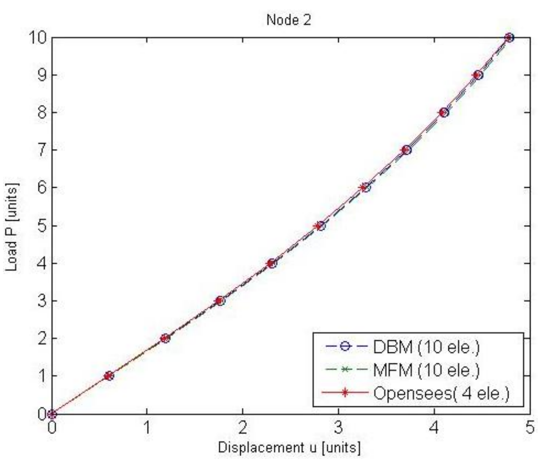


(a) Horizontal displacement component.

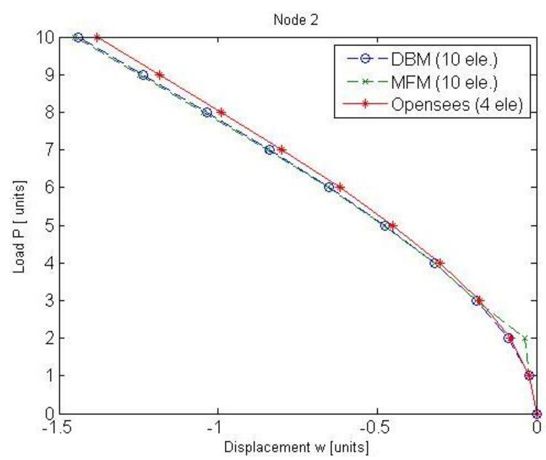


(b) Vertical displacement component.

Figure 5.6: Equilibrium path for the one story frame transverse load (Node 1).



(a) Horizontal displacement component.

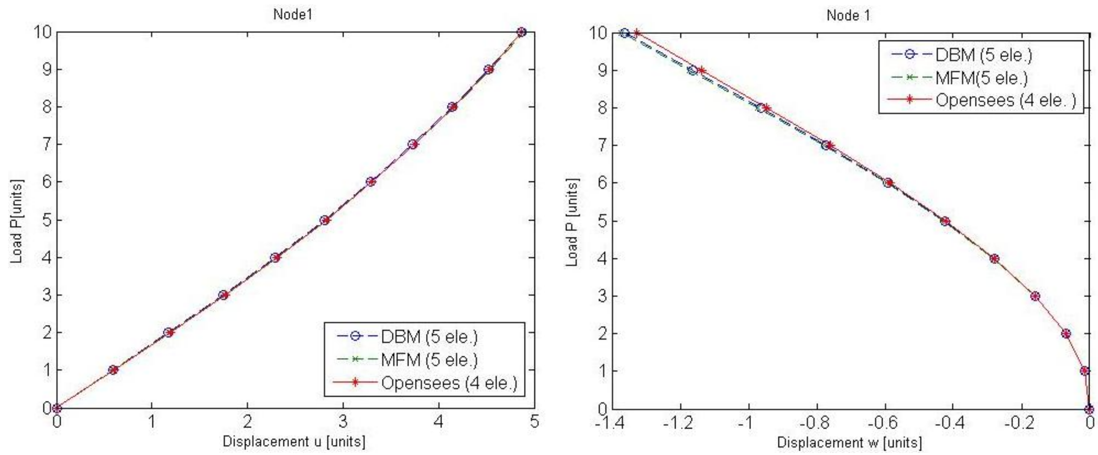


(b) Vertical displacement component.

Figure 5.7: Equilibrium path for the one story frame transverse load (Node 2).

Table 5.10: Convergence rate for one story frame analysis.

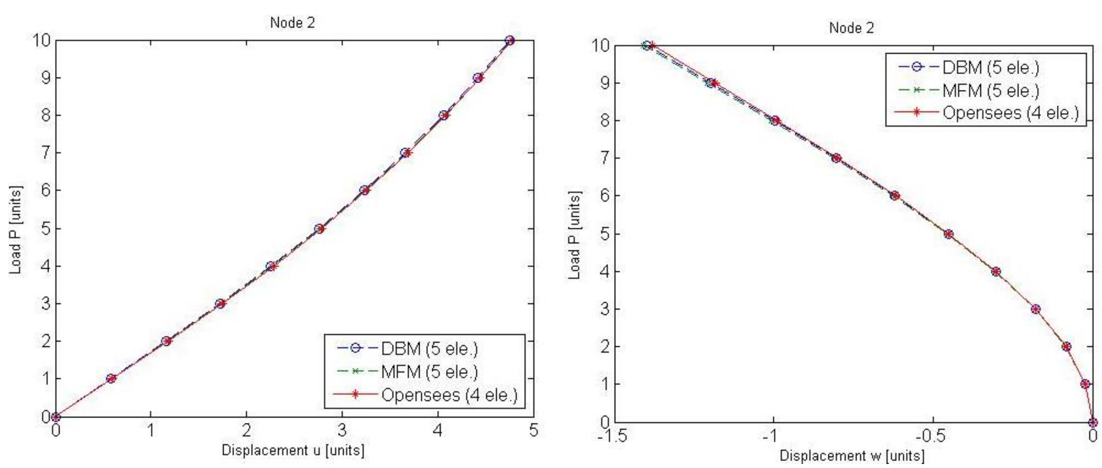
No. of iterations	Residual Norm (DBM)	Residual Norm (MFM)
0	1.66667	1.66667
1	11.13538	11.20696
2	0.15455	0.15509
3	0.00047	3.73×10^{-5}
4	1.36×10^{-5}	1.9×10^{-10}
5	3.79×10^{-8}	-
6	1.10×10^{-9}	-



(a) Horizontal displacement component.

(b) Vertical displacement component.

Figure 5.8: Equilibrium path for the one story frame transverse load (Node 1 and with 5 elements).



(a) Horizontal displacement component.

(b) Vertical displacement component.

Figure 5.9: Equilibrium path for the one story frame transverse load (Node 2 and with 5 elements).

6. CONCLUSION

This study proposed two models- displacement based and Mixed finite element method restricted to two-dimensional, small strain/large deformation theory and includes the effects of shear deformation. The proposed models works well in the finite deformation regime under suitable mesh discretization as shown by the examples. Mixed finite element method can adequately model the shear deformation without being get locked whereas in Displacement based method reduced quadrature rule is employed to get rid off locking phenomena in elements. In the mixed finite element method we used the independent interpolation functions for displacement, stresses and strains. With the help of the stationarity of the eq. (4.1.6), we obtained the section forces which is in equilibrium with the applied load. But since we used the linear interpolation functions for interpolating displacement field of the beam where the analytical solution for the displacement profile of the beam is cubic in nature, discretization of the member is required in order to converge to the exact solution provided by a theory of finite deformation. The target problems for the proposed force formulation works well in inelastic structural frames with small deformations where linear interpolation functions for displacement field can model accurately the displacement of the beam. It is observed that the symmetric tangent stiffness matrix is obtained in both models.

6.1 FUTURE WORK

1. Non- linear material behavior may be included by integrating the resultants for axial force, shear force and bending moment over the member cross- section through appropriate constitutive models.
2. The extension to three- dimensional direction.

3. The inclusion of distributed element loads by the addition of the exact internal force distribution function under the give element loads.

4. Extending the formulation for dynamic analysis, through the derivation of a consistent mass matrix.

5. Extending the formulation to curved beams.

A. APPENDIX

A.1 DERIVATION OF GEOMETRIC STIFFNESS MATRIX $\mathbf{K}_G$

The last term in eq. (3.1.20) can be expressed as the following using eq. (3.1.13) and eq. (3.1.15)

$$[\mathbf{B}^T\mathbf{F}]_{6\times 1} = \begin{Bmatrix} (1+u_{,X})\,N^d_{1,X}T + sin\,\beta\,N^d_{1,X}S + \cos\,\beta_{,X}\,N^d_{1,X}M \\ w_{,X}\,N^d_{1,X}T - \beta_{,X}\,N^d_{1,X}\,sin\,\beta\,M + cos\beta\,N^d_{1,X}S \\ \Lambda\,N^d_{1,X}M + \Lambda\,N^d_1 S - \Gamma\beta_{,X}\,N^d_1 M \\ (1+u_{,X})\,N^d_{2,X}T + sin\,\beta\,N^d_{2,X}S + \cos\,\beta_{,X}\,N^d_{2,X}M \\ w_{,X}\,N^d_{2,X}T - \beta_{,X}\,N^d_{2,X}\,sin\,\beta\,M + cos\beta\,N^d_{2,X}S \\ \Lambda\,N^d_{2,X}M + \Lambda\,N^d_2 S - \Gamma\beta_{,X}\,N^d_2 M \end{Bmatrix}_{6\times 1} = \{f_1,\ f_2,\dots f_6\}^T \tag{A.1}$$

$$[\mathbf{K_g}]_{6\times 6} = \int_L \frac{\partial(\mathbf{B}^T F)}{\partial\widetilde{u}}\,dX = \int_L \begin{bmatrix} \frac{\partial f_1}{\partial\widetilde{u}_1} & \frac{\partial f_1}{\partial\widetilde{w}_1} & \frac{\partial f_1}{\partial\widetilde{\beta}_1} & \frac{\partial f_1}{\partial\widetilde{u}_2} & \frac{\partial f_1}{\partial\widetilde{w}_2} & \frac{\partial f_1}{\partial\widetilde{\beta}_2} \\ \frac{\partial f_2}{\partial u_1} & & & & & \\ \vdots & & & & & \\ \frac{\partial f_6}{\partial u_1} & \dots & & \dots & & \frac{\partial f_6}{\partial\widetilde{\beta}_2} \end{bmatrix}_{6\times 6} dX \tag{A.2}$$

A.2 DERIVATION OF THE STIFFNESS MATRIX EXPRESSED IN EQ. (4.1.18)

Derivation of the stiffness matrix(eq. 4.1.18) is described considering two nodes per element in this section;

eq. (4.1.14) can be re-written as

$$\begin{bmatrix} \delta u_\alpha \\ \delta w_\alpha \\ \delta\beta_\alpha \\ \delta\widetilde{T} \\ \delta\widetilde{M_1} \\ \delta\widetilde{M_2} \\ \delta\widetilde{E_0} \\ \delta\widetilde{K_b} \\ \delta\widetilde{\Gamma} \end{bmatrix}^T \underbrace{\left\{ \begin{bmatrix} \int_L \{(1+u_{,X})\, N^d_{\alpha,X}\, T + sin\,\beta\, N^d_{\alpha,X}\, S + \cos\,\beta_{,X}\, N^d_{\alpha,X}\, M\,\}\, dX \\ \int_L \{w_{,X}\, N^d_{\alpha,X}\, T - \beta_{,X}\, N^d_{\alpha,X}\, sin\,\beta\, M + cos\beta\, N^d_{\alpha,X}\, S\}\, dX \\ \int_L \{\Lambda\, N^d_{\alpha,X}\, M + \Lambda\, N^d_\alpha\, S - \Gamma\beta_{,X}\, N^d_\alpha\, M\,\}\, dX \\ \int_L \{u_{,X} + \frac{1}{2}(u^2_{,X} + w^2_{,X}) - E_0\}\, dX \\ \int_L \{N_1{}^M\,(\,\Lambda\beta_{,X} - K_b) \,+\, [\,\frac{(1+u_{,X})\sin\beta + w_{,X}\cos\beta - \Gamma}{L}\,]\}\, dX \\ \int_L \{N_2{}^M\,(\,\Lambda\beta_{,X} - K_b) \,-\, [\,\frac{(1+u_{,X})\sin\beta + w_{,X}\cos\beta - \Gamma}{L}\,]\}\, dX \\ \int_L \{T^c - T\}\, dX \\ \int_L \{M^c - M\}\, dX \\ \int_L \{S^c - S\}\, dX \end{bmatrix} - \begin{bmatrix} [\,F_{ext}\,]_{6\times1} \\ [\phi]_{6\times1} \end{bmatrix} \right\}}_{ResidualVector} = 0 \tag{A.3}$$

For simplicity residual vector can be expressed as

$$ResidualVector = \{R_1,\, R_2, \ldots R_{12}\}^T \tag{A.4}$$

Matrix in the eq. (4.1.18) can be computed by taking the partial derivative of the Residual vector with respect to the element parameters $(u_\alpha, w_\alpha, \beta_\alpha, \widetilde{T}, \widetilde{M_1}, \widetilde{M_2}, \widetilde{E_0}, \widetilde{K_b}, \widetilde{\Gamma})$, where $\alpha = 1, 2$

$$\begin{bmatrix} K_g & H^T & 0 \\ H & 0 & -b^T \\ 0 & -b & k \end{bmatrix}_{12\times12} = \begin{bmatrix} \frac{\partial R_1}{\partial \widetilde{u}_\alpha} & \frac{\partial R_1}{\partial \widetilde{w}_\alpha} & \cdots & \frac{\partial R_1}{\partial \widetilde{\Gamma}} \\ \frac{\partial R_2}{\partial u_\alpha} & & & \\ \vdots & & & \\ & & & \\ \frac{\partial R_{12}}{\partial u_\alpha} & \cdots & & \frac{\partial R_{12}}{\partial \widetilde{\Gamma}} \end{bmatrix}_{12\times12} \tag{A.5}$$

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