Implicit Corotational Method

Theory and FEM implementation

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The Implicit Corotational Method (ICM)

ICM potentiality

- Recover objective nonlinear models
- Reuse the richness and complexity of linear theory
- Accuracy and easy FEM implementation

Availability of nonlinear models

- Geometrically exact (Simo, Antman)
- Galerkin approximation of 3D nonlinear continuum (Chen, Kim, Petrov, Cardona)
- Models based on corotational description (Nayfeh, Pai)

Availability of linear theory

- Saint Venánt's rod theory
- Vlasov's theory for thin walled beam
- Reissner's plate and classical theory for shell

Strain energy

Continuum description

The continuum is described using Biot's stress/strain. Mixed strain energy in terms of $\boldsymbol{\sigma}_b$ and ε_b is:

$$\Phi[\sigma, \varepsilon] := \int_V \left\{ \boldsymbol{\sigma}_b \cdot \varepsilon_b - \frac{1}{2}\boldsymbol{\sigma}_b \cdot \mathbf{C}^{-1}\boldsymbol{\sigma}_b \right\} dV$$

being $\mathbf{C}$ bilinear compliance operator assumed coincident with linear theory

Advantages

- Biot's stress/strain are the most suitable for reusing linear solution in non linear context
- Mixed format guaranty less nonlinear description (see Casciaro, Garcea)

Continuum kinematics

- Using a material description the deformation gradient of a point initially in $\mathbf{X}$ and with displacement $\mathbf{u}[\mathbf{X}]$ is:

$$\mathbf{F}[\mathbf{X}] \equiv \nabla \mathbf{x} = \mathbf{I} + \nabla \mathbf{u}$$

- From the decomposition theorem we have $\mathbf{F} = \mathbf{R}\mathbf{U}$ with $\mathbf{R}$ a rotation and $\mathbf{U} = \mathbf{U}^T$ a, small pure stretch $\|\mathbf{U} - \mathbf{I}\| \ll 1$.
- Material strain is unaffected by $\mathbf{R}$

Green-Lagrange strain

$$\varepsilon_g = \frac{1}{2}\left(\mathbf{U}^2 - \mathbf{I}\right) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})$$

Biot's strain

$$\varepsilon_b = \mathbf{U} - \mathbf{I} = \mathbf{R}^T\mathbf{F} - \mathbf{I}$$

- ε_g is related to $\mathbf{F}$ and $\nabla \mathbf{u}$ by a simple quadratic expression.
- $\varepsilon_b \approx \mathbf{E} \equiv \mathrm{sym}(\nabla \mathbf{u})$ when $\mathbf{R} \approx \mathbf{I}$.
- It is possible to make $\mathbf{R} \approx \mathbf{I}$ with an appropriate change of observer

Change of observer

Denoting with a bar quantities related to a new observer rotated of $\mathbf{Q} = \{\mathbf{i}_1, \mathbf{i}_2, \mathbf{i}_3\}$

- Kinematic quantities transform as

$$\nabla\bar{\mathbf{u}} := \mathbf{Q}^T\mathbf{F} - \mathbf{I} = \bar{\mathbf{R}}\mathbf{U} - \mathbf{I} \quad , \quad \bar{\mathbf{R}} = \mathbf{Q}^T\mathbf{R} \approx \mathbf{I} \quad , \quad \nabla\bar{\mathbf{u}} = \bar{\mathbf{E}} + \bar{\mathbf{W}}$$

$$\bar{\mathbf{E}} = \mathrm{sym}[\nabla\bar{\mathbf{u}}] \quad , \quad \bar{\mathbf{W}} = \mathrm{skew}[\nabla\bar{\mathbf{u}}]$$

- A corotational observer with $\mathbf{Q} = \mathbf{R}$ eliminate the rigid part of $\mathbf{F}$ exactly and $\bar{\mathbf{E}}$ become the Biot's strain ε_B:

$$\nabla\bar{\mathbf{u}} = \mathbf{U} - \mathbf{I} = \nabla\bar{\mathbf{u}}^T \quad \Longrightarrow \quad \varepsilon_b = \bar{\mathbf{E}}$$

- Material stress doesn't change under a change of observer $\bar{\boldsymbol{\sigma}} = \boldsymbol{\sigma}$.

- Biot's strain, assuming $\bar{\mathbf{R}} \approx \mathbf{I} + \bar{\mathbf{W}} + \cdots$, becomes

$$\varepsilon_b \approx \bar{\mathbf{E}} + \frac{1}{2}(\bar{\mathbf{E}}\bar{\mathbf{W}} - \bar{\mathbf{W}}\bar{\mathbf{E}} - \bar{\mathbf{W}}\bar{\mathbf{W}})$$

Reuse linear solution

- Recovering $\bar{\boldsymbol{\sigma}}_L[\mathbf{t}]$, $\bar{\mathbf{E}}_L[\bar{\mathbf{d}}]$ and $\bar{\mathbf{W}}_L[\bar{\mathbf{d}}]$ from linear theory, ICM assumes:

$$\boldsymbol{\sigma}_b \equiv \bar{\boldsymbol{\sigma}}_L[\mathbf{t}] \quad , \quad \bar{\mathbf{E}} \equiv \bar{\mathbf{E}}_L[\bar{\mathbf{d}}] \quad , \quad \bar{\mathbf{W}} \equiv \bar{\mathbf{W}}_L[\bar{\mathbf{d}}]$$

- In the frame of a Galerkin approximation, mixed energy becomes:

$$\Phi[\mathbf{t}, \mathbf{e}] := \int_s \left\{ \mathbf{t}^T \mathbf{e}[\bar{\mathbf{d}}] - \frac{1}{2} \mathbf{t}^T \mathbf{K}^{-1} \mathbf{t} \right\} ds$$

- Relationships between parameters $\bar{\mathbf{d}}$ in CR frame and $\mathbf{d}$ in global frame complete the kinematics. When $\bar{\mathbf{d}}$ collects displacements and rotations derivatives $\bar{\mathbf{d}} = \{\bar{\mathbf{u}}_{,s}\, , \, \bar{\mathbf{R}}_{,s}\}$, corotational relationships give

$$\bar{\mathbf{u}}_{,s} = \mathbf{Q}^T(\mathbf{e}_1 + \mathbf{u}_{,s}) - \mathbf{e}_1 \quad , \quad \bar{\mathbf{R}}_{,s} = \mathbf{Q}^T \mathbf{R}_{,s}$$

- CR frame defined by $\mathbf{Q}$ is choice so that $\bar{\mathbf{W}} \ll 1$

Planar beam with rectangular cross section

Linear solution for a rectangular section of area A and inertia J

- The nonzero components of $\bar{\boldsymbol{\sigma}}_L$ in terms of axial, shear and flexural action N, T, M are:

$$\bar{\sigma} = \frac{N}{A} - \frac{M}{J} y \; , \; \bar{\tau} = \frac{T}{A} \psi_{,y} [y] \quad \text{with} \int_S \psi \, dA = 0 \; , \; \int_S \psi_{,y} \; dA = A$$

- Kinematical solution gives $\nabla \bar{\mathbf{u}}_L = \bar{\mathbf{E}}_L + \bar{\mathbf{W}}_L$ with

$$\bar{\mathbf{E}}_L = \begin{bmatrix} \bar{\varepsilon} - \bar{\chi} y & \frac{\psi_{,y}}{2k} \bar{\gamma} \\ \text{sym.} & 0 \end{bmatrix} \qquad \bar{\mathbf{W}}_L = \begin{bmatrix} 0 & \bar{\alpha} - (1 - \frac{\psi_{,y}}{2k}) \bar{\gamma} \\ \text{skew} & 0 \end{bmatrix}$$

  $\bar{\alpha}$ defines the orientation of the local frame ($\bar{\alpha} = 0$ aligned with the section, $\bar{\alpha} = \bar{\gamma}$ with the axis) and can be selected to minimize $\bar{\mathbf{W}}_L$. Other quantities are

$$\bar{\varepsilon} = \bar{u}_{,s} \equiv \int_A \frac{\bar{u}_{x,s}}{A} dA, \quad \bar{\gamma} - \bar{\alpha} = \bar{v}_{,s} \equiv \int_A \frac{\bar{u}_{y,s}}{A} dA, \quad \bar{\chi} = \bar{\varphi}_{,s} = - \int_A \frac{y \, \bar{u}_x}{J} dA$$

Planar beam with rectangular cross section

Local model

- The strain energy performing the integration is

$$\Phi[\mathbf{t},\,\mathbf{e}] := (N\,\varepsilon + T\,\gamma + M\,\chi) - \frac{1}{2}\left(\frac{N^2}{EA} + \frac{k\,T^2}{G\,A} + \frac{M^2}{EJ}\right)$$

- Strains are

$$\bar{\alpha} = \bar{\gamma}/2\,,\ \begin{cases} \varepsilon := \bar{\varepsilon} + \dfrac{1}{48}\bar{\gamma}^2 \\ \gamma := \bar{\gamma} \end{cases} \quad , \quad \bar{\alpha} = 0\,,\ \begin{cases} \varepsilon := \bar{\varepsilon} + \dfrac{19}{48}\bar{\gamma}^2 \\ \gamma := \bar{\gamma} - \dfrac{1}{2}\,\bar{\varepsilon}\bar{\gamma} \end{cases}$$

- The elastic laws are $N = EA\,\varepsilon\ ,\ T = kGA\,\gamma\ ,\ M = EJ\,\chi$

Corotational kinematics

$$\bar{\varepsilon} = \bar{u}_{,s} = (1 + u_{,s}\,)\cos\alpha + v_{,s}\sin\alpha - 1 \quad , \quad \bar{\gamma} = \bar{v}_{,s} = -(1 + u_{,s}\,)\sin\alpha + v_{,s}\cos\alpha$$

corotational frame is globally defined by $\alpha = \varphi + \bar{\alpha}$.

Planar beam with rectangular cross section

Remarks

- Stress resultants $\bar{F}_x$ and $\bar{F}_y$ don't coincide exactly with N and T

$$\bar{F}_x = N \quad , \quad \bar{F}_y = T + \bar{\gamma} N/24 \quad , \quad \bar{M} = M$$

- Assuming the warping constant both models give

$$\begin{bmatrix} \varepsilon \\ \gamma \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} 1 + u_{,s} \\ v_{,s} \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \chi = \varphi_{,s}$$

  so obtaining the Antman measures when also $\alpha = \varphi$.

- Assuming $\bar{\gamma} := 0$ we obtain

$$\varepsilon = \sqrt{(1 + u_{,s})^2 + v_{,s}^2} - 1 \quad , \quad \chi = \frac{v_{,ss} + v_{,ss}\ u_{,s} - v_{,s}\ u_{,ss}}{1 + \varepsilon}$$

- All previous differences disappear when $\gamma \ll 1$.

Spatial beam with cross-shape section

Linear solution for a cross-shape section of area A and torsial inertia J

- The nonzero components of $\bar{\boldsymbol{\sigma}}_L$ in terms of axial and torsional action N, M are:

$$\bar{\sigma}_{xx} = \frac{N}{A}\,, \quad \bar{\sigma}_{xy} = (2\,\psi[y,z]_{,y} - z)\,\frac{M}{J_t}\,, \quad \bar{\sigma}_{xz} = (2\,\psi[y,z]_{,z} + y)\,\frac{M}{J_t}$$

- Kinematical solution gives $\nabla \bar{\mathbf{u}}_L = \bar{\mathbf{E}}_L + \bar{\mathbf{W}}_L$ with

$$\bar{\mathbf{E}}_L = \begin{bmatrix} \bar{\varepsilon} & (\psi_{,y} - \frac{1}{2}\;z)\bar{\chi} & (\psi_{,z} + \frac{1}{2}\;y)\bar{\chi} \\ (\psi_{,y} - \frac{1}{2}\;z)\bar{\chi} & 0 & 0 \\ (\psi_{,z} - \frac{1}{2}\;y)\bar{\chi} & 0 & 0 \end{bmatrix}$$

$$\bar{\mathbf{W}}_L = \begin{bmatrix} 0 & (\frac{1}{2}\;z + \psi_{,y}\,)\bar{\chi} & -(\frac{1}{2}\;y - \psi_{,z}\,)\bar{\chi} \\ -(\frac{1}{2}\;z + \psi_{,y}\,)\bar{\chi} & 0 & 0 \\ (\frac{1}{2}\;y - \psi_{,z}\,)\bar{\chi} & 0 & 0 \end{bmatrix}$$

with

$$\bar{\varepsilon} = \bar{u}_{,s} \quad , \quad \bar{\chi} = \bar{\varphi}_{,s}$$

Spatial beam with cross-shape section

Local model

- The strain energy performing the integration is

$$\Phi[\mathbf{t}, \mathbf{e}] := (N\,\varepsilon + M\,\chi) - \frac{1}{2}\left(\frac{N^2}{EA} + \frac{M^2}{GJ_t}\right)$$

- Strains are defined as

$$\varepsilon := \bar{\varepsilon} + \overbrace{\frac{1}{2}\,\frac{J_t\,\bar{\chi}^2}{A}}^{\text{Wagner's effect}} \quad , \quad \chi := \bar{\chi}$$

- Constitutive laws using Clapeyron equivalence is

$$N = EA\varepsilon \quad , \quad M = GJ_t\chi$$

Saint Venánt's rod theory

Stress

Considering a generic section $S[s]$ along the beam axis, and introducing the vectors collecting the strengths:

$$\mathbf{t}_\sigma := \{N_x,\ M_y,\ M_z\}^T \ , \quad \mathbf{t}_\tau := \{M_x,\ T_y,\ T_z\}^T$$

SV stress:

$$\bar{\boldsymbol{\sigma}}_L := \begin{bmatrix} \sigma & \boldsymbol{\tau}^T \\ \boldsymbol{\tau} & \mathbf{0} \end{bmatrix} \ , \quad \begin{cases} \sigma := \mathbf{D}_\sigma \mathbf{t}_\sigma \\ \boldsymbol{\tau} := \mathbf{D}_\tau \mathbf{t}_\tau \end{cases} \ , \quad (\nu = 0)$$

being

$$\mathbf{D}_\sigma := \left[1/A, \quad z/J_y, \quad -y/J_z\right] \ , \quad \mathbf{D}_\tau := \mathbf{D}_\tau[\psi]$$

Stress function ψ is evaluated numerically following:
A.S. Petrolo, R. Casciaro, '3D beam element based on Saint Venánt's rod theory', C & S, **82**, pp. 2471-2481, 2004.

Saint Venánt's rod theory

Kinematics

$$\bar{\mathbf{E}}_L := \mathrm{sym}[\nabla \bar{\mathbf{u}}_L] = \begin{bmatrix} \bar{\varepsilon} & \frac{1}{2}\left(\bar{\gamma} + \nabla w\right)^T \\ \frac{1}{2}\left(\bar{\gamma} + \nabla w\right) & \mathbf{0} \end{bmatrix}$$

and

$$\bar{\mathbf{W}}_L := \mathrm{skew}[\nabla \bar{\mathbf{u}}_L] = \begin{bmatrix} 0 & -\frac{1}{2}\left(\bar{\gamma} - \nabla w\right)^T \\ \frac{1}{2}\left(\bar{\gamma} - \nabla w\right) & \mathbf{0} \end{bmatrix}$$

being

$$\begin{cases} \bar{\varepsilon} := \mathbf{A}_\epsilon \bar{\boldsymbol{\varepsilon}}_\sigma \\ \bar{\gamma} := \mathbf{A}_\gamma \bar{\boldsymbol{\varepsilon}}_\tau \\ \nabla w := \mathbf{A}_w \bar{\boldsymbol{\varepsilon}}_\tau \end{cases}, \quad \boldsymbol{\varepsilon}_\sigma := \begin{Bmatrix} \bar{\varepsilon}_x \\ \bar{\chi}_y \\ \bar{\chi}_z \end{Bmatrix}, \quad \boldsymbol{\varepsilon}_\tau := \begin{Bmatrix} \bar{\chi}_x \\ \bar{\gamma}_y \\ \bar{\gamma}_z \end{Bmatrix}$$

with

$$\mathbf{A}_\epsilon := \begin{bmatrix} 1 & z & -y \end{bmatrix}, \quad \mathbf{A}_\gamma := \begin{bmatrix} -z & 1 & 0 \\ y & 0 & 1 \end{bmatrix}, \quad \mathbf{A}_w := \mathbf{A}_w[\psi]$$

Mixed Energy

- Strain energy in mixed form:

$$\Phi[\mathbf{t},\,\mathbf{e}] := \int_0^l \mathbf{t}^T \overbrace{\{\bar{\mathbf{e}} + \frac{1}{2}\boldsymbol{\Psi}[\bar{\mathbf{e}}]\bar{\mathbf{e}}\}}^{\mathbf{e}} - \frac{1}{2}\,\{\mathbf{t}^T\mathbf{K}^{-1}\mathbf{t}\}\,ds\,, \quad \begin{cases} \mathbf{t} := \{\mathbf{n},\,\mathbf{m}\}^T \\ \bar{\mathbf{e}} := \{\bar{\boldsymbol{\varepsilon}},\,\bar{\boldsymbol{\chi}}\}^T \end{cases}$$

- strengths and deformations :

$$\begin{cases} \mathbf{n} := \{N_x,\,T_y,\,T_z\}^T \\ \mathbf{m} := \{M_x,\,M_y,\,M_z\}^T \end{cases}, \quad \begin{cases} \bar{\boldsymbol{\varepsilon}} := \{\bar{\varepsilon}_x,\bar{\gamma}_y,\bar{\gamma}_z\}^T \\ \bar{\boldsymbol{\chi}} := \{\bar{\chi}_x,\bar{\chi}_y,\bar{\chi}_z\}^T \end{cases}$$

Corotational kinematics

- Compatibility equations

$$\bar{\boldsymbol{\varepsilon}} := \bar{\mathbf{u}}_{,x}\ ,\quad \bar{\boldsymbol{\chi}} := \bar{\boldsymbol{\varphi}}_{,x}$$

- Corotational change ($\bar{\mathbf{u}} \to \mathbf{u}$ and $\bar{\boldsymbol{\varphi}} \to \boldsymbol{\varphi}$) assuming $\mathbf{Q} \equiv \mathbf{R}[\boldsymbol{\varphi}]$

$$\bar{\mathbf{u}}_{,x} = \mathbf{R}[\boldsymbol{\varphi}]^T(\mathbf{u}_{,x} + \mathbf{e}_1) - \mathbf{e}_1\ ,\quad \mathrm{spin}[\bar{\boldsymbol{\varphi}}_{,x}] = \mathbf{R}[\boldsymbol{\varphi}]^T\mathbf{R}_{,x}[\boldsymbol{\varphi}]$$

FEM implementation - stress

Natural stress

$$\mathbf{n}_e := \frac{1}{l}\int_0^l \mathbf{n}\, ds\ , \quad \mathbf{m}_s := -(\mathbf{m}_j + \mathbf{m}_i)\ , \quad \mathbf{m}_e := (\mathbf{m}_j - \mathbf{m}_i)$$

Interpolation law

The axial/shear strengths and couples are assumed constant and linear on the element:

$$\mathbf{t} = \mathbf{D}_t^T\,\mathbf{t}_e\ , \quad \mathbf{D}_t = \begin{Bmatrix} \mathbf{D}_n \\ \mathbf{D}_m \end{Bmatrix}\ , \quad \mathbf{D}_n = \begin{bmatrix} \frac{1}{l}\mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \mathbf{D}_m = \begin{bmatrix} \mathbf{0} \\ -\frac{1}{2}\,\mathbf{I} \\ -\frac{1}{2}\,f_s\,\mathbf{I} \end{bmatrix}$$

being $\mathbf{t}_e = \{\mathbf{n}_e, \mathbf{m}_s, \mathbf{m}_e\}$ and $f_s = 1 - \frac{2s}{l}$.

FEM implementation - strain

Natural modes

$$\phi_m := \varphi|_{s=l/2}\ , \quad \phi_s := \frac{\varphi_i - \varphi_j}{2}\ , \quad \phi_e := \frac{\varphi_i + \varphi_j}{2}\ , \quad \phi_r := \frac{\mathbf{u}_j - \mathbf{u}_i}{l}$$

Interpolation law

Displacements and rotations vectors field are assumed to be linear and quadratic on the element:

$$\mathbf{d} = \mathbf{D}_d^T\,\mathbf{d}_e\ , \quad \mathbf{D}_d = \begin{Bmatrix} \mathbf{D}_u \\ \mathbf{D}_\varphi \end{Bmatrix}\ , \quad \mathbf{D}_u = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \\ \frac{1}{l}\mathbf{I} \end{bmatrix}\ , \mathbf{D}_\varphi = \begin{bmatrix} (1-f_e)\mathbf{I} \\ f_s\mathbf{I} \\ f_e\mathbf{I} \\ \mathbf{0} \end{bmatrix}$$

being $\mathbf{d}_e = \{\phi_m, \phi_s, \phi_e, \phi_r\}$ and $f_e = 1 - \frac{4\,s}{l} + \frac{4\,s^2}{l^2}$.

FEM implementation - strain energy

Discrete mixed energy

- Substituting interpolation laws into mixed energy

$$\Phi[\mathbf{t}_e, \mathbf{d}_e] = \mathbf{t}_e^T \varrho_e[\mathbf{d}_e] - \frac{1}{2}\,\mathbf{t}_e^T \mathbf{K}_e^{-1} \mathbf{t}_e$$

being

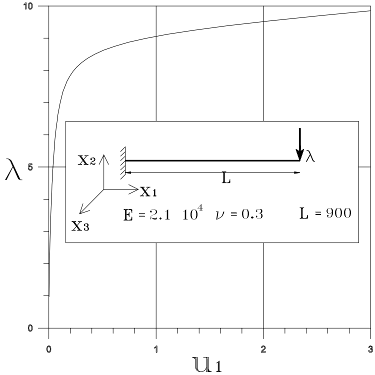
$$\varrho_e[\mathbf{d}_e] = \int_0^l \mathbf{D}_t^T \{\bar{\mathbf{e}}[\mathbf{d}_e] + \frac{1}{2}\,\boldsymbol{\Psi}[\mathbf{d}_e]\mathbf{d}_e\}\, ds\,, \quad \mathbf{K}_e^{-1} = \int_0^l \{\mathbf{D}_t^T \mathbf{K}^{-1} \mathbf{D}_t\}\, ds$$

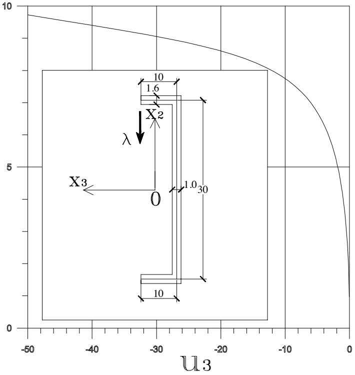
Energy derivatives

- Fréchet derivatives of discrete deformations $\varrho_e[\mathbf{d}_e]$ are required.
- Second and fourth order derivatives for Riks and Koiter analysis respectively.
- Algebra needs manipulation software (Maple, Mathematica $\cdots$)
- Asymptotic postbuckling FEM analysis using a corotational formulation, G. Garcea, A. Madeo, G. Zagari, R. Casciaro, IJSS

Numerical validation

Cantilever beam under shear force

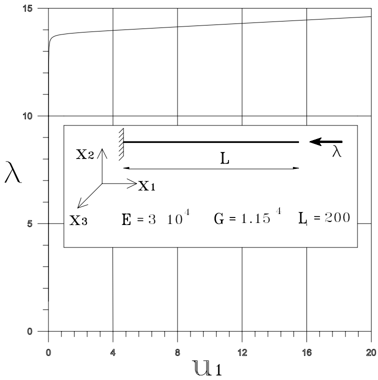


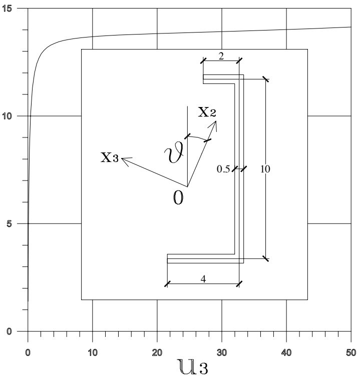


- F. Gruttmann, R. Sauer, W. Wagner, 'A geometrical nonlinear eccentric 3D-beam element with arbitrary cross sections', CMAME, 160, pp. 383-400, (1998).

Numerical validation

Compressed cantilever beam





- H.H. Chen, W.Y. Lin, K.M. Hsiao, 'Co-rotational finite element formulation for thin-walled beams with generic open section', CMAME, 195, pp. 2334-2370, (2006).

Nonlinear plate model

- The strains are collected in vectors $\bar{\epsilon} = [\bar{u}_{,1}\,, \bar{v}_{,2}\,, \bar{v}_{,1} + \bar{u}_{,2}\,]^T$, $\bar{\gamma} = [2\,\bar{w}_{,1}\,, 2\bar{w}_{,2}\,]^T$, $\bar{\chi} = [-\bar{\varphi}_{2,1}\,, \bar{\varphi}_{1,2}\,, 2(\bar{\varphi}_{1,1} - \bar{\varphi}_{2,2}\,)]^T$, being:

$$\bar{\epsilon} = \begin{bmatrix} \mathbf{i}_1^T \mathbf{a}_1 - 1 \\ \mathbf{i}_2^T \mathbf{a}_2 - 1 \\ \mathbf{i}_1^T \mathbf{a}_2 + \mathbf{i}_2^T \mathbf{a}_1 \end{bmatrix} \qquad \bar{\gamma} = \begin{bmatrix} \mathbf{i}_3^T \mathbf{a}_1 \\ \mathbf{i}_3^T \mathbf{a}_2 \end{bmatrix} \qquad \bar{\chi} = \begin{bmatrix} \mathbf{i}_3^T \mathbf{i}_{1,1} \\ \mathbf{i}_3^T \mathbf{i}_{2,2} \\ \mathbf{i}_3^T \mathbf{i}_{1,2} + \mathbf{i}_3^T \mathbf{i}_{2,1} \end{bmatrix}$$

  and $\mathbf{a}_1 = [1 + u_{,1}\,,\ v_{,1}\,,\ w_{,1}\,]^T$ and $\mathbf{a}_2 = [u_{,2}\,,\ 1 + v_{,2}\,,\ w_{,2}\,]^T$.

- Shear undeformable plate model, can be obtained assuming $\bar{\gamma} = \mathbf{0}$, i.e. $\mathbf{i}_1$ and $\mathbf{i}_2$ contained in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$.
- With same algebra and using the small strain hypothesis it is possible to show that:

$$\mathbf{i}_3^T \mathbf{i}_{i,j} \approx (\mathbf{a}_1 \wedge \mathbf{a}_2)^T \mathbf{a}_{i,j} \quad \text{with} \quad i, j = 1, 2.$$

$$\bar{\epsilon} = \begin{bmatrix} \mathbf{a}_1^T \mathbf{a}_1 - 1 \\ \mathbf{a}_2^T \mathbf{a}_2 - 1 \\ \mathbf{a}_1^T \mathbf{a}_2 + \mathbf{a}_2^T \mathbf{a}_1 \end{bmatrix} \quad , \quad \bar{\chi} = \begin{bmatrix} (\mathbf{a}_1 \wedge \mathbf{a}_2)^T \mathbf{a}_{1,1} \\ (\mathbf{a}_1 \wedge \mathbf{a}_2)^T \mathbf{a}_{2,2} \\ (\mathbf{a}_1 \wedge \mathbf{a}_2)^T (\mathbf{a}_{1,2} + \mathbf{a}_{2,1}\,) \end{bmatrix}$$

Nonlinear plate model

$$\bar{\boldsymbol{\epsilon}} = \left\{ \begin{array}{c} u_{,1} + \frac{1}{2}\left(u_{,1}^2 + v_{,1}^2 + w_{,1}^2\right) \\ v_{,2} + \frac{1}{2}\left(u_{,2}^2 + v_{,2}^2 + w_{,2}^2\right) \\ u_{,2} + v_{,1} + \left(u_{,1}\, u_{,2} + v_{,1}\, v_{,2} + w_{,1}\, w_{,2}\right) \end{array} \right\}$$

$$\bar{\boldsymbol{\chi}} = \left\{ \begin{array}{c} w_{,11} + \left(w_{,11}\, u_{,1} - u_{,11}\, w_{,1} - v_{,11}\, w_{,2} + w_{,11}\, v_{,2}\right) \\ + \left(u_{,2}\, v_{,11}\, w_{,1} + u_{,1}\, w_{,11}\, v_{,2} - v_{,2}\, u_{,11}\, w_{,1} \right. \\ \left. -\; u_{,1}\, v_{,11}\, w_{,2} - u_{,2}\, w_{,11}\, v_{,1} + v_{,1}\, u_{,11}\, w_{,2}\right) \\ \\ w_{,22} + \left(w_{,22}\, u_{,1} - u_{,22}\, w_{,1} - v_{,22}\, w_{,2} + w_{,22}\, v_{,2}\right) \\ + \left(u_{,2}\, v_{,22}\, w_{,1} + u_{,1}\, w_{,22}\, v_{,2} - v_{,2}\, u_{,22}\, w_{,1} \right. \\ \left. -\; u_{,1}\, v_{,22}\, w_{,2} - u_{,2}\, w_{,22}\, v_{,1} + v_{,1}\, u_{,22}\, w_{,2}\right) \\ \\ w_{,12} + \left(w_{,12}\, u_{,1} - u_{,12}\, w_{,1} - v_{,12}\, w_{,2} + w_{,12}\, v_{,2}\right) \\ + \left(u_{,2}\, v_{,12}\, w_{,1} + u_{,1}\, w_{,12}\, v_{,2} - v_{,2}\, u_{,12}\, w_{,1} \right. \\ \left. -\; u_{,1}\, v_{,12}\, w_{,2} - u_{,2}\, w_{,12}\, v_{,1} + v_{,1}\, u_{,12}\, w_{,2}\right) \end{array} \right\}$$

Nonlinear plate model

Technical plate models

- Green Lagrangian

$$\bar{\boldsymbol{\epsilon}} = \left\{ \begin{array}{c} u_{,1} + \frac{1}{2}\left(u_{,1}^2 + v_{,1}^2 + w_{,1}^2 \right) \\ v_{,2} + \frac{1}{2}\left(u_{,2}^2 + v_{,2}^2 + w_{,2}^2 \right) \\ u_{,2} + v_{,1} + \left(u_{,1}\, u_{,2} + v_{,1}\, v_{,2} + w_{,1}\, w_{,2} \right) \end{array} \right\} \quad \bar{\boldsymbol{\chi}} = \left\{ \begin{array}{c} w_{,11} \\ w_{,22} \\ w_{,12} \end{array} \right\}$$

- Simplified Green Lagrangian

$$\bar{\boldsymbol{\epsilon}} = \left\{ \begin{array}{c} u_{,1} + \frac{1}{2}\left(v_{,1}^2 + w_{,1}^2 \right) \\ v_{,2} + \frac{1}{2}\left(v_{,2}^2 + w_{,2}^2 \right) \\ u_{,2} + v_{,1} + w_{,1}\, w_{,2} \end{array} \right\} \quad \bar{\boldsymbol{\chi}} = \left\{ \begin{array}{c} w_{,11} \\ w_{,22} \\ w_{,12} \end{array} \right\}$$

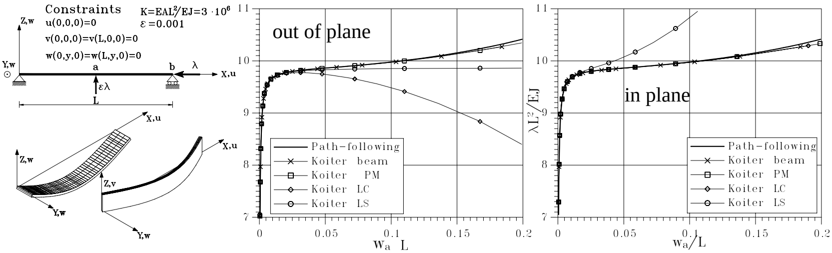
Numerical validation

- The numerical results refer to Koiter's asymptotic analysis of plate assemblages using Von–Karman plate model (*PM*).
- The results are compared with those obtained by the same code using technical models namely: *complete Green–Lagrange* (*LC*) and *simplified Green–Lagrange* (*LS*), previously used in KASP code; with analytical solutions and with path–following analysis.
- Further details regarding the model, the finite element formulation, and the asymptotic Koiter formulation can be found in

1. A.D.Lanzo, G.Garcea, Koiter's analysis of thin-walled structures by a finite element approach. *Int.J.Numer.Meth.Eng.*, **39**, 3007-3031, (1996).
2. G. Garcea, Mixed formulation in Koiter analysis of thin–walled beam. *Comp. Meth. in Appl. Mech. and Eng.*, **190**, 3369-3399. (2001).
3. G. Garcea , G.A. Trunfio, R. Casciaro, 'Path-following analysis of thin-walled structures and comparison with asymptotic post-critical solutions', *Int.J.Num.Meth.Eng.*, **55**, 73-100. (2002).

Numerical validation

Euler beam

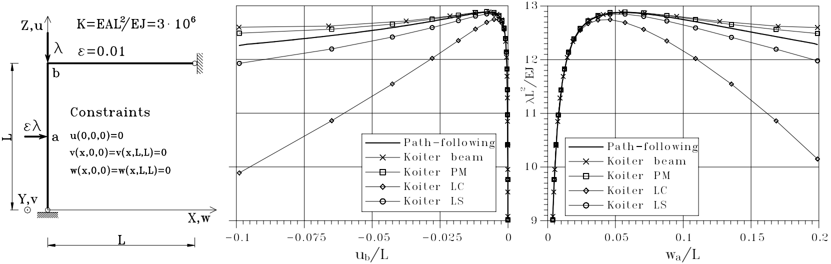


		Out plane			In plane			
	N.elem.	LC	LS	PM	LC	LS	PM	2D Beam(*)
λ_b	16	9.901	9.901	9.901	9.918	9.918	9.918	
	32	9.877	9.877	9.877	9.870	9.870	9.870	9.870
	64	9.872	9.872	9.871	9.867	9.870	9.870	
$\frac{\ddot{\lambda}_b}{2\lambda_b}$	16	-0.354	0.020	0.145	0.166	1.03	0.166	
	32	-0.375	0.000	0.125	0.126	1.00	0.126	0.125
	64	-0.375	0.000	0.125	0.125	1.00	0.125	

(*) Solution obtained by Antman model and exact interpolation functions

Numerical validation

Roorda frame

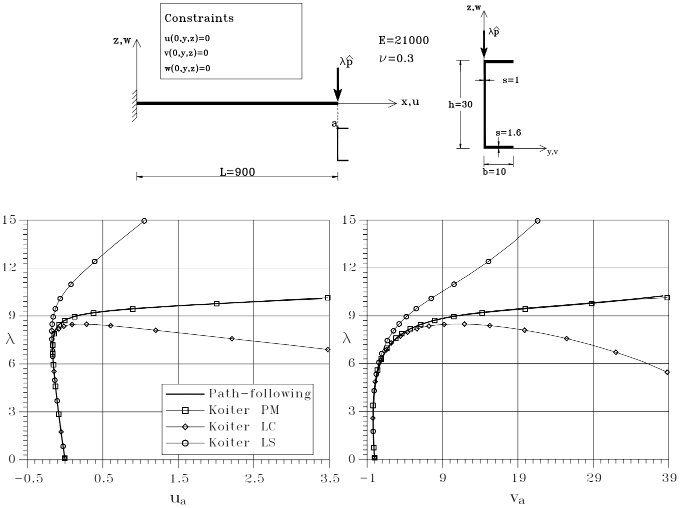


	elem.n°	LC	LS	PM	2D Beam(*)
λ_b	16	13.954	13.954	13.954	
	32	13.903	13.903	13.903	13.886
	64	13.890	13.890	13.890	
$\frac{\dot{\lambda}_b}{\lambda_b}$	16	0.3815	0.3815	0.3815	
	32	0.3808	0.3808	0.3807	0.3805
	64	0.3806	0.3806	0.3805	
$\ddot{\lambda}_b/2\lambda_b$	16	-0.6421	0.2178	0.4535	
	32	-0.7165	0.1434	0.3797	0.3787
	64	-0.7176	0.1422	0.3785	

(*)Solution obtained by Antman model and exact interpolation functions

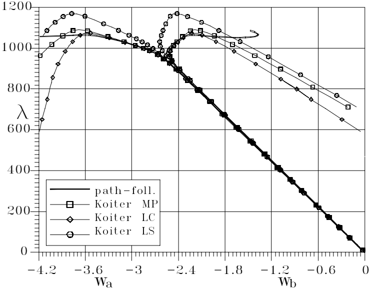
Numerical validation

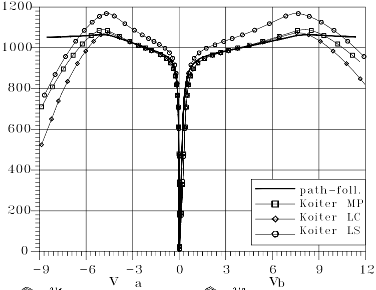
C beam

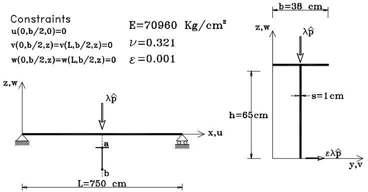


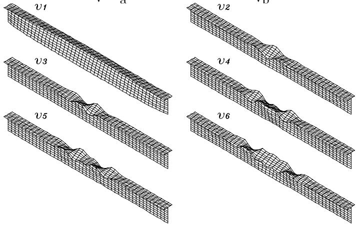
Numerical validation

T beam









Remarks

- Reuse in nonlinear context classical linear solution (Saint Venánt).
- Recover in automatic way a nonlinear beam model able to predict effect due to three dimensional behavior (Wagner effect).
- Improve standard beam models (Reissner-Simo).
- FEM implementation, although complex algebra is involved, is noticeable simplified by algebraic software.
- Generalization to other linear models is quite easy. For Vlasov's beam and plates the nonlinear models are already recovered.

References

- G. Garcea, A. Madeo, R. Casciaro, 'The implicit corotational method', submitted to CMAME.

Corotational formulation

CR idea

- FE motion is decomposed into a rigid and a relative deformation using a local CR frame that rotates and translates with the element.
- Assuming small the relative deformation it is possible to evaluate the strains using linear or simplified kinematical relationship.
- Strain energy is objective with respect to the element rigid body motion.

CR advantages

- The nonlinearity is transferred to the geometrically exact transformation between a fixed and a local corotational systems.
- With an appropriate selection of the CR frame, the accuracy increases with the refinement of the finite element mesh.
- Linear finite element library can be reused thereby avoiding the use of objective interpolation.

Strain energy

Linear model in CR frame

The strain will be a function of the small deformational displacement and rotation $\varepsilon = \varepsilon[\bar{\mathbf{d}}]$: we can use a *linear strain* relationship. Taking linear constitutive laws, we have:

$$\Phi_e[u] := \int_{\Omega_e} \left\{ \boldsymbol{\sigma} \cdot \varepsilon[\bar{\mathbf{d}}] - \frac{1}{2} \boldsymbol{\sigma} \cdot \mathbf{E}^{-1} \boldsymbol{\sigma} \right\} d\Omega_e$$

$\boldsymbol{\sigma} = \mathbf{E}\varepsilon$ is the stress, Ω_e the finite element domain.

FEM linear model in CR frame

Exploiting the element interpolation laws:

$$\Phi_e[\bar{\mathbf{u}}_e] = \mathbf{t}_e^T \varrho[\bar{\mathbf{d}}_e] - \frac{1}{2} \mathbf{t}_e^T \mathbf{K}_c^{-1} \mathbf{t}_e \quad \bar{\mathbf{u}}_e = \{\mathbf{t}_e, \bar{\mathbf{d}}_e\}^T$$

$\mathbf{t}_e$ being the vector of the element stress parameters and $\varrho = \mathbf{D}\bar{\mathbf{d}}_e$ the vector of the strains, linear function of $\bar{\mathbf{d}}_e$, $\mathbf{K}_c^{-1}$ is the Clapeyron compliance matrix.

Remarks

The CR formulation is based on two fundamental steps:

Geometric nonlinear relationship

Definition of the relation $\bar{\mathbf{d}}_e = \mathbf{g}[\mathbf{d}_e]$ between the element vector in the corotational $(\bar{\mathbf{d}}_e)$ and fixed $(\mathbf{d}_e)$ frames.

- The geometrical nonlinearities are essentially contained in this step.

Local modeling

Definition of the element strain energy in a simplified form in terms of CR finite element parameters.

- If using a linear strain model, this corresponds to a linear FEM modeling.
- We can reuse of the standard FEM technology in a nonlinear context.

modeling

3D rotations: main items

Parametrization

- Euler angles

$$\mathbf{R} = \mathbf{R}_1[\alpha]\mathbf{R}_2[\beta]\mathbf{R}_3[\gamma]$$

- Axial vector $\boldsymbol{\theta}$

$$\mathbf{R}[\boldsymbol{\theta}] = a_0[\theta]\mathbf{I} + a_1[\theta]\mathbf{W}[\boldsymbol{\theta}] + a_2[\theta]\mathbf{W}^2[\boldsymbol{\theta}]$$

with $\mathbf{W}[\boldsymbol{\theta}] = \text{spin}[\boldsymbol{\theta}]$, $\theta = \boldsymbol{\theta}^T\boldsymbol{\theta}$ and $a_i[\theta]$ scalar functions. In case of Rodriguez parametrization

$$a_0[\theta] = 1 \quad , \quad a_1[\theta] = \frac{\sin\theta}{\theta} \quad , \quad a_2[\theta] = \frac{1-\cos\theta}{\theta^2}$$

we can also express $\mathbf{R}[\boldsymbol{\theta}]$ as exponential map

$$\mathbf{R}[\boldsymbol{\theta}] = \exp[\boldsymbol{\theta}] = \sum_{0}^{\infty} \frac{1}{n!}\mathbf{W}^n[\boldsymbol{\theta}]$$

3D rotations: main items

Differentiation

- Multiplicative. Let be $\delta\boldsymbol{\Omega}$ an $\delta\boldsymbol{\Psi}$ spin axial vector's

$$\delta\mathbf{R} = \underbrace{\mathbf{R}\frac{d}{d\epsilon}\exp[\epsilon\delta\boldsymbol{\Omega}]}_{material} = \mathbf{R}\delta\boldsymbol{\Omega} \quad , \quad \delta\mathbf{R} = \underbrace{\frac{d}{d\epsilon}\exp[\epsilon\delta\boldsymbol{\Psi}]\mathbf{R}}_{spatial} = \delta\boldsymbol{\Psi}\mathbf{R}$$

- Additive

$$\delta\mathbf{R} = \frac{d}{d\epsilon}\mathbf{R}[\boldsymbol{\theta} + \epsilon\delta\boldsymbol{\theta}] = \mathbf{R}\,\mathrm{spin}\,[\mathbf{T}^T\delta\boldsymbol{\theta}]$$

with

$$\mathbf{T} = \mathbf{I} + \frac{1-\cos\theta}{\theta^2}\mathbf{W}[\theta] + \frac{\sin\theta - \theta}{\theta^3}\mathbf{W}[\theta]$$

Outcome

- Symmetry of tangent matrix
- Handing of incremental process in Rik's scheme
- Reliability of Koiter's analysis

Asymptotic analysis

- Alternative to path–following algorithms for analysis of nonlinear elastic structures.
- The equilibrium path is recovered in an approximate but synthetic fashion using an energy description.

Advantages

- Complete informations about the buckling and postbuckling behavior of the structure.
- Efficient imperfection sensitivity analysis in cases of multiple, nearly coincident, buckling loads.

Reliability

- The correct evaluation of strain energy variations require a geometrically exact models.
- Geometrically exact models are often too complex for FEM analysis or are unavailable.
- The corotational approach (CR) is a tool to obtain, starting from linear modeling, objective models.

Asymptotic analysis overview

Asymptotic expansion of the equilibrium path (perfect path)

$$\mathbf{u}[\lambda, \xi_1, \dots, \xi_m] = \lambda\hat{\mathbf{u}} + \sum_{i=1}^{m} \xi_i \dot{\mathbf{v}}_i + \frac{1}{2} \sum_{i,j=1}^{m} \xi_i \xi_j \ddot{\mathbf{w}}_{ij} + \cdots$$

1. Fundamental path $\hat{\mathbf{u}}$

$$\mathbf{K}_0 \hat{\mathbf{u}} = \hat{\mathbf{p}} \quad \text{with} \quad \mathbf{K}_0 = \mathbf{K}_t[0]$$

2. Bifurcation modes $\dot{\mathbf{v}}_i$

$$\mathbf{K}[\lambda]\mathbf{v} = \mathbf{0} \qquad \mathbf{K}[\lambda] = \mathbf{K}_t[\lambda\hat{\mathbf{u}}]$$

3. Secondary modes $\ddot{\mathbf{w}}_i$

$$\mathbf{K}_b \ddot{\mathbf{w}}_{ij} = \ddot{\mathbf{s}}[\mathbf{v}_i, \mathbf{v}_j] \quad \text{with}$$

with $\ddot{\mathbf{s}}$ defined by

$$\ddot{\mathbf{s}}[\mathbf{v}_i, \mathbf{v}_j]^T \delta\mathbf{w} = \Phi_b''' \dot{v}_i \dot{v}_j \delta w$$

Asymptotic analysis overview

Equilibrium path (imperfect path)

$$-\mu_k[\lambda] - (\lambda - \lambda_k)\xi_k + \frac{1}{2}\sum_{i,j=1}^{m} \xi_i\xi_j\mathcal{A}_{ijk} + \frac{1}{6}\sum_{i,j,h=1}^{m} \xi_i\xi_j\xi_h\mathcal{B}_{ijhk} = 0\ , \quad k = 1, \ldots, m$$

where

$$\mathcal{A}_{ijk} = \Phi_b''' \dot{v}_i\dot{v}_j\dot{v}_k \quad , \quad \mathcal{B}_{ijhk} = \Phi_b'''' \dot{v}_i\dot{v}_j\dot{v}_h\dot{v}_k - \Phi_b''(\ddot{w}_{ij}\ddot{w}_{hk} + \ddot{w}_{ih}\ddot{w}_{jk} + \ddot{w}_{ik}\ddot{w}_{jh})$$

and

$$\mu_k[\lambda] = \mu_k^i[\lambda] + \mu_k^g[\lambda] + \mu_k^l[\lambda]$$

$$\mu_k^i[\lambda] = -\frac{1}{2}\lambda^2\Phi_b'''\hat{u}\hat{u}\dot{v}_k$$

$$\mu_k^g[\lambda] = -\lambda\,\Phi_b'''\hat{u}\tilde{u}\dot{v}_k \quad , \quad \mu_k^l[\lambda] = \lambda\,\tilde{p}\,\dot{v}_k$$

Corotational derivatives

- Strain energy is expressed in mixed form to avoid locking

$$\Phi[\mathbf{u}] = \sum_e \Phi_e[\mathbf{u}_e]; \quad \Phi_e[\mathbf{u}_e] = \mathbf{t}_e^T \mathbf{D}\mathbf{g}[\mathbf{d}_e] - \frac{1}{2}\mathbf{t}_e^T \mathbf{K}_c^{-1}\mathbf{t}_e$$

- Strain energy variations, are written using a fourth order Taylor expansion of $\mathbf{g}[\mathbf{d}_e]$ starting from the reference configuration ($\mathbf{d}_e = \mathbf{0}$):

$$\begin{aligned}\mathbf{g}[\mathbf{d}_e] = {} & \mathbf{g}_1[\mathbf{d}_e] + \frac{1}{2}\mathbf{g}_2[\mathbf{d}_e, \mathbf{d}_e] \\ & + \frac{1}{6}\mathbf{g}_3[\mathbf{d}_e, \mathbf{d}_e, \mathbf{d}_e] + \frac{1}{24}\mathbf{g}_4[\mathbf{d}_e, \mathbf{d}_e, \mathbf{d}_e, \mathbf{d}_e] + \cdots\end{aligned}$$

  $\mathbf{g}_n$ are n–multilinear symmetric forms which express the nth Fréchet variations of $\mathbf{g}[\mathbf{d}_e]$.

Energy variations

Letting $\mathbf{u}_{ke} = \{\mathbf{t}_{ke}, \mathbf{d}_{ke}\}^T$ a variation of the element configuration vector

Element second variation

$$\Phi_e^{''}\mathbf{u}_{1e}\mathbf{u}_{2e} = \mathbf{t}_{1e}^T\mathbf{D}\mathbf{g}_1[\mathbf{d}_{2e}] + \mathbf{t}_{2e}^T\mathbf{D}\mathbf{g}_1[\mathbf{d}_{1e}] - \mathbf{t}_{1e}^T\mathbf{K}_c^{-1}\mathbf{t}_{2e} + \mathbf{t}_{0e}^T\mathbf{D}\mathbf{g}_2[\mathbf{d}_{1e}, \mathbf{d}_{2e}]$$

Using the equivalences $\mathbf{d}_{1e}^T\mathbf{G}[\mathbf{t}_{0e}]\mathbf{d}_{2e} = \mathbf{t}_{0e}^T\mathbf{D}\mathbf{g}_2[\mathbf{d}_{1e}, \mathbf{d}_{2e}]$, $\mathbf{L}_1\mathbf{d}_{je} = \mathbf{g}_1[\mathbf{d}_{je}]\mathbf{d}_{1e}^T$ the Hessian is:

$$\Phi_e^{''}\mathbf{u}_{1e}\mathbf{u}_{2e} = \mathbf{u}_{1e}^T\mathbf{H}_e\mathbf{u}_{2e} \quad , \quad \mathbf{H}_e = \begin{bmatrix} -\mathbf{K}_c^{-1} & \mathbf{D}\mathbf{L}_1 \\ \mathbf{L}_1^T\mathbf{D}^T & \mathbf{G}[\mathbf{t}_{0e}] \end{bmatrix}$$

Third and fourth variations

$$\begin{aligned} \Phi_e^{'''}\mathbf{u}_{1e}\mathbf{u}_{2e}\mathbf{u}_{3e} &= \mathbf{t}_{1e}^T\mathbf{D}\mathbf{g}_2[\mathbf{d}_{2e}, \mathbf{d}_{3e}] + \mathbf{t}_{2e}^T\mathbf{D}\mathbf{g}_2[\mathbf{d}_{1e}, \mathbf{d}_{3e}] \\ &+ \mathbf{t}_{3e}^T\mathbf{D}\mathbf{g}_2[\mathbf{d}_{1e}, \mathbf{d}_{2e}] + \mathbf{t}_{0e}^T\mathbf{D}\mathbf{g}_3[\mathbf{d}_{1e}, \mathbf{d}_{2e}, \mathbf{d}_{3e}] \end{aligned}$$

The fourth variation can be obtain in the same fashion

The beam local modeling

Beam interpolation

- Interpolation functions satisfy the linear problem for zero body forces.
- Argyris natural modes are used $\mathbf{d}_{ce} = \{\phi_{cr}, \phi_{ce}, \phi_{cs}\}^T$

$$\phi_{cr} = \frac{\mathbf{d}_{cj} - \mathbf{d}_{ci}}{\ell} \quad \phi_{ce} = \frac{\varphi_{ci} + \varphi_{cj}}{2} \quad \phi_{cs} = \frac{\varphi_{ci} - \varphi_{cj}}{2}$$

Beam mixed energy

$$\boldsymbol{\varrho}_e := \left[\phi_{cr1}, \quad \phi_{ce2} \quad \phi_{ce3}, \quad \phi_{cs1}, \quad \phi_{cs2}, \quad \phi_{cs3}\right]^T = \mathbf{D}\mathbf{d}_{ce}$$

$$\mathbf{t}_e := \left[n_e, \quad m_{2e}, \quad m_{3e}, \quad m_{1s}, \quad m_{2s}, \quad m_{3s}\right]^T$$

$$\mathbf{K}_c = \mathrm{diag}\left[EA\ell, \frac{12EJ_2}{\ell(1+\beta_3)}, \frac{12EJ_3}{\ell(1+\beta_2)}, \frac{4GJ_1}{\ell}, \frac{4EJ_2}{\ell}, \frac{4EJ_3}{\ell}\right]$$

with $\beta_2 = \frac{12EJ_3}{GA_2\ell^2}$ and $\beta_3 = \frac{12EJ_2}{GA_3\ell^2}$.

Beam corotational derivatives

CR trasformation for beam

Geometric relationship

$$\mathbf{d}_{ce} = \mathbf{g}_e[\mathbf{d}_e] \quad \text{with} \quad \mathbf{g}_e[\mathbf{d}_e] \equiv \{\mathbf{g}_r\ ,\ \mathbf{g}_e\ ,\ \mathbf{g}_s\}^T$$

where

$$\mathbf{g}_r = \mathbf{Q}_e^T(\mathbf{e}_1 + \phi_r) - \mathbf{e}_1\ , \quad \mathbf{g}_e = \frac{\mathbf{g}_i + \mathbf{g}_j}{2}\ , \quad \mathbf{g}_s = \frac{\mathbf{g}_i - \mathbf{g}_j}{2}$$

and

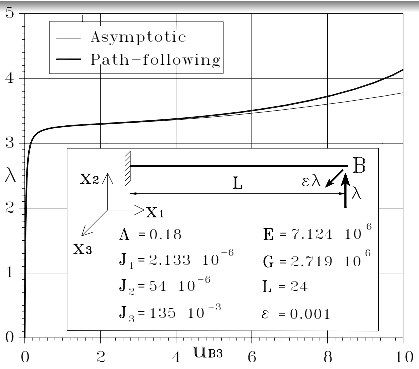
$$\mathbf{g}_k = \log\left[\mathbf{Q}_e^T \mathbf{R}[\varphi_k]\right] \quad k = i, j$$

Remarks

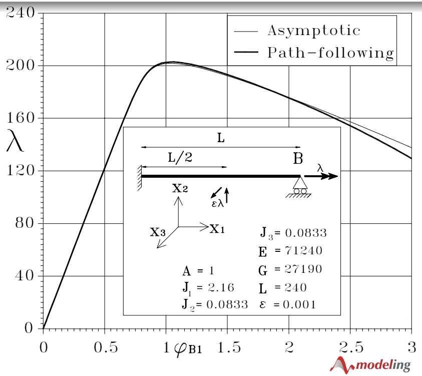
- $\mathbf{Q}$ can be defined as $\mathbf{R}[\phi_e]$ or using a more complex secant frame.
- Algebra needs manipulation software (MAPLE, MATHEMATICA..).

Numerical results

Narrow cantilever beam

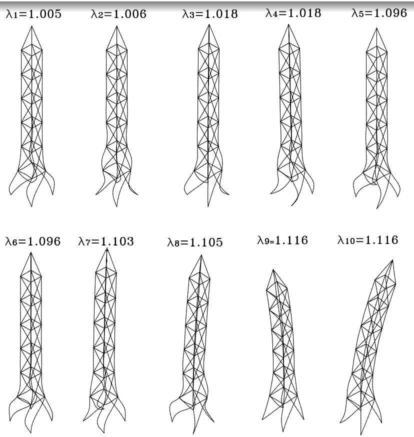


Cable hockling test

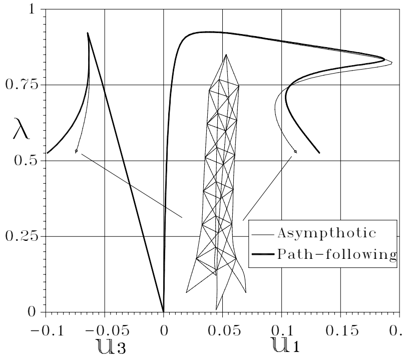


Multimodal test: 3D tower

Buckling modes



Equilibrium path



Quadratic local model

The quadratic local model is defined as coherent second–order expansion of the Simo-Antman strain measure

$$\begin{cases} \bar{\boldsymbol{\epsilon}} = \bar{\mathbf{d}}_{,s} - \bar{\mathbf{W}}(\mathbf{e}_1 + \bar{\mathbf{d}}_{,s}) + \dfrac{1}{2}\bar{\mathbf{W}}^2\mathbf{e}_1 \\ \bar{\boldsymbol{\chi}} = \bar{\boldsymbol{\varphi}}_{,s} - \dfrac{1}{2}\bar{\mathbf{W}}\bar{\boldsymbol{\varphi}}_{,s} \end{cases}$$

with $\bar{\mathbf{W}} = \mathrm{spin}\,[\bar{\boldsymbol{\varphi}}]$.
After some algebra we obtain internal work becomes

$$\int_0^{\ell} \left\{ \mathbf{N}^T \bar{\boldsymbol{\epsilon}} + \mathbf{M}^T \bar{\boldsymbol{\chi}} \right\} \, ds = \mathbf{t}_e^T \boldsymbol{\varrho}_l + \frac{1}{2}\bar{\mathbf{d}}_e^T \boldsymbol{\Psi}[\mathbf{t}_e]\bar{\mathbf{d}}_e$$

$\boldsymbol{\varrho}_l$ being coincident with linear model and

$$\boldsymbol{\Psi}[\mathbf{t}_e] = \sum_{j=1}^{6} t_{ej}\boldsymbol{\Psi}_j$$

Quadratic local model

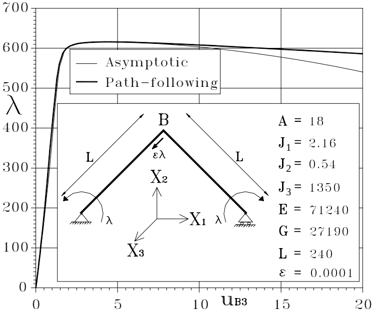
$\boldsymbol{\Psi}_j$ with $j = 1 \cdots 6$

$$\boldsymbol{\Psi}_1 = -\begin{bmatrix} 6\,\frac{\mathbf{W}_1^2}{5} & -\frac{\mathbf{W}_1}{5} & \mathbf{0}_3 \\ \frac{\mathbf{W}_1}{5} & \frac{\mathbf{W}_1^2}{5} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \frac{\mathbf{W}_1^2}{3} \end{bmatrix} \quad \boldsymbol{\Psi}_2 = \begin{bmatrix} -\mathbf{P}_{13} & -\mathbf{d}_{21} & \mathbf{0}_3 \\ -\mathbf{d}_{12} & \frac{\mathbf{P}_{13}}{2} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & -\frac{\mathbf{P}_{13}}{6} \end{bmatrix} \quad \boldsymbol{\Psi}_3 = \begin{bmatrix} \mathbf{P}_{12} & -\mathbf{d}_{31} & \mathbf{0}_3 \\ -\mathbf{d}_{13} & -\frac{\mathbf{P}_{12}}{2} & \mathbf{0}_3 \\ \mathbf{0}_3 & \mathbf{0}_3 & \frac{\mathbf{P}_{12}}{6} \end{bmatrix}$$

$$\boldsymbol{\Psi}_4 = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{W}_1^2 \\ \mathbf{0}_3 & \mathbf{0}_3 & -\frac{\mathbf{W}_1}{2} \\ -\mathbf{W}_1^2 & \frac{\mathbf{W}_1}{2} & \mathbf{0}_3 \end{bmatrix} \quad \boldsymbol{\Psi}_5 = \begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & -\mathbf{d}_{21} \\ \mathbf{0}_3 & \mathbf{0}_3 & \frac{\mathbf{P}_{13}}{2} \\ -\mathbf{d}_{12} & \frac{\mathbf{P}_{13}}{2} & \mathbf{0}_3 \end{bmatrix} \quad \boldsymbol{\Psi}_6 = -\begin{bmatrix} \mathbf{0}_3 & \mathbf{0}_3 & \mathbf{d}_{31} \\ \mathbf{0}_3 & \mathbf{0}_3 & \frac{\mathbf{P}_{12}}{2} \\ \mathbf{d}_{13} & \frac{\mathbf{P}_{12}}{2} & \mathbf{0}_3 \end{bmatrix}$$

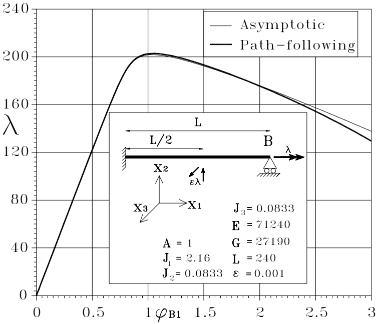
where $\mathbf{W}_1 = \text{spin}[\mathbf{e}_1]$, $\mathbf{d}_{hk} = \mathbf{e}_h\mathbf{e}_k^T = \mathbf{d}_{kh}^T$ and $\mathbf{P}_{hk} = \mathbf{d}_{hk} + \mathbf{d}_{kh}$.

Accuracy



	loc. mod.	*linear* 8*e*	*linear* 16*e*	*quadratic* 8*e*	*quadratic* 16*e*	*ref.*
λ_b	Sec	651.7	629.3	638.3	626.2	622.3
	Mid	656.3	630.3	638.3	626.2	"
$\mathcal{B}$	Sec	−348.7	−294.36	−313.5	−286.6	−277.7
	Mid	−340.96	−292.35	−311.7	−286.5	"

Accuracy



	loc. mod.	*linear* 8*e*	16*e*	*quadratic* 8*e*	16*e*	*ref.*
λ_b	Sec	241.1	226.7	222.7	222.2	222.2
	Mid	248.8	228.2	229.0	222.2	"
$\mathcal{B}$	Sec	−126.69	−100.40	−93.65	−92.97	−92.56
	Mid	−130.53	−100.86	−86.40	−92.49	"

Remarks

- The approach allows a geometrically coherent, locking–free, nonlinear modeling to be obtained, starting from a standard linear finite element local discretization.
- Although complex algebra is necessary in the derivation of the strain energy derivatives in an explicit form, it could be noticeably simplified by the use algebraic manipulators.
- The application to beam, characterized by a strong sensitivity to kinematical coherency, have been chosen as suitable to test the reliability of the approach. The formulation is however general and can be easily extended to other structural models.

References

- G. Garcea, A. Madeo, G. Zagari, R. Casciaro, 'Asymptotic post-buckling FEM analysis using corotational formulation', Accepted IJSS.

Develops

- Curved beams and shells
- Composite beam/plates (Saint Venánt like solutions)
- Plasticity and dynamics

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