Department of Structural Mechanics
Faculty of Engineering
University of Pavia - Italy

Duality in the Geometrically Exact Analysis of Three-Dimensional Framed Structures

Hugo A.F.A. Santos
(Post-Doctoral Fellow)

Dipartimento di Meccanica Strutturale
Università degli Studi di Pavia

Pavia, October 12 2009

Outline

- Introduction
- Objectives and Scope
- Reissner-Simo Beam Theory - BVP
- Primal Variational Problem
- Dual Variational Problem
- Dual FE Formulation
- Dual Analysis Method - *A Posteriori* Error Estimation
- Numerical Applications
- Closure

Displacement-based finite element formulations

- are on the basis of most of the finite element models used in computer analysis of structures;
- assume the configuration variables as primary unknowns;
- lead to approximate *kinematically admissible solutions* in which *stress discontinuities may occur across element boundaries*; stress 'averaging' procedures are required;
- are well developed for both linear and nonlinear analyses.

Equilibrium-based finite element formulations

- are less common than the displacement-based finite element formulations;
- lead to approximate *statically admissible solutions*;
- have a special appeal for practical design engineers due to the exact transmission of stresses across interelement boundaries, thus avoiding the need for 'averaging' procedures;
- are not well studied in the context of the geometrically nonlinear analysis of framed structures.

Objectives and Scope

To present, in the framework of the quasi-static linear elastic analysis of geometrically exact framed structures modeled using the three-dimensional Reissner-Simo beam theory:

- Two dual energy-based variational formulations: one (Primal) derived from the well known Principle of Stationary Total Potential Energy, and the other (Dual) resulting from the Principle of Stationary Total Complementary Energy;

- An equilibrium-based (hybrid-mixed) finite element formulation relying on a modified Principle of Complementary Energy;

- A duality based method in which both primal and dual variational problems are studied in conjunction.

Kinematical Considerations

- The deformed geometry of a beam is described by the centroidal axis and the set of orientations of cross-sections;
- Only initially straight beam configurations and initially undistorted cross-sections are assumed;
- The geometric shape of the cross-sections is assumed to be arbitrary and constant along the beam;
- The cross-sections are assumed to suffer only rigid body motions during deformation;
- The beam theory is valid for arbitrarily large displacements and rotations - **Geometrically Exact Beam Theory**.

Kinematical Considerations (Cont.)

- The deformed configuration of a beam is described by the position of the line of centroids of the cross-sections and also the rotations of the cross-sections;
- The rotations of the cross-sections are described using the Euler-Rodrigues formula, which is assumed to be parameterized through the total rotation vector as follows:

$$\mathbf{Q} = \mathbf{I} + \frac{\sin\theta}{\theta}\mathbf{\Theta} + \frac{1-\cos\theta}{\theta^2}\mathbf{\Theta}^2$$

where $\mathbf{\Theta} = Skew(\boldsymbol{\theta})$ and $\theta = \|\boldsymbol{\theta}\|$

Reissner-Simo Beam BVP (Material Form)

- **Differential Equations**

$$
\begin{array}{ll}
\text{Equilibrium} & \left\{ \mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q} = \mathbf{0}, \text{ in } \Omega \right. \\
\text{Elasticity} & \left\{ \boldsymbol{\sigma}^r = \dfrac{\partial W(\varepsilon^r(\mathbf{d}))}{\partial \varepsilon^r}, \text{ in } \Omega \right. \\
\text{Compatibility} & \left\{ \varepsilon^r = \varepsilon^r(\mathbf{d}), \text{ in } \Omega \right.
\end{array}
$$

- **Neumann Boundary Conditions:** $n\mathbf{H}\boldsymbol{\sigma}^r = \bar{\mathbf{q}}$, on Γ_N
- **Dirichlet Boundary Conditions:** $\mathbf{d} = \bar{\mathbf{d}}$, on Γ_D

Remark: If the strain energy $W(\varepsilon^r)$ is differentiable and convex, by means of the Legendre transformation, the constitutive relations can be alternatively established using the format

$$\varepsilon^r = \frac{\partial W_c(\boldsymbol{\sigma}^r)}{\partial \boldsymbol{\sigma}^r}, \text{ in } \Omega$$

Principle of Stationary Total Potential Energy

- Let $\mathcal{U}_k$ and $\mathcal{V}_k$ be the kinematically admissible function spaces

$$\mathcal{U}_k = \{\mathbf{d} \in \mathcal{H}^1(\Omega) |\ \mathbf{d} = \bar{\mathbf{d}} \text{ on } \Gamma_D\}$$

$$\mathcal{V}_k = \{\delta\mathbf{d} \in \mathcal{H}^1(\Omega) |\ \delta\mathbf{d} = \mathbf{0} \text{ on } \Gamma_D\}$$

- The total potential energy associated with vector $\mathbf{d}$ is the one-field functional $\Pi_p(\mathbf{d}) : \mathcal{U}_k(\Omega) \to \mathcal{R}$ given by

$$\Pi_p(\mathbf{d}) = \int_\Omega [W(\varepsilon^r(\mathbf{d})) - \mathbf{q} \cdot \mathbf{d}]\ dS - [\bar{\mathbf{q}} \cdot \mathbf{d}]_{\Gamma_N}$$

- **Principle of Stationary Total Potential Energy (PSTPE):** vector $\mathbf{d} \in \mathcal{U}_k$ is a solution of the BVP iff $\delta\Pi_p = 0\ \forall \delta\mathbf{d} \in \mathcal{V}_k$, i.e., a beam is in equilibrium iff its total potential energy takes a stationary value for all kinematically admissible displacement fields.

Hybrid-Multi-Field Variational Principles

- The PSTPE can be generalized by means of the Lagrangian multiplier method leading to a **Generalized Variational Principle (GVP)**;
- The GVP can afterwards be particularized into different **Hybrid-Multi-Field Principles**, *e.g.*:
  - Principles of Hu-Washizu;
  - Principles of Hellinger-Reissner;
  - Principle of Total Complementary Energy, etc.

Principle of Stationary Total Complementary Energy

- Let $\mathcal{U}_s$ and $\mathcal{V}_s$ be the statically admissible function spaces

$$\mathcal{U}_s = \{(\boldsymbol{\sigma}^r, \mathbf{d}) \in (\mathcal{H}^1(\Omega)\times\mathcal{H}^1(\Omega))|\ \mathcal{T}_e^r(\mathbf{d})\boldsymbol{\sigma}^r + \mathbf{q} = \mathbf{0} \text{ in } \Omega \text{ and } n\mathbf{H}\boldsymbol{\sigma}^r - \bar{\mathbf{q}} = \mathbf{0} \text{ on } \Gamma_N\}$$

$$\mathcal{V}_s = \{(\delta\boldsymbol{\sigma}^r, \mathbf{d}) \in \mathcal{H}^1(\Omega)\times\mathcal{H}^1(\Omega)|\ \mathcal{T}_e^r(\mathbf{d})\delta\boldsymbol{\sigma}^r = \mathbf{0} \text{ in } \Omega \text{ and } n\mathbf{H}\delta\boldsymbol{\sigma}^r = \mathbf{0} \text{ on } \Gamma_N\}$$

- The complementary energy associated with $(\boldsymbol{\sigma}^r, \mathbf{d})$ is the 2-field functional $\Pi_c : \mathcal{U}_s(\Omega) \to \mathcal{R}$ given by

$$\Pi_c(\boldsymbol{\sigma}^r, \mathbf{d}) = \int_0^L [W_c(\boldsymbol{\sigma}^r) - \boldsymbol{\sigma}^r\cdot\varepsilon^r(\mathbf{d}) + \boldsymbol{\sigma}^r\cdot\mathcal{T}_c^r(\mathbf{d})\mathbf{d}]\, dS - [n\mathbf{H}\boldsymbol{\sigma}^r\cdot\bar{\mathbf{d}}]_{\Gamma_D}$$

- **Principle of Stationary Total Complementary Energy (PSTCE):** the pair $(\boldsymbol{\sigma}^r, \mathbf{d}) \in \mathcal{U}_s$ is a solution of the BVP iff $\delta\Pi_c = 0\ \forall(\delta\boldsymbol{\sigma}^r, \mathbf{d}) \in \mathcal{V}_s$.

Hybrid-Mixed Complementary Energy

- If the equilibrium equations are assumed to be relaxed within the framework of the PSTCE, the following hybrid-mixed complementary energy $\Pi_c^g : \chi(\Omega) \to \mathcal{R}$ can be obtained

$$\Pi_c^g(\boldsymbol{\sigma}^r, \mathbf{d}, \mathbf{d}^\Gamma) = \sum_{b=1}^{B} \int_{\Omega_b} [W_c(\boldsymbol{\sigma}_b^r) - \boldsymbol{\sigma}_b^r \cdot \varepsilon_b^r(\mathbf{d}_b) + \mathbf{q}_b \cdot \mathbf{d}_b] \, d\Omega_b$$

$$+ [\bar{\mathbf{q}} \cdot \mathbf{d}^\Gamma]_{\Gamma_N \cup \Gamma_{int}} + [n\mathbf{H}\boldsymbol{\sigma}^r \cdot (\mathbf{d} - \mathbf{J}_N \mathbf{d}^\Gamma)]_{\Gamma_N \cup \Gamma_{int}} + [n\mathbf{H}\boldsymbol{\sigma}^r \cdot (\mathbf{d} - \mathbf{J}_D \bar{\mathbf{d}})]_{\Gamma_D}$$

- $\mathbf{J}_N$ and $\mathbf{J}_D$ represent transformation matrices mapping global vectors (matrices) onto local element vectors (matrices) defined on $\Gamma_N \cup \Gamma_{int}$ and Γ_D, respectively;

- The functions in class $\chi(\Omega)$ consist of pairs $(\boldsymbol{\sigma}_b^r, \mathbf{d}_b) \in \mathcal{H}^0(\Omega_b) \times \mathcal{H}^1(\Omega_b)$, with $1 \leq b \leq B$, and a real-valued vector $\mathbf{d}^\Gamma$ defined on $\Gamma_N \cup \Gamma_{int}$.

Hybrid-Mixed Complementary Variational Principle

- The variational (weak) problem $\delta \Pi_c^g = 0, \ \forall (\delta \boldsymbol{\sigma}^r, \delta \mathbf{d}, \delta \mathbf{d}^\Gamma) \in \chi(\Omega)$ is formally equivalent to the following system of Euler-Lagrange equations

$$\begin{aligned} \mathcal{T}_e^r(\mathbf{d}_b)\boldsymbol{\sigma}_b^r + \mathbf{q}_b &= \mathbf{0} \text{ in } \Omega_b \\ \varepsilon_b^r(\boldsymbol{\sigma}_b^r) - \varepsilon_b^r(\mathbf{d}_b) &= \mathbf{0} \text{ in } \Omega_b \\ \bar{\mathbf{q}} - n\mathbf{J}_N^T \mathbf{H} \boldsymbol{\sigma}^r &= \mathbf{0} \text{ in } \Gamma_N \cup \Gamma_{int} \\ \mathbf{d} - \mathbf{J}_N \mathbf{d}^\Gamma &= \mathbf{0} \text{ in } \Gamma_N \cup \Gamma_{int} \\ \mathbf{d} - \mathbf{J}_D \bar{\mathbf{d}} &= \mathbf{0} \text{ on } \Gamma_D \end{aligned}$$

with $1 \leq b \leq B$.

Approximations

- *Element variables*:

$$\boldsymbol{\sigma}^{r^h} = \begin{bmatrix} \mathbf{n}^r \\ \mathbf{m}_i^r + (\mathbf{m}_j^r - \mathbf{m}_i^r)\frac{S}{L} \end{bmatrix}, \quad \mathbf{d}^h = \begin{bmatrix} \mathbf{u}_i + (\mathbf{u}_j - \mathbf{u}_i)\frac{S}{L} \\ \boldsymbol{\theta} \end{bmatrix}$$

- *Nodal variables*: $\mathbf{d}^{\Gamma}$ (generalized displacements)

Remarks:

- As the approximate displacements are one degree greater than the approximate rotations, this formulation is capable of representing zero shear solutions and is, thus, completely free from shear locking;
- Using these approximations, the formulation can provide solutions that satisfy the equilibrium differential equations in strong form, as well as the stress continuity conditions (when assuming zero distributed loads);
- Furthermore, the necessary and sufficient condition for solvability of the discrete linearized system of equations is fulfilled either for a single element or a patch of elements with appropriate boundary conditions ($n_{\sigma^r} \geq n_d - n_r$).

Linearized Global System of Equations

- The linearized global system of equations can be stated as

$$\mathbf{r}(\mathbf{p}) + \mathbf{T}(\mathbf{p})\Delta\mathbf{p} = \mathbf{0} \ ,$$

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}_{\sigma^r} \\ \mathbf{p_d} \end{bmatrix}, \ \mathbf{T} = \begin{bmatrix} \mathbf{F} & \mathbf{A}^T \\ \mathbf{A} & \mathbf{K}_c \end{bmatrix}$$

$$\mathbf{K}_{eq} = \mathbf{A}\mathbf{F}^{-1}\mathbf{A}^T - \mathbf{K}_c$$

(for the classification of the stability of the equilibrium)

- $\mathbf{F} = \frac{\partial^2 \Pi_c^g}{\partial \mathbf{p}_{\sigma^r} \partial \mathbf{p}_{\sigma^r}}$ - flexibility matrix
- $\mathbf{A} = \frac{\partial^2 \Pi_c^g}{\partial \mathbf{p_d} \partial \mathbf{p}_{\sigma^r}}$ - equilibrium matrix; $\mathbf{A}^T$ - compatibility matrix
- $\mathbf{K}_c = \frac{\partial^2 \Pi_c^g}{\partial \mathbf{p_d} \partial \mathbf{p_d}}$ - stiffness matrix

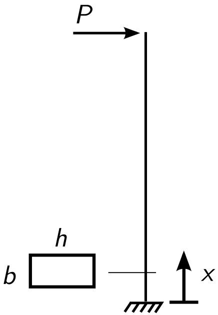
Fully Linear Case (FLC) vs Geom. Nonlinear Case (GNC)

FLC	**GNC**
$\Pi_p(\mathbf{d})$ is convex	$\Pi_p(\mathbf{d})$ is nonconvex
$\Pi_c(\boldsymbol{\sigma}^r)$ is concave	$\Pi_c(\boldsymbol{\sigma}^r, \mathbf{d})$ is a saddle functional
$\epsilon \geq (\epsilon_k, \epsilon_s)$	$\epsilon \; \substack{\geq \\ <} \; ?(\epsilon_k, \epsilon_s)$*

* Extremum conditions of Π_p and Π_c are required (Nobel and Sewell 1972, Gao and Strang 1989)

- $\epsilon = \left|\overline{\Pi}_p - \overline{\Pi}_c\right|$, $\epsilon_k = \left|\overline{\Pi}_p - \Pi_p\right|$, $\epsilon_s = \left|\Pi_c - \overline{\Pi}_c\right|$
- $\overline{\Pi}_p = \inf_{\mathbf{d}\in\mathcal{U}_k} \Pi_p(\mathbf{d})$, $\overline{\Pi}_c = \sup_{\boldsymbol{\sigma}^r\in\mathcal{U}_s} \Pi_c(\boldsymbol{\sigma}^r)$

Problem Definition



$$P = 2.5 \times 10^{-5}$$

$$E = 1 \times 10^{4}$$

$$\nu = 0.2$$

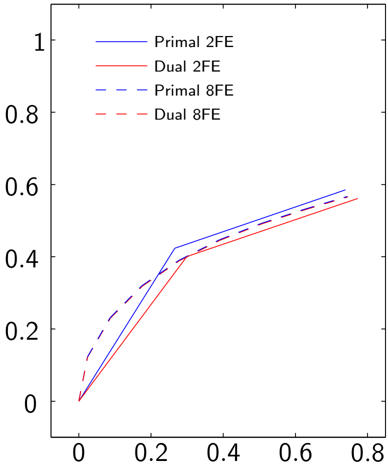
$$\alpha = \frac{5}{6}$$

$$L = 1$$

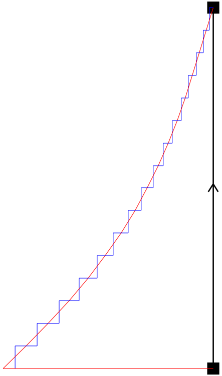
$$h = 0.01$$

$$b = 0.005$$

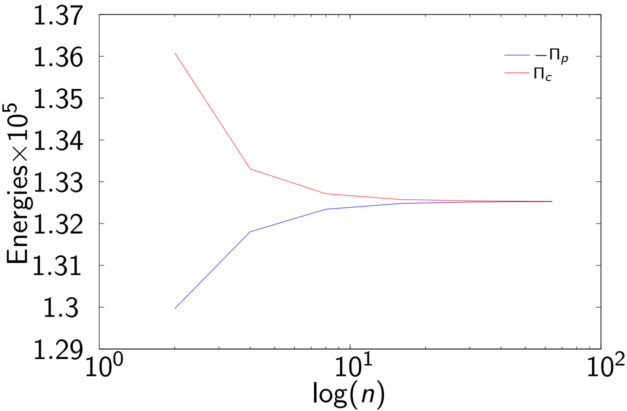
Deformed Configurations



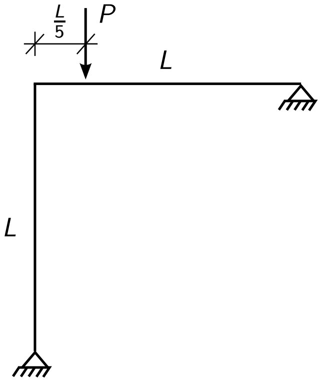
Diagrams of moments for $P = 2.5 \times 10^{-5}$ (16FE)



Energies


1.37
1.36
1.35
1.34
1.33
1.32
1.31
1.3
1.29
Energies×10^5
10^0
10^1
10^2
log(n)
$-\Pi_p$
Π_c

Problem Definition



$P = 50000$

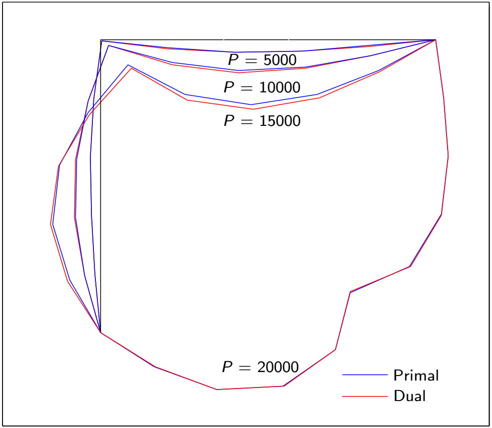
$EI = 1.44 \times 10^7$

$EA = 4.32 \times 10^7$

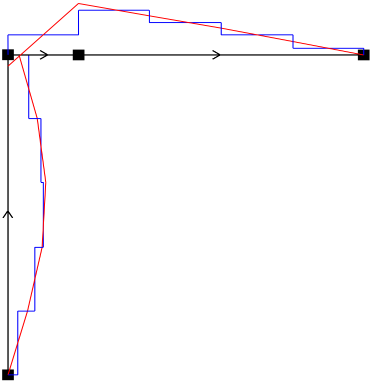
$GA' = 1.66 \times 10^7$

$L = 120$

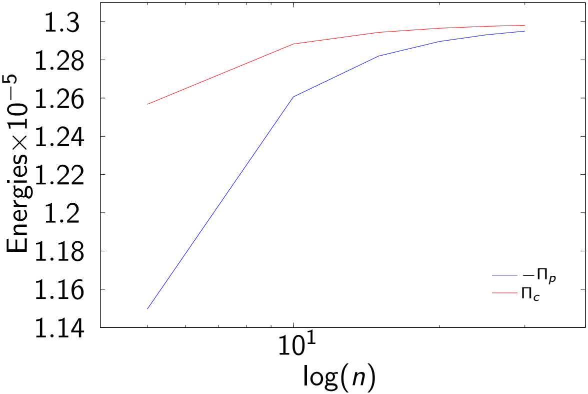
Deformed Configurations (5FE per leg)



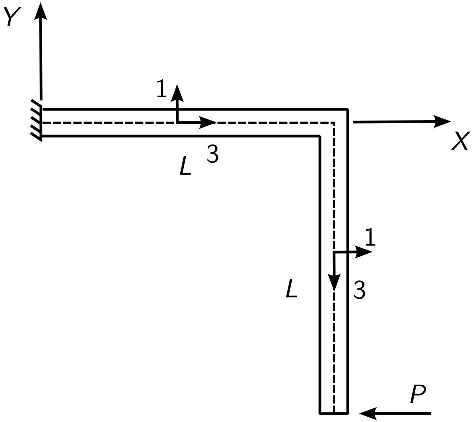
Diagrams of moments for $P = 10000$ (5FE per leg)



Energies for $P = 15000$



Problem Definition



$P = 2$

$E = 71240$

$\nu = 0.31$

$\alpha = \frac{5}{6}$

$L = 240$

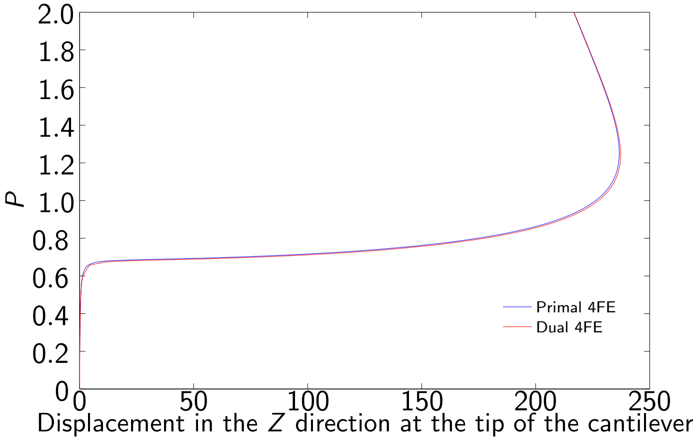
$h_1 = 30$

$h_2 = 0.6$

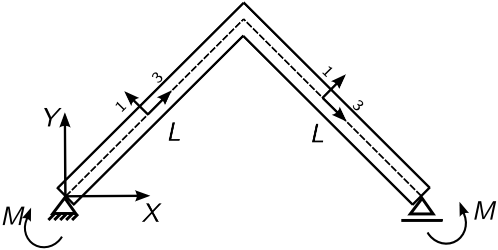
Deformed Configurations

Lateral Torsion Buckling

Equilibrium Paths

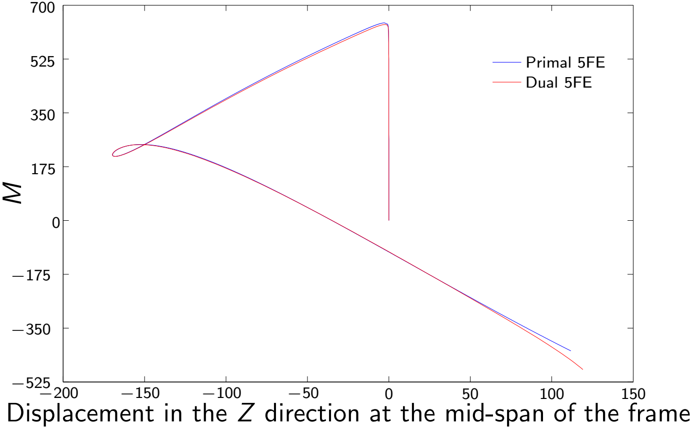


Problem Definition

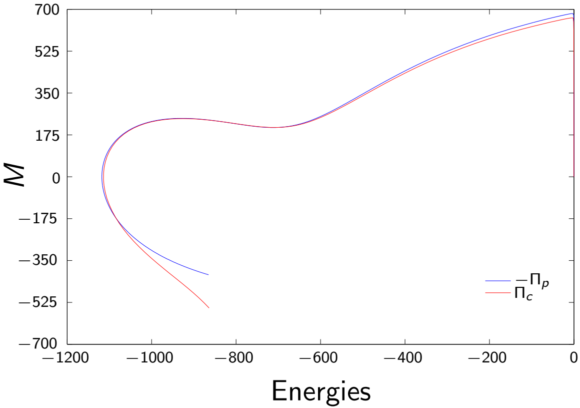


$$M = 700$$
$$E = 71240$$
$$\nu = 0.31$$
$$\alpha = \frac{5}{6}$$
$$L = 240$$
$$h_1 = 30$$
$$h_2 = 0.6$$

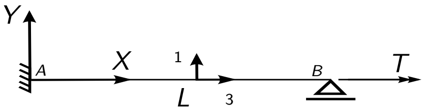
Equilibrium Paths



Energies (3FE per leg)



Problem Definition



$T = 270$

$E = 71240$

$\nu = 0.31$

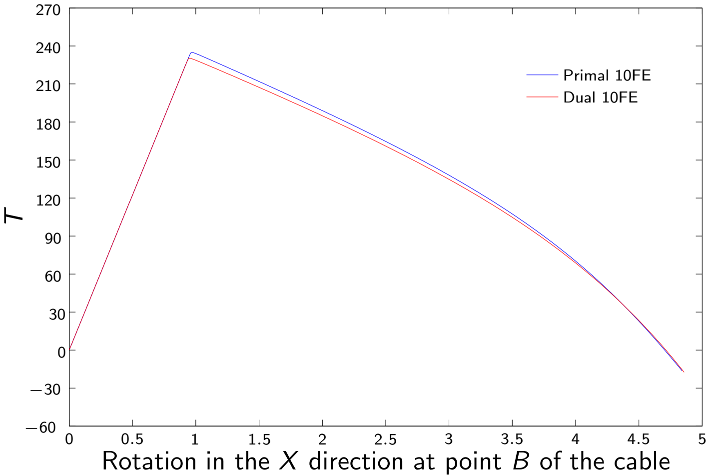
$L = 240$

$A = 1$

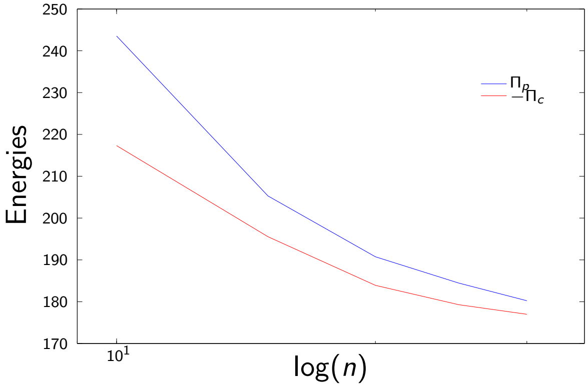
$I1 = I2 = 0.0833$

$J = 2.16$

Equilibrium Paths



Energies for $T = 210$



Conclusions

- The present hybrid-mixed FE formulation, established within the framework of the geometrically exact (Reissner-Simo) analysis of 3D framed structures, is:
  - variationally consistent;
  - completely free from shear locking;
  - capable of producing statically admissible approximate solutions;
- The present duality based method opens a new way on a posteriori error estimation and on possible bounding aspects within the framework of geometrically nonlinear analysis of framed structures.

Future Developments

- Consider higher-order polynomial sets of approximate functions within the dual FE formulation (p-type refinement schemes);
- Consider initially curved beam elements within the framework of the dual formulation;
- Incorporate general cross-sectional in-plane changes and out-of-plane warping phenomena within the framework of the dual formulation;
- Include physical nonlinearities within the framework of the dual formulation;

Future Developments (Cont.)

- Extend the dual formulation to shells and membranes;
- Derive (hybrid-) mixed FE formulations from other (hybrid-) multi-field variational principles;
- Investigate, from a mathematical point of view, the numerical stability of the present dual FE formulation;
- Investigate alternative error estimation methods which can provide guaranteed upper bounds of the exact error of the approximate solutions (considering both global and local quantities of interest).

Department of Structural Mechanics

Faculty of Engineering
University of Pavia - Italy

Duality in the Geometrically Exact Analysis of Three-Dimensional Framed Structures

Hugo A.F.A. Santos
(Post-Doctoral Fellow)

Dipartimento di Meccanica Strutturale
Università degli Studi di Pavia

Pavia, October 12 2009