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The Modified Finite Particle Method in the context of meshless methods

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Outline

- Introduction
- Literature review on meshless methods
- Modified Finite Particle Method (MFPM)
    - Application to elasticity
- MFPM for incompressible elasticity
    - Discretization with collocation methods: numerical difficulties
    - Alternative formulations of incompressile elasticity equations
    - Applications
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    - Solution of linear and non linear problems

Numerical methods are computational algorithms used to find the approximated solution of (algebraic, ordinary differential, partial differential) equations for which an analytical solution is not available

Here we focus on numerical algorithms for **partial differential equations**

Numerical methods for partial differential equations can be classified in

- **Collocation methods:** a solution is found by discretizing the partial differential equation in the strong form
- **Variational methods:** a solution is found by discretizing a weak formulation of the partial differential equation

Numerical methods: classification

Numerical methods for partial differential equations can be subdivided in:

- **Particle methods:** The domain is discretized into particles with physical characteristics (mass, velocity, energy, …)
- ***Non*-particle methods:** The domain is discretized into nodes or elements with no physical characteristics

and in:

- **Mesh-based methods:** The nodes of the domain have a predetermined connectivity
- **Meshless methods:** The connectivity among nodes is based on the current reciprocal positions

Meshless methods

In the present work we focus on **meshless methods**

Interesting features of meshless methods: since there is no rigid connectivity among nodes, such methods can easily model problems in which large deformations are involved, such as fast solid dynamics, explosions, fluid-dynamics

In fluid dynamics, particularly, meshless methods permit the study of fluid flows using a **Lagrangian framework**, that is, the fluid motion is studied through the motion of each fluid particle. This approach is convenient when studying problems of fluid-structure interaction (FSI), or fluid-dynamics problems with moving boundaries

The Smoothed Particle Hydrodynamics (SPH)

The first meshless method introduced in the literature is the **Smoothed Particle Hydrodynamics (SPH)**, introduced by **Lucy** (1977) and **Gingold and Monaghan** (1977) for the study of astrophysical problems.

The starting point of the approximation is

$$f(x_i) = \int_\Omega f(x)\delta(x - x_i)dx$$

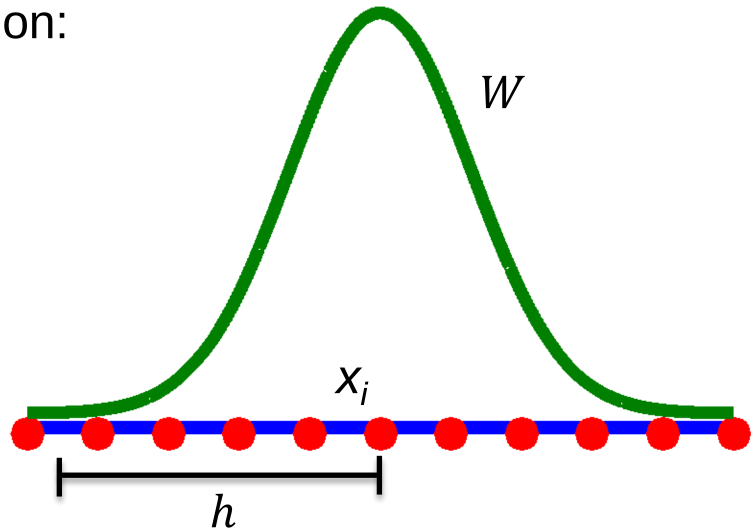
$\delta(x - x_i)$ **Dirac Delta** distribution, centered in x_i

- Approximation of the Dirac Delta distribution:

$$\delta(x - x_i) \simeq W(x - x_i, h)$$

- Function approximation

$$f(x_i) = \int_\Omega f(x)W(x - x_i, h)dx$$



$W(x - x_i, h)$ **kernel function**

h **smoothing length**

Smoothed Particle Hydrodynamics (SPH)

- Properties of the kernel function

$$\int_\Omega W(x-x_i,h)dx = 1$$

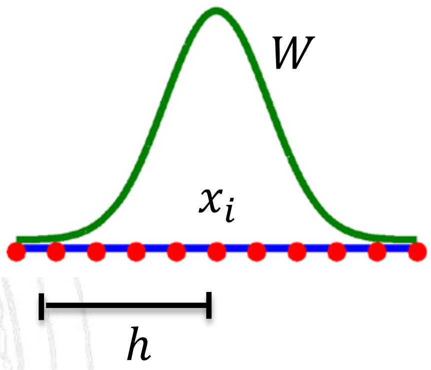
Unity

$$\begin{cases} W(x-x_i,h) \neq 0 & |x-x_i| < h \\ W(x-x_i,h) = 0 & |x-x_i| \geq h \end{cases}$$

Compact support

$$\lim_{h\to 0} W(x-x_i,h) = \delta(x-x_i)$$

Dirac Delta property



- Approximation of first order derivative

$$f'(x_i) = \left[f(x)W(x-x_i,h)\right]_{-\infty}^{+\infty} - \int_\Omega f(x)W'(x-x_i,h)dx$$

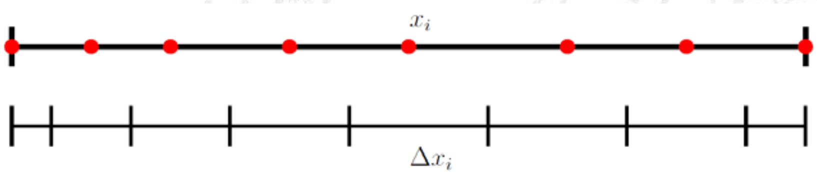
Here the first term is neglected due to the shape of $W(x-x_i,h)$

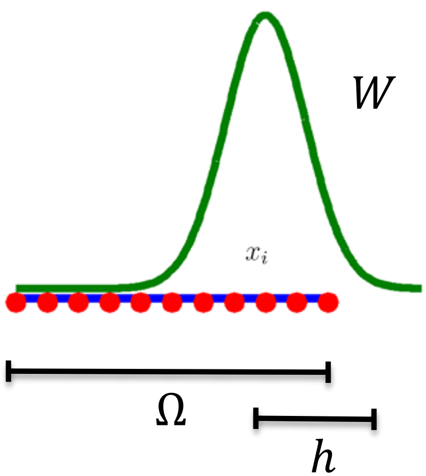
- Discrete approximation for **derivatives of order *n***

$$f^{(n)}(x_i) = (-1)^n \sum_j f(x_j) W^{(n)}(x_j - x_i) \Delta x_j$$

Integrals are discretized and **replaced by summation**

Δx_i subdomain of the particle in x_i





PROBLEM !!!

What about particles near to the boundary domain?

When the smoothing length exceeds the boundary of the domain, the smoothing function is said **not completely developed.** In these cases the derivative approximation is not accurate.

- **Reproducing Kernal Particle Method (Liu *et al,* 1995)**

$$K_i(x) = C_i(x) W(x - x_i)$$

- **Corrective smmothed Particle Method (Chen *et al,* 1999)**

$$f(x_i) \simeq \frac{\int_\Omega f(x) W(x - x_i, h)\, dx}{\int_\Omega W(x - x_i, h)\, dx}$$

- **Modified Smoothed Particle Method (Zhang and Batra, 2004)**

$$\begin{pmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{pmatrix} \begin{pmatrix} f(x_i) \\ f'(x_i) \end{pmatrix} = \begin{pmatrix} \int_\Omega f(x) W_i(x)\, dx \\ \int_\Omega f(x) W_i'(x)\, dx \end{pmatrix}$$

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Meshless methods based on shape functions

Radial Basis Collocation (Franke, 1998; Buhmann, 2003)

$$f(\boldsymbol{x}) = \sum_j a_j \phi_j(r_{ij}) \qquad r_{ij} = \sqrt{(\boldsymbol{x_i} - \boldsymbol{x_j}) \cdot (\boldsymbol{x_i} - \boldsymbol{x_j})}$$

Radial basis functions

$$\phi(r) = e^{-r^2/c^2} \qquad \phi(r) = (r^2 + c^2)^{n-3/2} \qquad \phi(r) = r\log(r)$$

Local max-ent shape functions (Arroyo, 2006, 2007)

$$f_\beta(\mathbf{x}, \mathbf{p}) \equiv \beta U(\mathbf{x}, \mathbf{p}) - H(\mathbf{p})$$

Pareto optima

Shape function

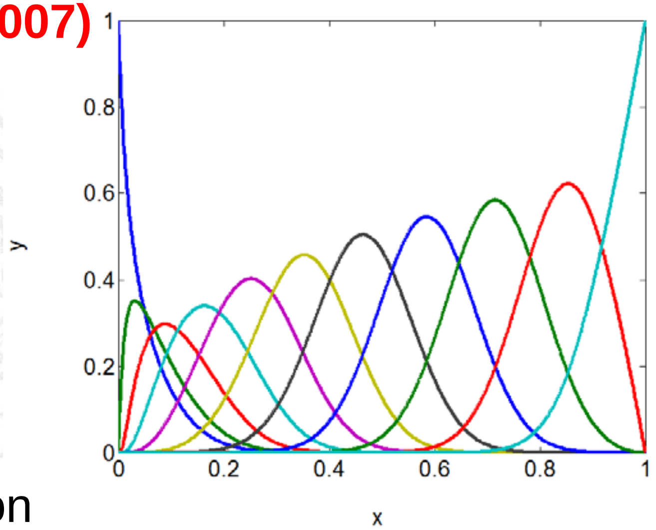
$$p_{\beta a}(\mathbf{x}) = \frac{1}{Z(\mathbf{x}, \boldsymbol{\lambda}^*(\mathbf{x}))} \exp[-\beta|\mathbf{x} - \mathbf{x}_a|^2 + \boldsymbol{\lambda}^*(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_a)]$$

$$Z(\mathbf{x}, \boldsymbol{\lambda}) \equiv \sum_{a=1}^{N} \exp[-\beta|\mathbf{x} - \mathbf{x}_a|^2 + \boldsymbol{\lambda}^*(\mathbf{x}) \cdot (\mathbf{x} - \mathbf{x}_a)]$$

Partition function

$$\boldsymbol{\lambda}^*(\boldsymbol{x}) = \arg\min \log Z(\mathbf{x}, \boldsymbol{\lambda})$$

Lagrangian multiplier



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Modified Finite Particle Method (MFPM)

The Modified Finite Particle Method is a **collocation, meshless, local** method based on projection functions.

Collocation method: we approximate partial differential equations in their strong form. This leads to avoid problems related to numerical integration of functions.

Meshless method: there is no *a priori* connectivity among nodes. The idea is to approximate functions and derivatives based only on the reciprocal position between nodes.

Local method: derivative approximations are based on nodes in a neighborhood of each collocation point.

Modified Finite Particle Method (MFPM)

Consider the **two dimensional** Taylor series of a function $u(\boldsymbol{x})$ centered in a point $\boldsymbol{x}_i$

$$u(\mathbf{x}) - u(\mathbf{x}_i) \quad = \frac{\partial u}{\partial x}(\mathbf{x}_i)(x - x_i) + \frac{\partial u}{\partial y}(\mathbf{x}_i)(y - y_i) +$$

$$\frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_i)(x - x_i)^2 + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(y - y_i)^2 + \frac{\partial^2 u}{\partial x \partial y}(\mathbf{x}_i)(x - x_i)(y - y_i)$$

- **5** unknown derivatives have to be approximated
- **5 projection functions** $W_\alpha^i = W_\alpha(\mathbf{x} - \mathbf{x}_i)$ for $\alpha = 1, \ldots, 5$ are introduced

$$W_1^i = x - x_i; \qquad W_2^i = y - y_i; \qquad W_3^i = (x - x_i)^2;$$

$$W_4^i = (y - y_i)^2; \qquad W_5^i = (x - x_i)(y - y_i)$$

- **No unity** property is required
- **No compact support** property is required
- **No Dirac Delta property** is required

Modified Finite Particle Method (MFPM)

- Taylor series is projected on the projection functions W_α^i

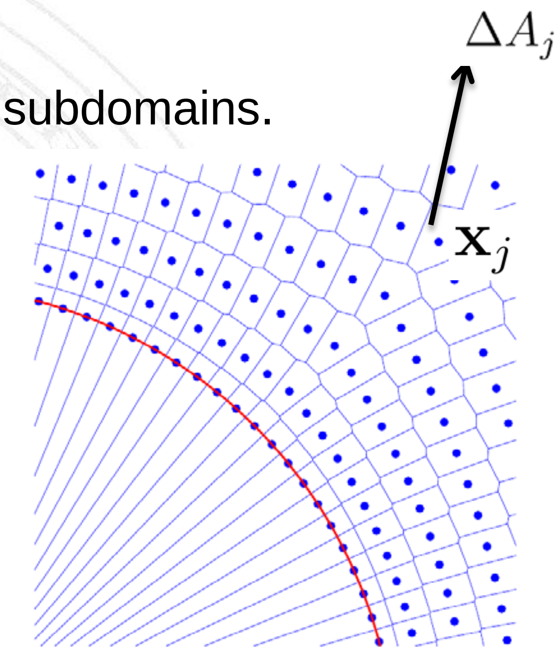
$$
\begin{aligned}
\int_\Omega [u(\mathbf{x}) - u(\mathbf{x}_i)] W_\alpha^i \, d\Omega \quad &= \frac{\partial u}{\partial x}(\mathbf{x}_i) \int_\Omega (x - x_i) W_\alpha^i \, d\Omega + \frac{\partial u}{\partial y}(\mathbf{x}_i) \int_\Omega (y - y_i) W_\alpha^i \, d\Omega \\
&+ \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_i) \int_\Omega \frac{1}{2}(x - x_i)^2 W_\alpha^i \, d\Omega + \frac{\partial^2 u}{\partial y^2}(\mathbf{x}_i) \int_\Omega \frac{1}{2}(y - y_i)^2 W_\alpha^i \, d\Omega \\
&+ \frac{\partial^2 u}{\partial x \partial y}(\mathbf{x}_i) \int_\Omega (x - x_i)(y - y_i) W_\alpha^i \, d\Omega \qquad \alpha = 1, \dots, 5
\end{aligned}
$$

- In matrix form

$$
\boldsymbol{A}_i \begin{pmatrix} \partial u(\mathbf{x}_i)/\partial x \\ \partial u(\mathbf{x}_i)/\partial y \\ \partial^2 u(\mathbf{x}_i)/\partial x^2 \\ \partial^2 u(\mathbf{x}_i)/\partial y^2 \\ \partial^2 u(\mathbf{x}_i)/\partial x \partial y \end{pmatrix} = \begin{pmatrix} \int_\Omega [u(\mathbf{x}) - u(\mathbf{x}_i)] W_1^i \, d\Omega \\ \int_\Omega [u(\mathbf{x}) - u(\mathbf{x}_i)] W_2^i \, d\Omega \\ \int_\Omega [u(\mathbf{x}) - u(\mathbf{x}_i)] W_3^i \, d\Omega \\ \int_\Omega [u(\mathbf{x}) - u(\mathbf{x}_i)] W_4^i \, d\Omega \\ \int_\Omega [u(\mathbf{x}) - u(\mathbf{x}_i)] W_5^i \, d\Omega \end{pmatrix}
$$

Modified Finite Particle Method (MFPM)

- Integrals have to be discretized.
- Voronoi procedure is used to divide the domain in subdomains.
- Each particle $\boldsymbol{x}_j$ is assigned a subdomain ΔA_j
- Integrals are replaced by summation



$$\boldsymbol{A}_i \begin{pmatrix} \partial u(\mathbf{x}_i)/\partial x \\ \partial u(\mathbf{x}_i)/\partial y \\ \partial^2 u(\mathbf{x}_i)/\partial x^2 \\ \partial^2 u(\mathbf{x}_i)/\partial y^2 \\ \partial^2 u(\mathbf{x}_i)/\partial x \partial y \end{pmatrix} = \begin{pmatrix} \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)] W_1^{ij} \Delta A_j \\ \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)] W_2^{ij} \Delta A_j \\ \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)] W_3^{ij} \Delta A_j \\ \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)] W_4^{ij} \Delta A_j \\ \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)] W_5^{ij} \Delta A_j \end{pmatrix}$$

$$W_\alpha^{ij} = W_\alpha(\mathbf{x}_j - \mathbf{x}_i)$$

- Inverting this system, we get derivative approximations

PROBLEMS

- The Voronoi tessellation procedure is **time consuming**
- **Numerical errors** introduced in the approximation of integrals

Modified Finite Particle Method: novel formulation

- Consider the Taylor series expansion of $u(\boldsymbol{x})$ centered $\boldsymbol{x}_i$
- Evaluate $u(\boldsymbol{x}_j) - u(\boldsymbol{x}_i)$ in a set of **supporting nodes** $\boldsymbol{x}_j$.

$$
\begin{aligned}
u(\mathbf{x}_j) - u(\mathbf{x}_i) \; &= \frac{\partial u}{\partial x}(\mathbf{x}_i)(x_j - x_i) + \frac{\partial u}{\partial y}(\mathbf{x}_i)(y_j - y_i) \\
&+ \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(\mathbf{x}_i)(x_j - x_i)^2 + \frac{1}{2}\frac{\partial^2 u}{\partial y^2}(y_j - y_i)^2 \\
&+ \frac{\partial^2 u}{\partial x \partial y}(\mathbf{x}_i)(x_j - x_i)(y_j - y_i)
\end{aligned}
$$

- Collect the evaluations in a vector $\mathbf{q}^i = \{u(\mathbf{x}_j) - u(\mathbf{x}_i)\}, j = 1, \ldots, N_{supp}$ where N_{supp} is the number of the nodes supporting $\boldsymbol{x}_i$.

- Evaluate **five** projection functions $W_\alpha^i = W_\alpha(\mathbf{x} - \mathbf{x}_i)$ in the same supporting nodes $\boldsymbol{x}_j$

- Collect the evaluations of $W_\alpha{}^i$ in **five** vectors $\mathbf{W}_\alpha^i = \{W_\alpha^{ij}\}, j = 1, \ldots, N_{supp}$ where $W_\alpha^{ij} = W_\alpha(\boldsymbol{x}_j - \boldsymbol{x}_i)$

Modified Finite Particle Method: novel formulation

- Scalarly multiply $\mathbf{W}_\alpha^i \cdot \mathbf{q}^i$

$$\sum_j^{N_{supp}} [u(\mathbf{x}) - u(\mathbf{x}_i)]W_\alpha^{ij} = \frac{\partial u}{\partial x}(\mathbf{x}_i) \sum_j^{N_{supp}} (x_j - x_i)W_\alpha^{ij} + \frac{\partial u}{\partial y}(\mathbf{x}_i) \sum_j^{N_{supp}} (y_j - y_i)W_\alpha^{ij}$$
$$+ \frac{\partial^2 u}{\partial x^2}(\mathbf{x}_i) \sum_j^{N_{supp}} \frac{1}{2}(x_j - x_i)^2 W_\alpha^{ij} + \frac{\partial^2 u}{\partial y^2}(\mathbf{x}_i) \sum_j^{N_{supp}} \frac{1}{2}(y_j - y_i)^2 W_\alpha^{ij}$$
$$+ \frac{\partial^2 u}{\partial x \partial y}(\mathbf{x}_i) \sum_j^{N_{supp}} (x_j - x_i)(y_j - y_i)W_\alpha^{ij} \qquad \alpha = 1, \ldots, 5$$

- In matrix form

$$\boldsymbol{A}_i \begin{pmatrix} \partial u(\mathbf{x}_i)/\partial x \\ \partial u(\mathbf{x}_i)/\partial y \\ \partial^2 u(\mathbf{x}_i)/\partial x^2 \\ \partial^2 u(\mathbf{x}_i)/\partial y^2 \\ \partial^2 u(\mathbf{x}_i)/\partial x \partial y \end{pmatrix} = \begin{pmatrix} \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)]W_1^{ij} \\ \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)]W_2^{ij} \\ \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)]W_3^{ij} \\ \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)]W_4^{ij} \\ \sum_j [u(\mathbf{x}_j) - u(\mathbf{x}_i)]W_5^{ij} \end{pmatrix}$$

- Inverting the system, derivative approximations are obtained

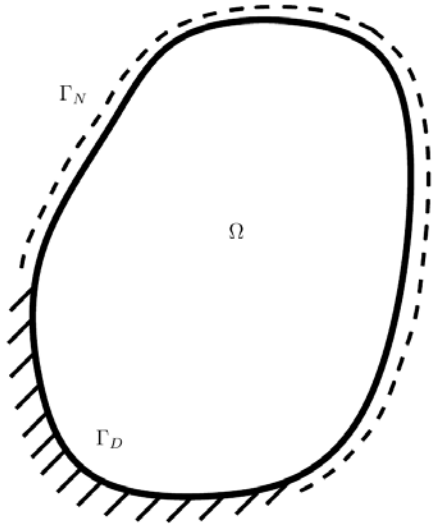
ADVANTAGES

- **No integral discretization** is needed
- **No Voronoi tessellation** is required

Consider a linear elastic body Ω subjected to internal forces $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x})$, constrained displacements $\bar{\boldsymbol{u}} = \bar{\boldsymbol{u}}(\boldsymbol{x})$ on the Dirichlet boundary Γ_D and tractions $\bar{\boldsymbol{t}} = \bar{\boldsymbol{t}}(\boldsymbol{x})$ on the Neumann boundary Γ_N.

Dynamic equilibrium equations

$$\begin{cases} \rho \dfrac{\partial^2 \boldsymbol{s}}{\partial t^2} = \nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} & \boldsymbol{x} \in \Omega \\ \boldsymbol{\sigma n} = \bar{\boldsymbol{t}}(t) & \boldsymbol{x} \in \Gamma_N \\ \boldsymbol{s} = \bar{\boldsymbol{s}}(t) & \boldsymbol{x} \in \Gamma_D \\ \boldsymbol{s}\big|_{t=0} = \boldsymbol{s}_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega \\ \dfrac{\partial \boldsymbol{s}}{\partial t}\Big|_{t=0} = \dot{\boldsymbol{s}}_0(\boldsymbol{x}) & \boldsymbol{x} \in \Omega \end{cases}$$

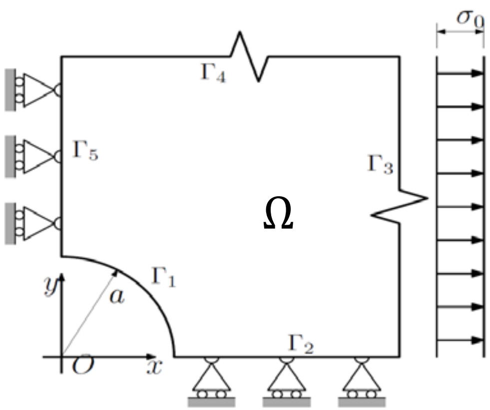


Spatial discretization of the problem

$$\begin{pmatrix} \rho \ddot{\hat{\boldsymbol{u}}} \\ \rho \ddot{\hat{\boldsymbol{v}}} \\ \rho \ddot{\hat{\boldsymbol{w}}} \end{pmatrix} = \begin{pmatrix} \hat{\boldsymbol{K}}_{11} & \hat{\boldsymbol{K}}_{12} & \hat{\boldsymbol{K}}_{13} \\ \hat{\boldsymbol{K}}_{21} & \hat{\boldsymbol{K}}_{22} & \hat{\boldsymbol{K}}_{23} \\ \hat{\boldsymbol{K}}_{31} & \hat{\boldsymbol{K}}_{32} & \hat{\boldsymbol{K}}_{33} \end{pmatrix} \begin{pmatrix} \hat{\boldsymbol{u}} \\ \hat{\boldsymbol{v}} \\ \hat{\boldsymbol{w}} \end{pmatrix} + \begin{pmatrix} \boldsymbol{b}_x \\ \boldsymbol{b}_y \\ \boldsymbol{b}_z \end{pmatrix}$$

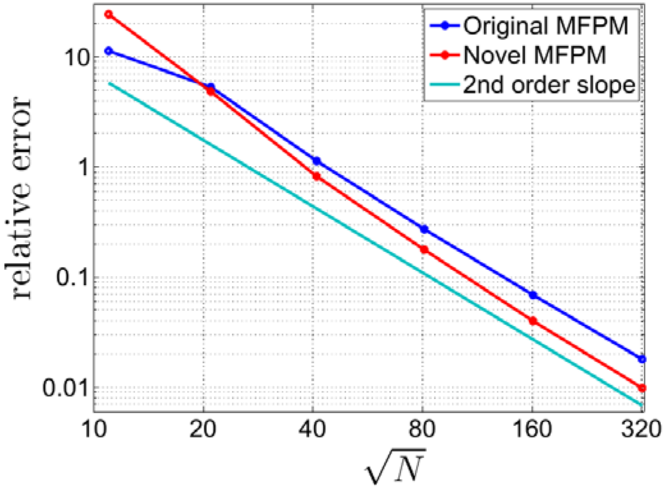
$$\boxed{\begin{aligned} \hat{\boldsymbol{K}}_{11} &= (\lambda + 2\mu)\boldsymbol{D}_{xx} + \mu(\boldsymbol{D}_{yy} + \boldsymbol{D}_{zz}) \\ \hat{\boldsymbol{K}}_{22} &= (\lambda + 2\mu)\boldsymbol{D}_{yy} + \mu(\boldsymbol{D}_{xx} + \boldsymbol{D}_{zz}) \\ \hat{\boldsymbol{K}}_{33} &= (\lambda + 2\mu)\boldsymbol{D}_{zz} + \mu(\boldsymbol{D}_{xx} + \boldsymbol{D}_{yy}) \\ \hat{\boldsymbol{K}}_{12} &= \hat{\boldsymbol{K}}_{21} = (\lambda + \mu)\boldsymbol{D}_{xy} \\ \hat{\boldsymbol{K}}_{13} &= \hat{\boldsymbol{K}}_{31} = (\lambda + \mu)\boldsymbol{D}_{xz} \\ \hat{\boldsymbol{K}}_{23} &= \hat{\boldsymbol{K}}_{32} = (\lambda + \mu)\boldsymbol{D}_{yz} \end{aligned}}$$

Traction of an infinitely extended plate with a circular hole

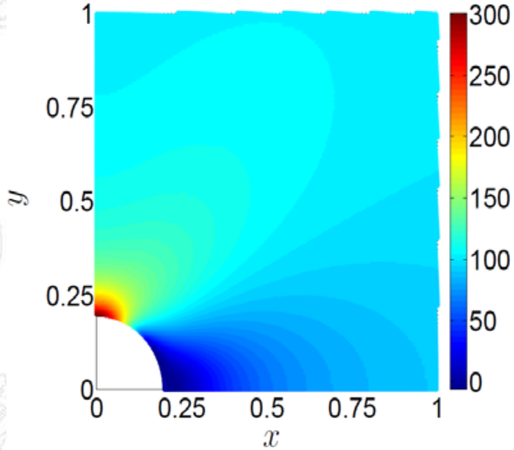


Geometry and boundary conditions
for a bar under impulsive traction

$$
\begin{cases}
\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0 & \boldsymbol{x} \in \Omega \\
\boldsymbol{\sigma n} = \boldsymbol{0} & \text{on } \Gamma_1 \text{ and } \Gamma_4 \\
\boldsymbol{\sigma n} \cdot \boldsymbol{t} = 0 \quad \text{and} \quad \boldsymbol{s} \cdot \boldsymbol{n} = 0 & \text{on } \Gamma_2 \text{ and } \Gamma_5 \\
\boldsymbol{\sigma n} = [\sigma_0 \quad 0]^T & \text{on } \Gamma_3
\end{cases}
$$



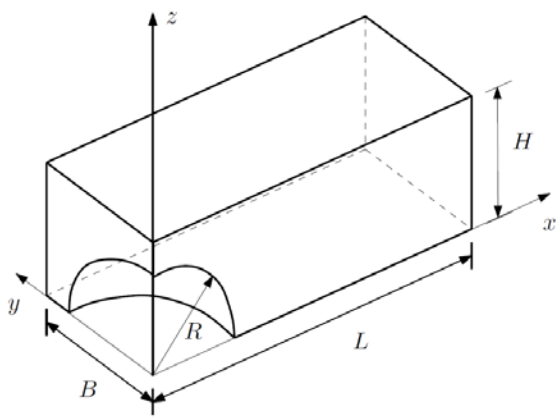
Convergence diagram of the error in stress for the original and novel formulation



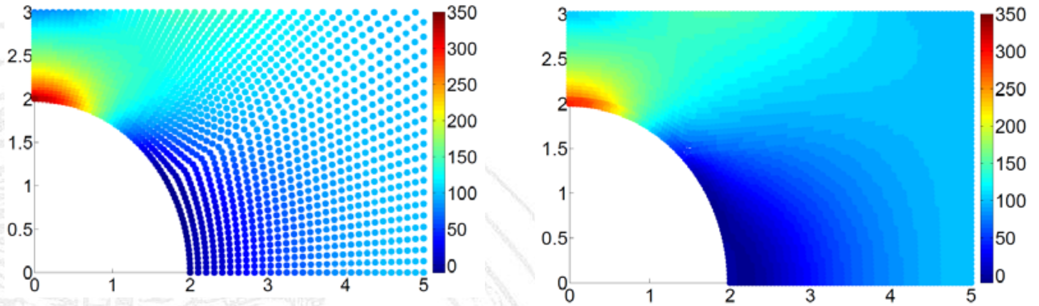
Stress component σ_{xx} obtained with MFPM

- Second order convergence of the error in both the original and novel formulation
- The error costant is reduced in the novel formulation

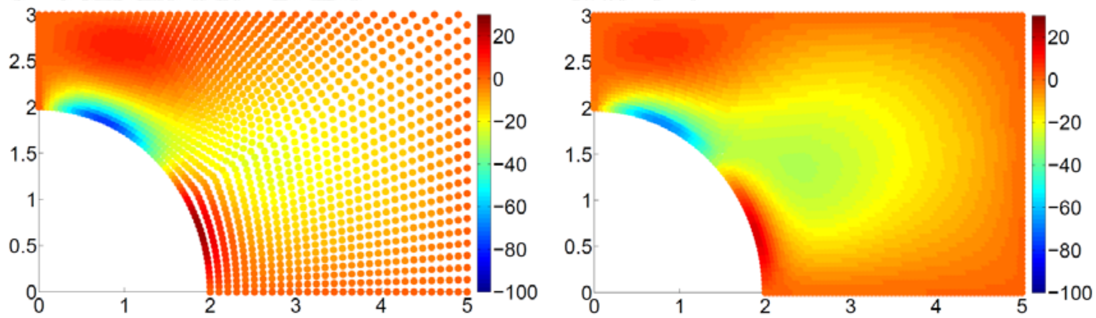
3D problem: parallelepiped with spherical hole under traction



Geometry of a parallelepiped with spherical bore with its simmetry planes



Stress component σ_{xx} on the simmetry plane xz using MFPM (left) and using an overkilled Finite Element discretization (right)

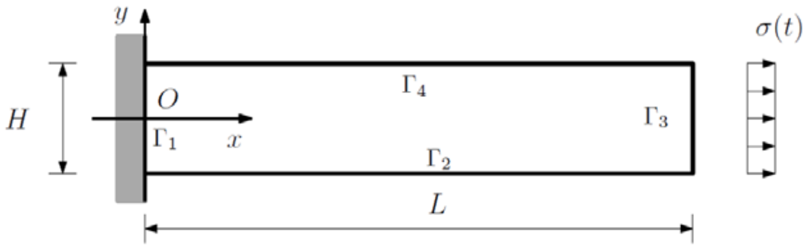


Stress component τ_{xy} on the simmetry plane xz using MFPM (left) and using an overkilled Finite Element discretization (right)

- The parallelepiped is under traction in the x direction.
- $x = 0$, $y = 0$, $z = 0$ are symmetry planes

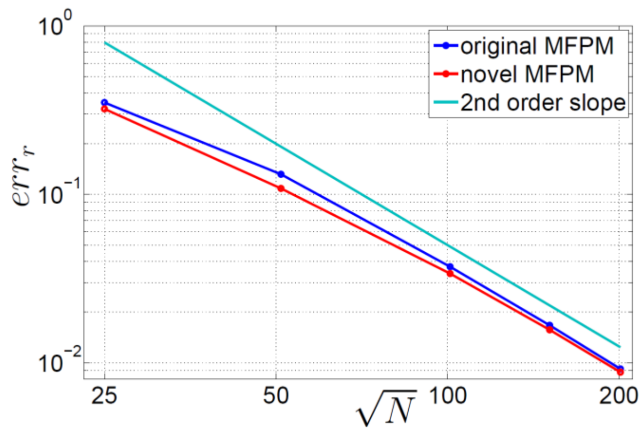
- The novel formulation is easily extensible to 3D, while the original formulation implies 3D Voronoi tessellation that is difficoult to code and extremely time consuming

Dynamics: 2d bar under impulsive load

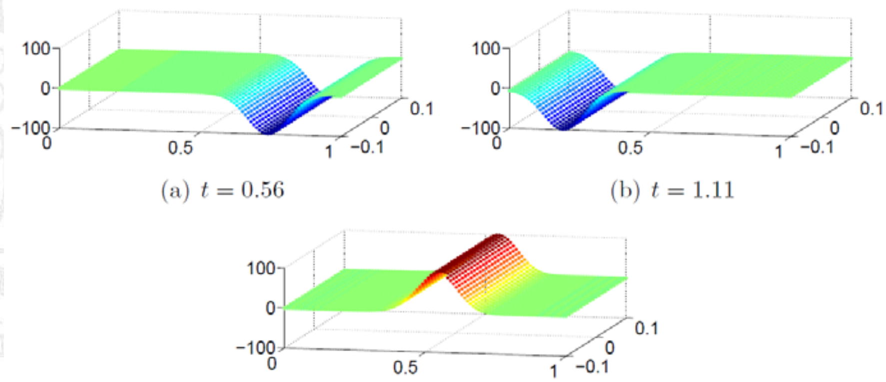


$$
\begin{cases}
u = 0 & \text{and} & v = 0 & \text{on } \Gamma_1 \\
\sigma_{yy} = 0 & \text{and} & \tau_{xy} = 0 & \text{on } \Gamma_2 \text{ and } \Gamma_4 \\
\sigma_{xx} = \sigma(t) & \text{and} & \tau_{xy} = 0 & \text{on } \Gamma_3
\end{cases}
$$

Geometry and boundary conditions for a bar under impulsive traction



Convergence diagram of the error in stress for the original and novel formulation



(a) $t = 0.56$

(b) $t = 1.11$

(c) $t = 2.76$

Stress component σ_{xx} in three time instant. The stress wave propagation can be appreciated

Also in the case of dynamics, second order spatial accuracy is achieved both for original and novel formulation.

References about MFPM in elasticity

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The equation that model the dynamics of an **incompressible** (solid or fluid) body is

$$\rho\, \boldsymbol{a} = -\nabla p + \mu \Delta \boldsymbol{u} + \boldsymbol{b}$$

ρ	density	$\boldsymbol{a}$	material acceleration
p	pressure	μ	elastic shear modulus
$\boldsymbol{b}$	body loads		

The variable $\boldsymbol{u}$ is the body displacement field in the solid case, and is the body velocity field in the fluid case.

The incompressibility condition is $\nabla \cdot \boldsymbol{u} = 0$

Lagrangian point of view: the observer follows body particles during thier motion.

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t}$$

Eulerian point of view: the observer is in a fixed position during the fluid motion

$$\mathbf{a} = \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{c} \cdot \nabla \mathbf{u}$$

$\mathbf{c} = \mathbf{u} - \mathbf{u}_{ref}$ **convective velocity**

Stokes and Navier Stokes Equations

Stokes equations

$$\rho\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \mu\Delta\mathbf{u} + \mathbf{b}$$

$$\nabla \cdot \mathbf{u} = 0$$

- Describe **highly-viscous flows** in an Eulerian framework
- Describe all flows in a Lagrangian framework

Navier-Stokes equations

$$\rho\frac{\partial \mathbf{u}}{\partial t} + \rho\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + \mu\Delta\mathbf{u} + \mathbf{b}$$

$$\nabla \cdot \mathbf{u} = 0$$

- Describe all flows in an Eulerian framework
- Are **non linear** equations

Stokes and Navier-Stokes Equations are **parabolic problems** with a **constraint equation**. For its solution, **one initial condition is required** on the whole domain;
two boundary conditions are required at each boundary

Possible boundary conditions

- Dirichlet boundary conditions (known velocity at the boundary)
- Neumann boundary conditions (known traction at the boundary)

- **No boundary conditions** are required for the constraint equation

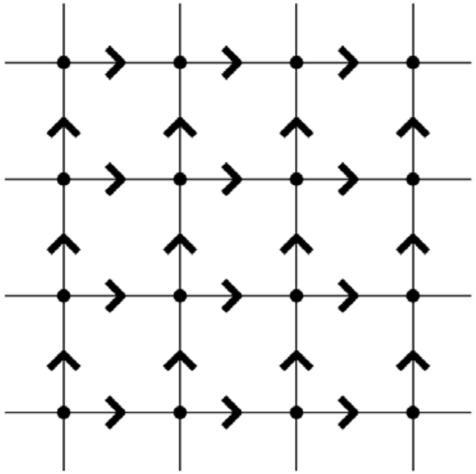
Stokes and Navier Stokes Equations: discretization

The same interpolation of velocity and pressure leads to a well known **instability** of the pressure field.

This instability has been first studied by **Brezzi** (1974), which estabilished a condition that has to be respected in order to avoid pressure instability. This condition is known as **LBB condition** or **inf-sup condition.**

In the field of the Finite Element Method, usually velocity is discretized using **quadratic interpolation**, and pressure using **linear interpolation.**

In the field of Finite Difference Method, the difficulty is overcome using **staggered grids**, that is, velocity and pressure are discretized in different nodes, and also the equilibrium equations and incompressibility constrains are collocated in different points.



A staggered grid

Staggered grids **cannot be used** in meshless methods, where nodes could be randomly distributed.

Stokes Equations – alternative formulations

Different formulations should be used in order to solve incompressibility equations on non staggered grids.

$$-\nabla p + \mu \Delta \mathbf{u} = -\mathbf{b}$$
$$\nabla \cdot \mathbf{u} = 0$$

S1 (Sani, 2006)

$$-\nabla p + \mu \Delta \mathbf{u} = -\mathbf{b}$$
$$\Delta p = \nabla \cdot \mathbf{b}$$
$$(-\nabla p + \mu \Delta \mathbf{u}) \cdot \mathbf{n} = -\mathbf{b} \cdot \mathbf{n} \quad on\ \partial\Omega$$

S5 (Wang&Liu, 2000)

$$\mathbf{u} = \mathbf{a} - \nabla \phi$$
$$p = -\Delta \phi$$

$$\mu \Delta \mathbf{a} = -\mathbf{b}$$
$$\nabla \cdot \mathbf{a} - \Delta \phi = 0$$
$$\partial \phi / \partial \mathbf{n} = 0 \quad on\ \partial\Omega$$

S2 (Sani, 2006)

$$-\nabla p + \mu \Delta \mathbf{u} = -\mathbf{b}$$
$$\Delta p - \mu \nabla \cdot (\Delta \mathbf{u}) = \nabla \cdot \mathbf{b}$$

No BC for the constraint equation

S3 (Sani, 2006)

$$-\nabla p + \mu \Delta \mathbf{u} = -\mathbf{b}$$
$$\Delta p = \nabla \cdot \mathbf{b}$$
$$\nabla \cdot \mathbf{u} = 0 \quad on\ \partial\Omega$$

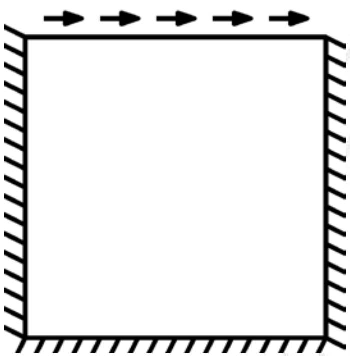
S4 (Brezzi&Douglas, 1988)

$$-\nabla p + \mu \Delta \mathbf{u} = -\mathbf{b}$$
$$\nabla \cdot \mathbf{u} + \varepsilon \Delta p = 0$$
$$\nabla \cdot \mathbf{u} = 0 \quad on\ \partial\Omega$$

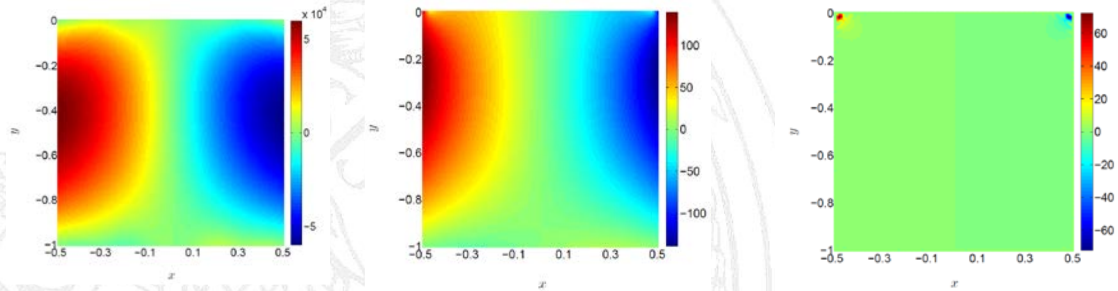
In formulations **S1**, **S2** and **S3** the incompressibility is **not explicitly imposed**!

Formulations S1, S2 and S3 do not impose directly the incompressibility constrain, but a derived one.

- Lid-driven cavity flow (Stokes case)



$\nabla \cdot \boldsymbol{u}$

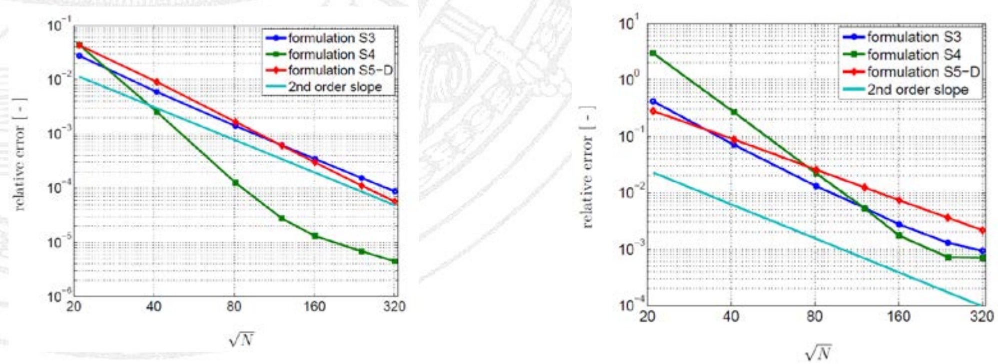


S1 S2 S3

From the above figures we see that only in formulation S3 the constraint is respected. For this reason, formulations S1 and S2 are not further investigated

Problem of a square with polynomial solution

$$u(x,y) = 20xy^3$$
$$v(x,y) = 5(x^4 - y^4)$$
$$p(x,y) = (60x^2y - 20y^3 + C)$$



Convergence diagrams for displacements and pressure

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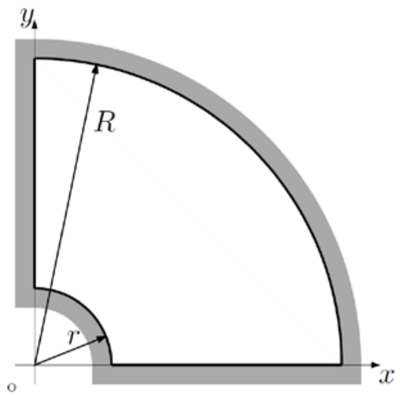
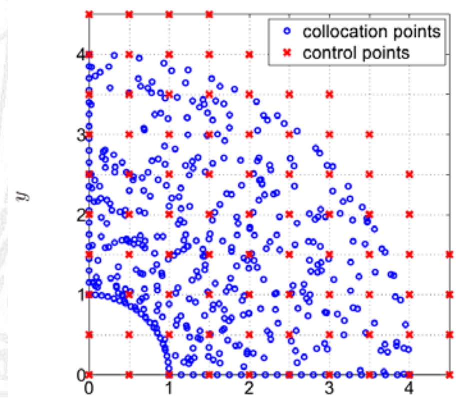
Modified Finite Particle Method: extended formulation

Two different sets of points

- **Collocation points** $\boldsymbol{x} = [x, y]^T$: the points where function and derivatives are computed
- **Control points** $\boldsymbol{\xi} = [\xi, \eta]^T$: the auxiliary points used for approximating function and derivatives in the collocation points





The derivative approximation technique starts from the Taylor series of $u(\boldsymbol{x})$, centered in $\boldsymbol{x}_i$, evaluated in $\boldsymbol{\xi}_j$:

$$
\begin{aligned}
u(\boldsymbol{\xi}_j) \; &= u(\mathbf{x}_i) + D_x u(\mathbf{x}_i)(\xi_j - x_i) + D_y u(\mathbf{x}_i)(\eta_j - y_i) + \frac{1}{2} D^2_{xx} u(\mathbf{x}_i)(\xi_j - x_i)^2 \\
&+ \frac{1}{2} D^2_{yy} u(\mathbf{x}_i)(\eta_j - y_i)^2 + D^2_{xy} u(\mathbf{x}_i)(\xi_j - x_i)(\eta_j - y_i)
\end{aligned}
$$

6 unknowns: $u(x_i)$, $D_\mathrm{x} u(x_i)$,$D_\mathrm{y} u(x_i)$, $\mathrm{D}_\mathrm{xx} u(x_i)$, $D_\mathrm{yy} u(x_i)$, $D_\mathrm{xy} u(x_i)$

MFPM function and derivative approximation/2

Introduce 6 known **projection functions** $W_\alpha^i = W_\alpha(\boldsymbol{\xi} - \boldsymbol{x_i})$ and evaluate them in the same points $\boldsymbol{\xi}_j$.

Multiply left- and right-hand sides of Equation (1) in each $\boldsymbol{\xi}_j$ by the evaluations $W_\alpha^{ij} = W_\alpha(\boldsymbol{\xi}_j - \boldsymbol{x_i})$, and sum all terms, obtaining **6 equations** of the type

$$
\begin{aligned}
& u_i \sum_j W_\alpha^{ij} + D_x u_i \sum_j (\xi_j - x_i) W_\alpha^{ij} + D_y u_i \sum_j (\eta_j - y_i) W_\alpha^{ij} + \\
& + \frac{1}{2} D_{xx}^2 u_i \sum_j (\xi_j - x_i)^2 W_\alpha^{ij} + \frac{1}{2} D_{yy}^2 u_i \sum_j (\eta_j - y_i)^2 W_\alpha^{ij} + \\
& + D_{xy}^2 u_i \sum_j (\xi_j - x_i)(\eta_j - y_i) W_\alpha^{ij} = \sum_j u(\boldsymbol{\xi}_j) W_\alpha^{ij} \qquad \alpha = 1, ..., 6
\end{aligned}
$$

Matrix form

$$
\mathbf{A}^i \begin{pmatrix} u(\mathbf{x}_i) \\ D_x u(\mathbf{x}_i) \\ D_y u(\mathbf{x}_i) \\ D_{xx}^2 u(\mathbf{x}_i) \\ D_{yy}^2 u(\mathbf{x}_i) \\ D_{xy}^2 u(\mathbf{x}_i) \end{pmatrix} = \begin{pmatrix} \sum_j u(\boldsymbol{\xi}_j) W_1^{ij} \\ \sum_j u(\boldsymbol{\xi}_j) W_2^{ij} \\ \sum_j u(\boldsymbol{\xi}_j) W_3^{ij} \\ \sum_j u(\boldsymbol{\xi}_j) W_4^{ij} \\ \sum_j u(\boldsymbol{\xi}_j) W_5^{ij} \\ \sum_j u(\boldsymbol{\xi}_j) W_6^{ij} \end{pmatrix}
$$

Compact form

$$
\mathbf{A}^i \mathbf{D}(u_i) = \mathbb{W}^i \mathbf{u}
$$

If we assume to know the values of $u(\boldsymbol{\xi})$ collected in $\mathbf{u}$ we can retrieve the approximations of function and derivatives collected in $\mathbf{D}(u_i)$ in $\boldsymbol{x}_i$

Stokes problem

$$\nabla p = \mu\, \Delta \boldsymbol{u} + \boldsymbol{b}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

MFPM discretization of the Stokes problem

$$\begin{bmatrix} \mu\mathbf{L} & 0 & -\mathbf{D}_x \\ 0 & \mu\mathbf{L} & -\mathbf{D}_y \\ \mathbf{D}_x & \mathbf{D}_y & 0 \end{bmatrix} \begin{pmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{v}} \\ \hat{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \hat{\mathbf{f}}_x \\ \hat{\mathbf{f}}_y \\ \mathbf{0} \end{pmatrix} \Rightarrow \boldsymbol{K}\,\boldsymbol{d} = \boldsymbol{f}$$

$\mathbf{L}$, $\mathbf{D}_x$, $\mathbf{D}_y$: **MFPM** discrete differential operators

The above algebraic system is completed with suitable discrete boundary condition on the velocity or on the boundary outward stress.

When collocation points $\boldsymbol{x}$ and control points $\boldsymbol{\xi}$ coincide, the linear system is a square system, that can be solved through direct inversion.

Unfortunately, if collocation points coincide with control nodes, the solution of linear system exhibits unphysical obscillations of the pressure, known in the literature as **pressure checkerboard instability**.

Least Square Residual Method

The problem is discretized using a number of **collocation points higher than** the number of **control points**. The corresponding linear system is **overdetermined**. Its solution is approximated through an **error minimization.**

Global error $$E = (\boldsymbol{Kd} - \boldsymbol{f})^T(\boldsymbol{Kd} - \boldsymbol{f})$$

Minimization $$\frac{\partial E}{\partial \boldsymbol{d}} = \boldsymbol{0} \Rightarrow \boldsymbol{K}^T\boldsymbol{Kd} = \boldsymbol{K}^T\boldsymbol{f}$$

The present error definition is the sum of the squared errors coming from the equilibrium equations, from the incompressibility constraint, and from Dirichlet and Neumann boundary conditions. They in general contribute differently to the total squared error.

Hu et al (2007) and **Chi et al. (2014)** proposed a **weighted squared error**, based on the balance of different components of the error

Weighted error $$E_w = (\boldsymbol{Kd} - \boldsymbol{f})^T\boldsymbol{A}(\boldsymbol{Kd} - \boldsymbol{f})$$

$\boldsymbol{A}$: diagonal matrix composed by the squared weights associated to the different error components

Weighted error minimization

$$\boldsymbol{K}^T \boldsymbol{A}\, \boldsymbol{K}\, \boldsymbol{d} = \boldsymbol{K}^T \boldsymbol{A}\, \boldsymbol{f}$$

The present linear system is a well-posed square system, which gives the control nodal values $\hat{\boldsymbol{u}}$, $\hat{\boldsymbol{v}}$, and $\hat{\boldsymbol{p}}$ from which it is possible, using the MFPM approximation procedure, to retrieve the evaluations of $u(\boldsymbol{x})$, $v(\boldsymbol{x})$, and $p(\boldsymbol{x})$ in the collocation points.

The weights are selected on the basis of the **scale** of each equation

- **Equilibrium equations**: $o(\mu \partial u/\partial x^2) \simeq o(\mu/h^2)$
- **Incompressibility constraints**: $o(\partial u/\partial x) \simeq o(1/h)$
- **Dirichlet boundary conditions**: $o(u) \simeq o(1)$
- **Neumann boundary conditions**: $o(\mu \partial u/\partial x) \simeq o(\mu/h)$

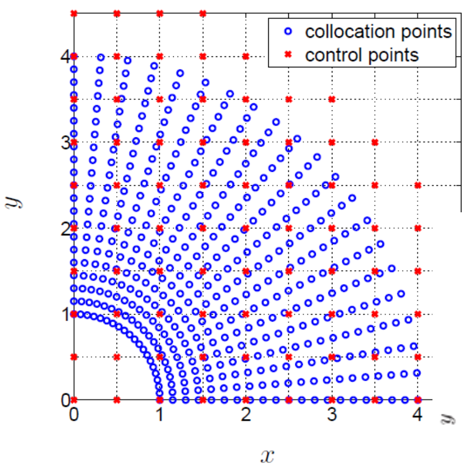
To restore the same order of magnitude of equations, the weights are

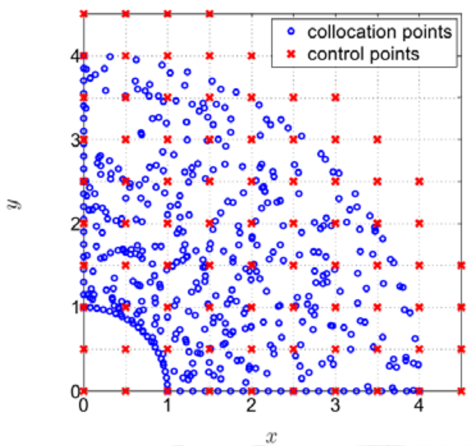
$$\alpha_{eq} = 1; \qquad \alpha_{inc} = \mu/h; \qquad \alpha_{Dir} = \mu/h^2; \qquad \alpha_{Neum} = 1/h$$

Quarter of annulus under polynomial body loads

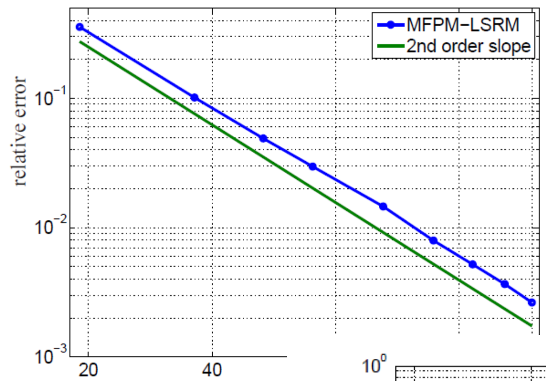
Stokes flow in a quarter of annulus with exact solution

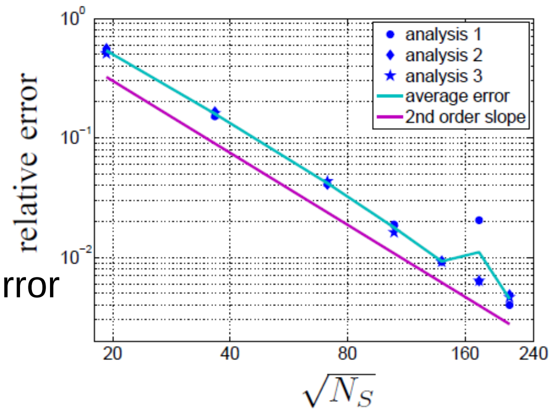
$$\begin{cases} u = 10^{-6} x^2 y^4 (x^2+y^2-16)(x^2+y^2-1)(5x^4+18x^2y^2-85x^2+13y^4+80-153y^2) \\ v = -2\cdot 10^{-6} x y^5 (x^2+y^2-16)(x^2+y^2-1)(5x^4-51x^2+6x^2y^2-17y^2+16+y^4) \end{cases}$$





Collocation and control nodes distributions





Convergence diagram of the error

Navier-Stokes Equations

Stationary Navier-Stokes equations:

$$\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \mu \, \Delta \boldsymbol{u} + \boldsymbol{b}$$

$$\nabla \cdot \boldsymbol{u} = 0$$

The **MFPM/LSRM** procedure is extended to the non-linear case, and the solution of a problem is found through the Newton Raphson algorithm.

Iterative linearization process $$\boldsymbol{K}^{\boldsymbol{k}}(\boldsymbol{d}) \, \Delta \boldsymbol{d}^{k} = \boldsymbol{R}^{\boldsymbol{k}}$$

Since the present system is **overdetermined** ($N_c > N_s$), a **LSRM procedure** is required to approximate the solution increment at each step.

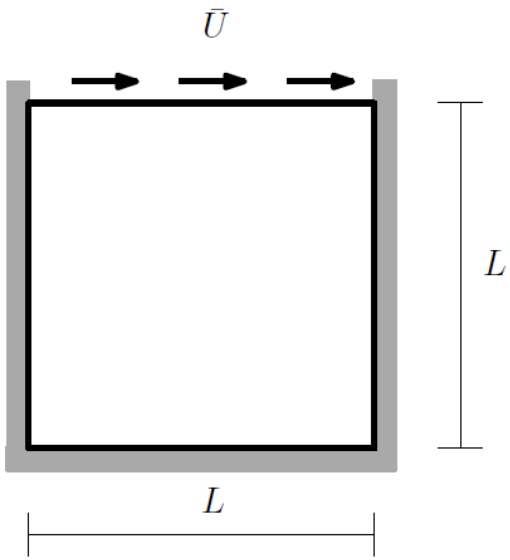
Iteration weighted squared error $$E_w^k = \left(\boldsymbol{K}^k \, \Delta \boldsymbol{d}^k - \boldsymbol{R}^k\right)^T \boldsymbol{A}^k \left(\boldsymbol{K}^k \, \Delta \boldsymbol{d}^k - \boldsymbol{R}^k\right)$$

Minimization $$(\boldsymbol{K}^k)^T \boldsymbol{A}^k \boldsymbol{K}^k \Delta \boldsymbol{d}^k = (\boldsymbol{K}^k)^T \boldsymbol{A}^k \boldsymbol{R}^k$$

Solution until convergence

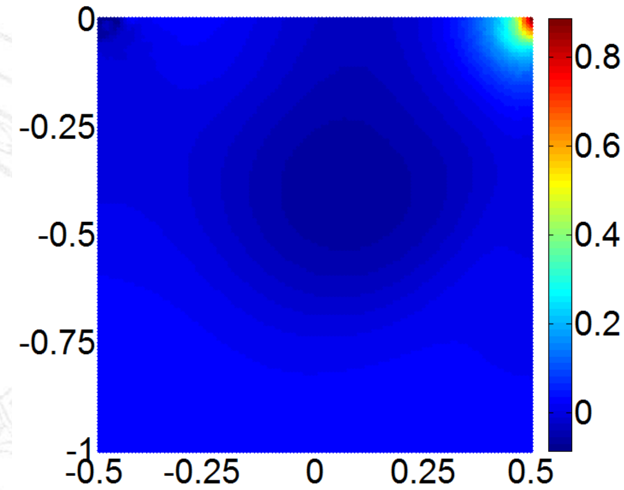
Selected weights $$\alpha_i = A_{ii}^k = \left(\frac{3N_i}{\sum_{j=1}^{N_i} K_{ij}^k}\right)^2$$

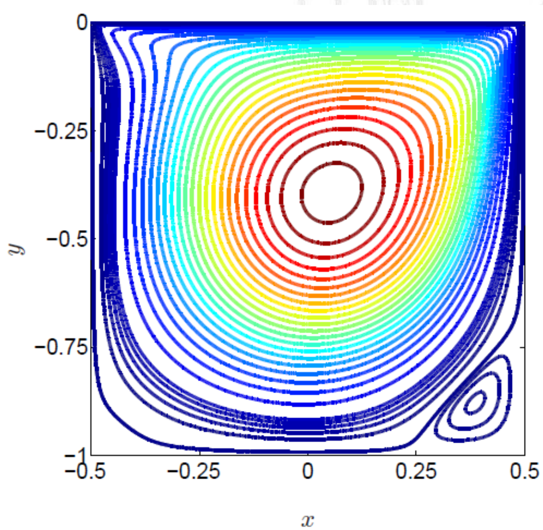
Application: lid-driven cavity



Domain and boundary condition for the lid-driven cavity problem

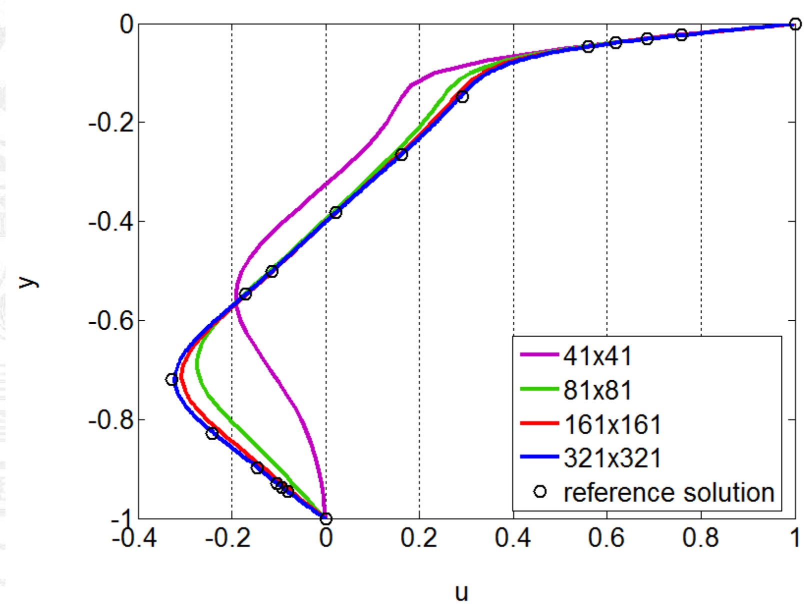
Pressure contour plot





Vorticity contour plot

$$Re = 400$$



Horizontal velocity profile at x=0 and comparison with **Ghia et al.** (1982)

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- **Asprone, D., F. Auricchio, A. Montanino, and A. Reali**. Solution of the stationary Stokes and Navier-Stokes equations using the Modified Finite Particle Method in the framework of a Least Square Residual Method. *To be submitted*

Conclusion

- In this thesis we study meshless method, and apply and extend the Modified Finite Particle Method to compressible linear elasticity, incompressibile elasticity, fluid dynamics.

- For linear elastic problems, in the tests where an analytical solution is available, MFPM shows correct second-order accuracy.

- When approaching incompressible elasticity, MFPM, similarly to other numerical methods, suffers from spurious oscillations of the pressure, unless the so-called *inf-sup condition* is respected, or alternative formulations are used.

- An extended formulation of the MFPM has been introduced and used in combination with a Least Square Residual Method for the solution of Stokes and Navier-Stokes problem. The method is more robust with respect to the original one when approaching problems with complicated geometries and/or randomly distributed set of nodes, both in the linear and in the non-linear case.

Additional activities

International conferences

- Particle-Based Methods III – Stuttgart, September 18-20, 2013
- SPH and Particular Methods for Fluids and Fluid Structure Interaction – Lille, January 21-22, 2015 - coauthor
- SPHeric workshop 2015 – Parma, June 16-18, 2015
- Particle-Based Methods IV – Barcelona, September 28-30, 2015

Teaching activities

- Elementi di Meccanica Computazionale – Teaching assistant – A.A 2013-2014, 2014-2015
- Scienza delle costruzioni B – Tutor – A.A. 2012-2013, 2013-2014, 2014-2015, 2015-2016

Collaboration with companies

- Design and verification of steel doors under blast load, according to the American standard UFC-3-340

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