PhD defense: IUSS XXVI Ciclo

On the use of anisotropic triangles in an "immersed" finite element approach with application to fluid-structure interaction problems

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Context: aortic valve-blood fluid-structure interaction

Figure: MRI showing motion of an aortic valve

Context: numerical methods

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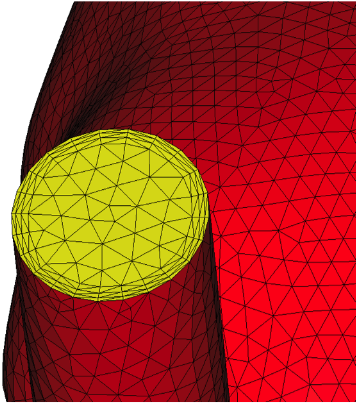
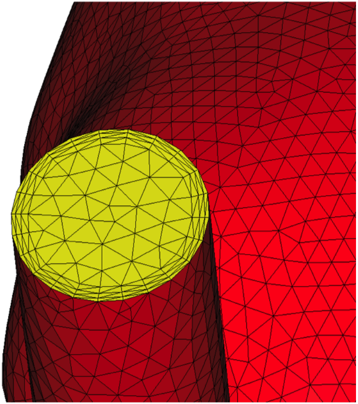


Context: numerical methods



Outline

- **Introduction: a 1D interface toy problem**
  - Finite elements
  - Error estimates
- **A 2D Extension: A locally anisotropic remeshing for Stokes**
  - Incompressible Stokes problem
  - Locally anisotropic remeshing
  - Mixed finite elements and inf-sup stability
  - A Smallest Generalized Eigenvalue test
  - Some Inf-sup stability numerical results on anisotropic triangles
- **FSI Application: a thin hinged rigid leaflet**
  - Coupled problem
  - On the finite element spaces choice
  - Numerical tests: validation & inf-sup stability issues
- **Conclusions**
- **Computational hemodynamics**

Immersed methods:
a 1D presentation

1D PROBLEM: A 1D INTERFACE TOY PROBLEM

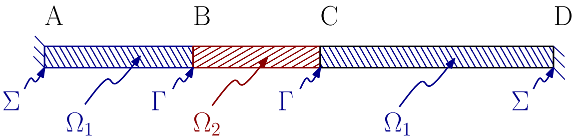


Figure: $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$

1D Problem: A 1D interface toy problem

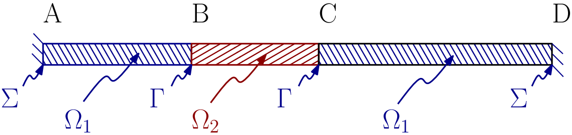


Figure: $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$

- **Finite elements**
  Let $u_h(x) = \sum_j^N \hat{u}_j \phi_j(x)$ with $V_h = \text{Span}(\phi_j)$ such that
  $V_h \subset \left\{ u \in C^0(\Omega) \middle| u_{|\partial\Omega} = 0 \right\}$
- **Galerkin method**
  Find $u_h \in V_h(\Omega)$ such that

$$\int_\Omega \alpha u_h' v_h' \mathrm{d}x = \int_\Omega f v_h \mathrm{d}x \qquad \forall v_h \in V_h$$

with

$$\alpha = \begin{cases} \alpha_1 & \text{on } \Omega_1 \\ \alpha_2 & \text{on } \Omega_2 \end{cases} \quad \& \quad f = \begin{cases} f_1 & \text{on } \Omega_1 \\ f_2 & \text{on } \Omega_2 \end{cases}$$

1D Problem: Error estimates

Pick an arbitrary mesh over Ω and:

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$$\begin{cases} ||u - u_h||_1 \lesssim h^1 \\ ||u - u_h||_0 \lesssim h^2 \end{cases}$$

where $||\cdot||_{1,\Omega}$ and $||\cdot||_{0,\Omega}$ are H^1 and L^2 norms over Ω, respectively.

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where $||\cdot||_{1,\Omega}$ and $||\cdot||_{0,\Omega}$ are H^1 and L^2 norms over Ω, respectively.

But if we have, e.g.: $\alpha_1 \in \mathbb{R}^+$ and $\alpha_2 \in \mathbb{R}^+$ with $\alpha_1 \neq \alpha_2$ then $\alpha \notin C^0$ thus $u \notin C^1(\Omega)$ but we only have $u \in C^0(\Omega)$ as a consequence

$$\begin{cases} ||u - u_h||_1 \lesssim h^{1/2} \\ ||u - u_h||_0 \lesssim h^1 \end{cases}$$

Numerical tests

- **Geometry**: $A = 0,\ B = e,\ C = 1 + \pi,\ D = 6$
- **Loads**: $f_e = 1$ on $]A, D[$ and $f_2 = 1$ on $]B, C[$

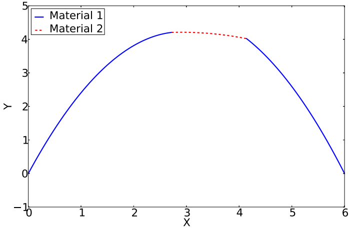
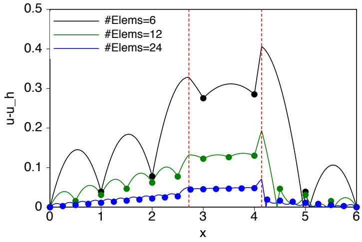
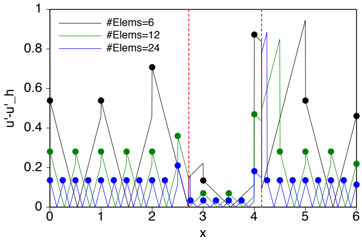


Figure: Analytical solutions: $\alpha_1 = 1$ and $\alpha_2 = 4$

1D Problem: numerical tests

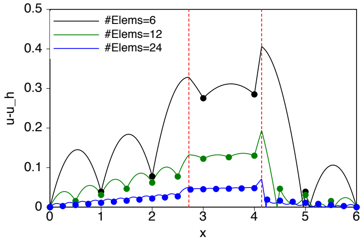


Error $u - u_h$



Error $u' - u'_h$

1D Problem: numerical tests



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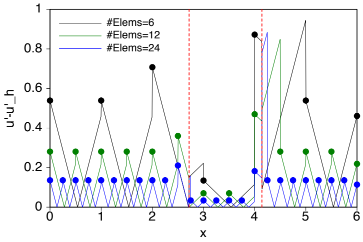

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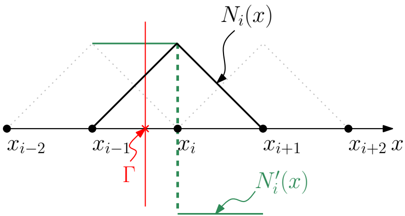
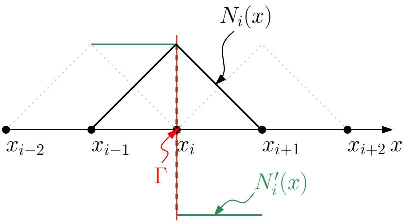


Unfitted mesh



Fitted mesh

Second order "Immersed" approaches for interface problems

- **Immersed Interface Method (IIM)** LeVeque 1994
  - Build shape functions embedding interface constraints
  - **Pros** High order, no added dofs
  - **Cons** Requires ad-hoc interface shape functions construction
- **eXtended Finite Element Method (XFEM)** Hansbo 2002, Wall 2008
  - Build Finite element spaces allowing discontinuities inside elements: weakly couple fluid and solid materials
  - **Pros** High order
  - **Cons** integration issues, weak coupling issues, added dofs
- **Local refinement** van Loon 2004, Ilinca 2010
  - Local mesh refinement around interface
  - **Pros** High order, strong interface constraints enforcement
  - **Cons** Smoothing operation (to avoid distorted elements), added degrees of freedom

Not an exhaustive list!

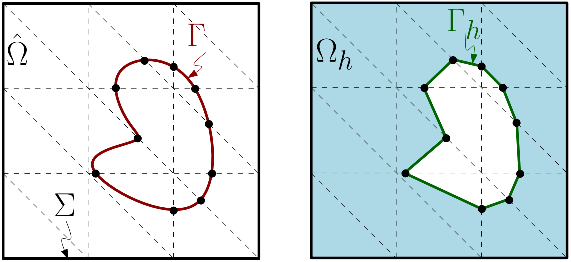
2D extension: Refinement with anisotropic elements

Boundary reconstruction & integration domain

- **Step 1:** define an extended mesh
- **Need:** reconstruct boundary to properly detect integration domain

$\boxed{\Omega_h\text{: integration domain}}$

- Necessity of integration over Ω_h pointed out in **?**



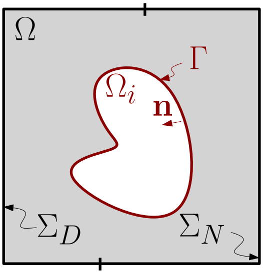
- $\bullet$: intersection points of the immersed boundary with mesh
- Γ_h: reconstructed immersed boundary
- Ω_h: integration domain with $\partial\Omega_h = \Sigma \cup \Gamma_h$

Incompressible Stokes: continuous strong form

- **Need:** impose Dirichlet BCs on immersed boundary Γ

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{0} \quad \text{in } \Omega & \text{Balance of momentum} \\ \operatorname{div}(\mathbf{u}) = 0 \quad \text{in } \Omega & \text{Incompressibility} \\ \partial \mathbf{u}/\partial \mathbf{n} - p\mathbf{n} = \mathbf{0} \quad \text{on } \Sigma_N & \text{Neumann BC} \\ \mathbf{u} = \mathbf{g} \quad \text{on } \Sigma_D & \text{External Dirichlet BC} \\ \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma := \partial\Omega / \partial\hat{\Omega} & \text{Immersed Dirichlet BC} \end{cases}$$

  - $\mathbf{g}$: suitable given function
  - $\overline{\Sigma_D \cup \Sigma_N} = \partial\hat{\Omega}$ and $\Sigma_D \cap \Sigma_N = \emptyset$
  - $\mathbf{n}$ outward normal



- Neumann BC on Γ easier (weakly imposed)

Proposed approach: a locally anisotropic remeshing

- ✓ **Idea:** consider new DOFS on the immersed boundary
- ✓ **Idea:** subdivide elements cut by immersed boundary
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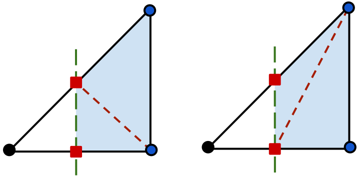
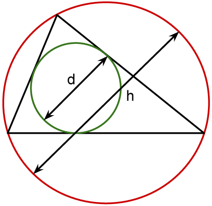


Two possible choices



Element ratio $\sigma = h/d$

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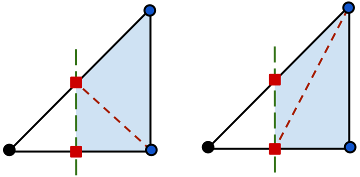

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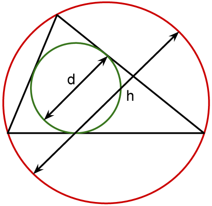

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Given an element ratio σ, an element is distorted when σ is large

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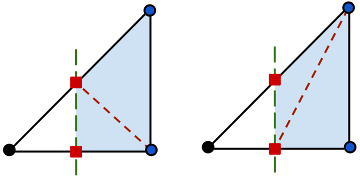

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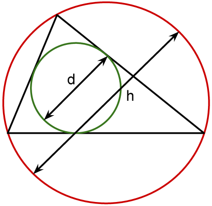


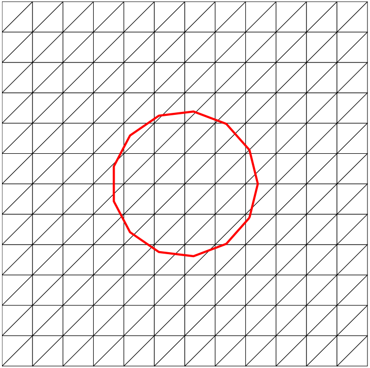
Element ratio $\sigma = h/d$

- ■ **Problem:** avoid distorted elements as much as possible

> Given an element ratio σ, an element is distorted when σ is large

- ✓ **Idea:** choose the pair of triangles with best element ratios
- ✓ **Actually:** Delaunay triangulation leads to best choice

A subdivision example



Original mesh $\hat{\mathcal{T}}$ & circle Γ

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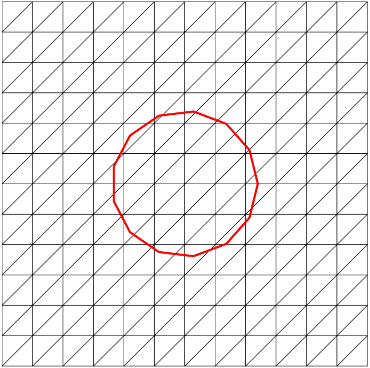

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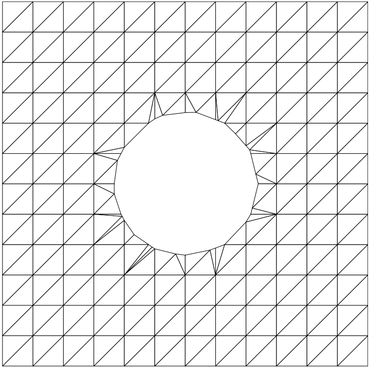

Refined mesh $\mathcal{T}_r$

Clearly the method may induce a mesh with distorted elements

Mixed FEM: an algebraic presentation

- **Algebraic (Galerkin) system**

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}^T \\ \mathbf{B} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{p}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{g} \end{Bmatrix}$$

- **2D incompressible Stokes** problem

$$\begin{cases} \mathbf{A}|_{ij} = \displaystyle\int_{\Omega_h} \nabla \mathbf{N}_i : \nabla \mathbf{N}_j d\Omega_h & \forall (i,j) \in \{1,\dots,n\} \times \{1,\dots,n\} \\ \mathbf{B}|_{ij} = -\displaystyle\int_{\Omega_h} M_i \operatorname{div}(\mathbf{N}_j) d\Omega_h & \forall (i,j) \in \{1,\dots,m\} \times \{1,\dots,n\} \end{cases}$$

with $\mathbf{N}$ velocity shape functions, M pressure shape functions

Conditions for a non-singular system

Discrete inf-sup condition required to properly solve previous algebraic system

- $\exists \beta > 0$ (independent of h) such that

$$\max_{\mathbf{v} \neq \mathbf{0} \in \mathbb{R}^n} \frac{\mathbf{v}^T \mathbf{B}^T \mathbf{q}}{||\mathbf{v}||_A} \geq \beta ||\mathbf{q}||_Q \qquad \forall \mathbf{q} \in \mathbb{R}^m$$

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- Error estimate (**?**)

$$||\mathbf{u} - \hat{\mathbf{u}}|| \lesssim ||\mathbf{f}|| + \beta^{-1} ||\mathbf{g}||$$
$$||\mathbf{p} - \hat{\mathbf{p}}|| \lesssim \beta^{-1} ||\mathbf{f}|| + \beta^{-2} ||\mathbf{g}||$$

with $\mathbf{u}$ exact solution at nodes and for suitable norms

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with $\mathbf{u}$ exact solution at nodes and for suitable norms

Problem: if $\beta \to 0$ as element ratio $\sigma \to \infty$, then

- errors are not bounded
- pressure error deteriorates faster than velocity error

A numerical measure for inf-sup constant

- **Goal:** numerically show if a finite element approximation is stable for large element distortions
- Discrete inf-sup constant given by **square root of lowest positive eigenvalue** of generalized eigensystem (e.g., **?**)

$$\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T\mathbf{q} = \beta_h^2\mathbf{Q}\mathbf{q},$$

with *pressure mass matrix* $\mathbf{Q}$ defined as

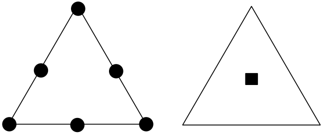
$$\mathbf{Q}|_{ij} = \int_{\Omega_h} M_i M_j \, d\Omega_h$$

- **Remark on conditioning** (e.g., **?**)

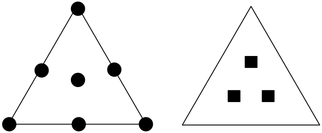
$$\kappa(\mathbf{B}\mathbf{A}^{-1}\mathbf{B}^T) \lesssim \beta_h^{-2}\kappa(\mathbf{Q})$$

where κ condition number

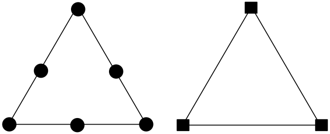
2D Problem: Mixed finite elements in consideration



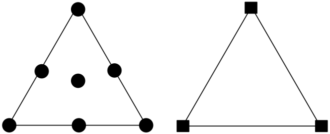
$\mathbf{P}_2/P_0$



$\mathbf{P}_2^+/P_1^d$



$\mathbf{P}_2/P_1$

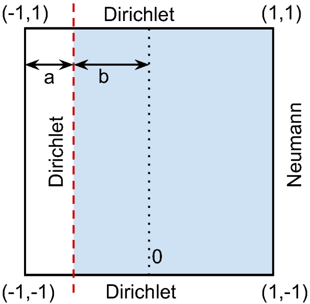


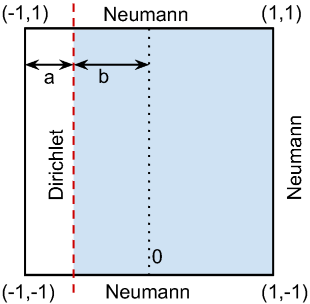
$\mathbf{P}_2^+/P_1$

$\boxed{\text{All elements are inf-sup stable under isotropic mesh distortion}}$

A Smallest Generalized Eigenvalue test

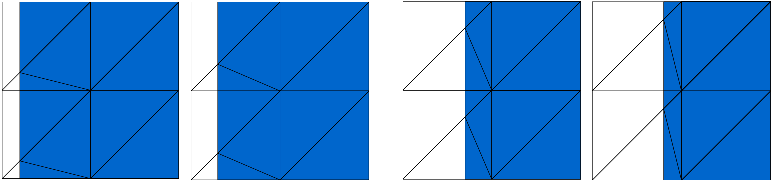
Test D **Test N**





Remeshing examples:

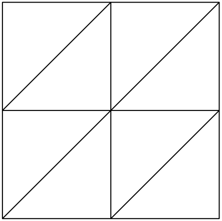
$a \to 0$ $b \to 0$



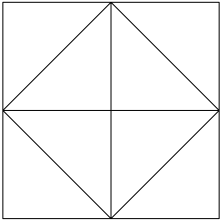
$a = 0.2$ $a = 0.3$ $b = 0.3$ $b = 0.2$

Test and results

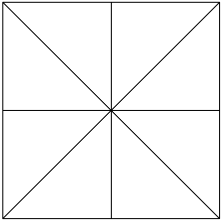
Mesh 1



Mesh 2



Mesh 3



Test D

Mesh:		1	2	3
$a \to 0$	$\mathbf{P}_2/P_0$	P	P	P
	$\mathbf{P}_2/P_1$	P	P	P
	$\mathbf{P}_2^+/P_1$	P	P	P
	$\mathbf{P}_2^+/P_1^d$	2	2	2
$b \to 0$	$\mathbf{P}_2/P_0$	2	2	1
	$\mathbf{P}_2/P_1$	1	2	P
	$\mathbf{P}_2^+/P_1$	P	P	P
	$\mathbf{P}_2^+/P_1^d$	5	5	4

Test N

Mesh		1	2	3
$a \to 0$	$\mathbf{P}_2/P_0$	P	P	P
	$\mathbf{P}_2/P_1$	P	P	P
	$\mathbf{P}_2^+/P_1$	P	P	P
	$\mathbf{P}_2^+/P_1^d$	2	2	2
$b \to 0$	$\mathbf{P}_2/P_0$	1	P	1
	$\mathbf{P}_2/P_1$	P	P	P
	$\mathbf{P}_2^+/P_1$	P	P	P
	$\mathbf{P}_2^+/P_1^d$	2	1	2

P if inf-sup stable, number of spurious modes otherwise

Comments

- **Stability for distorted triangular elements is important**

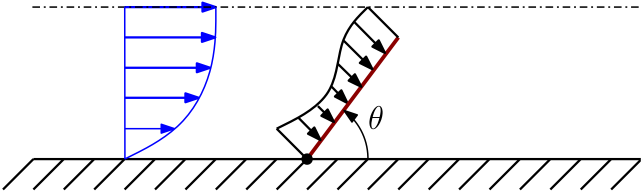
Element	**numerics**	**theory**	**reference**
$\mathbf{P}_2^+/P_1$	stable	proof?	Apel 2003
$\mathbf{P}_2/P_1$ (Taylor-Hood)	not stable		Apel 2003
$\mathbf{P}_1^+/P_1$ (MINI)	not stable		Russo 1996
$\mathbf{P}_2/P_0$	not stable		(despite Apel 2004)
$\mathbf{P}_2^+/P_1^d$	not stable		

- **Stabilization:** specific stabilization may be used for distorted elements

Application to an Fluid-Structure Interaction problem:

an hinged rigid leaflet

Application problem



Hinged rigid leaflet under fluid load.

FSI Application: Coupled system

Fluid **Solid** **Coupling**

Problem: Let Ω be the fluid domain, Γ the leaflet, & $\Sigma_D \cup \Sigma_N = \partial\Omega$
Find u, p, θ such that

$$\rho_f \left(\frac{\partial u}{\partial t} + u \cdot \nabla u \right) - \operatorname{div}(\mu(\nabla u + \nabla^T u)) + \nabla p = f \qquad \text{in } \Omega \backslash \Gamma$$

$$\operatorname{div}(u) = 0 \qquad \text{in } \Omega \backslash \Gamma$$

$$u = b_D \qquad \text{on } \Sigma_D$$

$$-pn + \mu(\nabla u + \nabla^T u)n = b_N \qquad \text{on } \Sigma_N$$

$$u(x,0) = u_i(x) \qquad \text{in } \Omega$$

$$I \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \tau$$

$$u = r \frac{\mathrm{d}\theta}{\mathrm{d}t} n^+ \qquad \text{on } \Gamma$$

$$\tau = \int_\Gamma r [\![pn^+ - \mu(\nabla u + \nabla^T u)n^+]\!] \cdot n^+$$

FSI Application: Coupled system

Fluid **Solid** **Coupling terms**

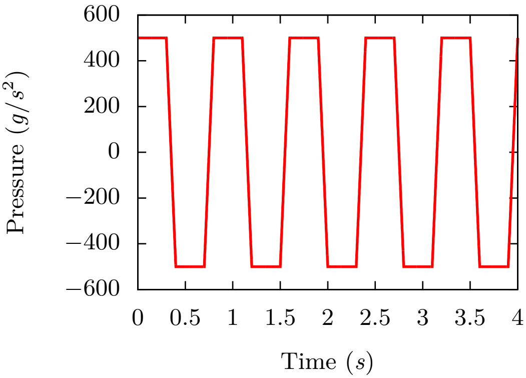
- **Unknowns**
  - $\mathbf{u}$ Fluid velocity around leaflet
  - $\mathbf{g}$ Fluid velocity on leaflet (subscript l restriction on the leaflet)
  - p Fluid pressure
  - θ Leaflet angle
- **Global system**

$$\begin{bmatrix} \mathbf{A} & \mathbf{A}_l & \mathbf{D}^T & \mathbf{0} \\ \mathbf{0} & \mathbf{C} & \mathbf{0} & \mathbf{l}_\theta \\ D & \mathbf{D}_l & \mathbf{0} & \mathbf{0} \\ \mathbf{l}_u & \mathbf{l}_l & \mathbf{l}_p & s \end{bmatrix} \begin{Bmatrix} \hat{\mathbf{u}} \\ \hat{\mathbf{g}} \\ \hat{p} \\ \theta \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{b}}_u \\ \hat{\mathbf{b}}_g \\ \hat{0} \\ \lambda \end{Bmatrix}$$

  - Matrix A: Stiffness, Mass, & Convection (Picard fixed point) terms
  - Matrix D: Divergence terms
  - Scalars s and λ: solid
  - Vectors $\mathbf{l}_u$, $\mathbf{l}_u$, $\mathbf{l}_p$: Conservation of momentum
  - Matrix $\mathbf{C}$ & vector $\mathbf{l}_\theta$: Kinematic constraint

FSI Application: Numerical test 1

- Domain $[-3, 3] \times [0, 1]$ cm^2 discretized
- Fluid $\nu = 0.03\ \mathsf{cm}^2 \cdot \mathsf{s}^{-1}$
- Solid $L = 0.8$ cm, rotation around $(0, 0)$, $I = 0.51$ g^2, $\theta_0 = \pi/2$
- BCs: top: no-slip; bottom: no-slip; outflow: free-stress; inflow:

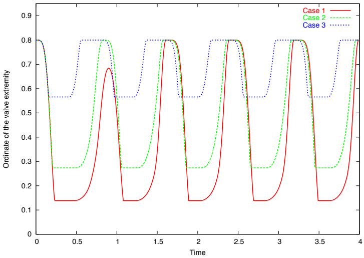


Pressure inflow BC

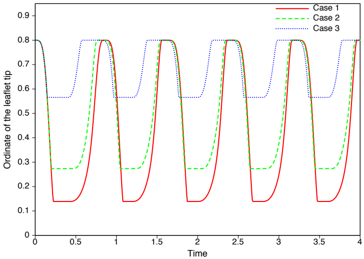
FSI Application: Video Numerical Test 1

Speed (top) & Pressure (bottom) (using $\mathbf{P}_2^+/P_1$)

FSI Application: Validation Numerical Test 1



Extracted from Causin 2005

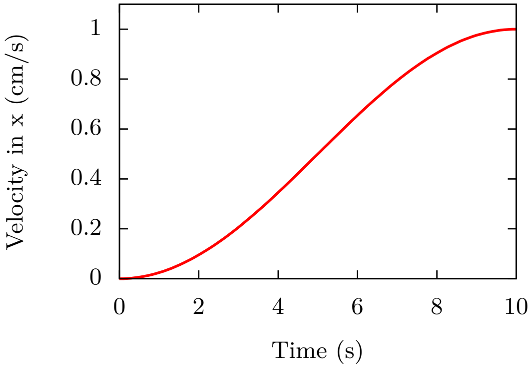


using $\mathbf{P}_2^+/P_1$

- **Case 1:** $\theta \in [10^\circ, 90^\circ]$
- **Case 2:** $\theta \in [20^\circ, 90^\circ]$
- **Case 3:** $\theta \in [45^\circ, 90^\circ]$

FSI Application: Numerical Test 2: massless leaflet

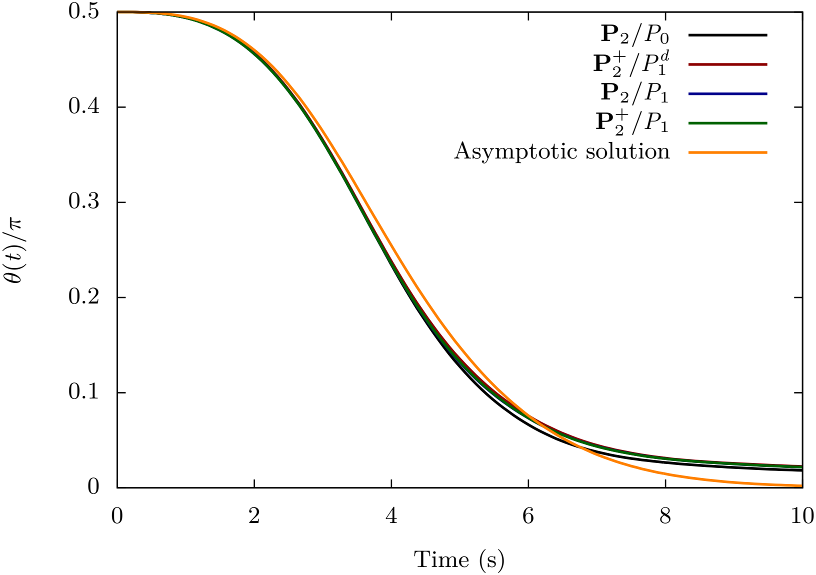
- Fluid domain: $[-1, 6] \times [0, 1]$
- Fluid: $\nu = 0.001$ (Reynolds ≈ 1000)
- Solid: L=0.999, rotation around $(0, 0)$, $\theta_0 = \pi/2$
- BCs: bottom: no-slip; top: symmetric; outflow: free-stress; inflow:



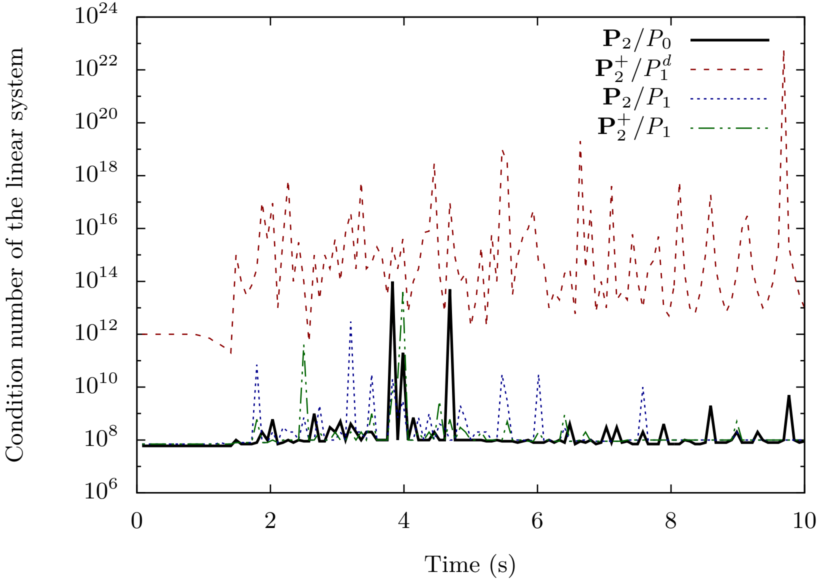
Velocity inflow BC

FSI Application: Numerical Test 2 elements distortion

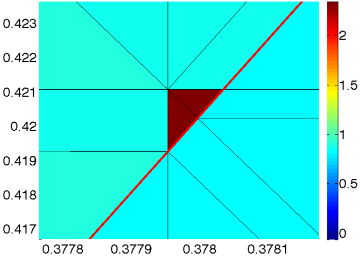
FSI Application: Numerical Test 2 leaflet motion



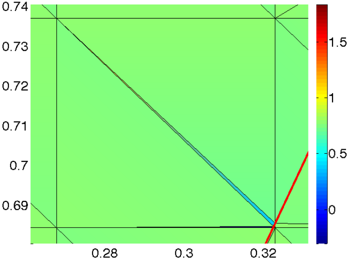
FSI Application: Numerical Test 2 conditioning



FSI Application: Numerical Test 2 spurious modes



$\mathbf{P}_2/P_0$



$\mathbf{P}_2^+/P_1^d$

Conclusions

Proposed an immersed approach using an anisotropic remeshing

- Studied inf-sup stability issues with a series of numerical tests
- Showed that *as is*, mixed elements with discontinuous pressure not suitable
- Application to a simple fluid-structure interaction problem
- Stabilization strategies: proofs of concept

Further researches

- Lower order schemes: $\mathbf{P}_1^+/P_1$
- Conditioning issues
- Extension to 3D
- The Virtual Element Method
- ...

Principally associated articles

- **On the 1D part:**

  F. Auricchio, D. Boffi, L. Gastaldi, A. Lefieux, and A. Reali. A study on unfitted 1d finite element methods.
  Computers and Mathematics with Applications, 2014.

- **On the 2D part:**

  F. Auricchio, F. Brezzi, A. Lefieux, and A. Reali. An “immersed” finite element method based on a locally anisotropic remeshing for the incompressible stokes problem.
  Computer Methods In Applied Mechanics and Engineering, 2014.

- **On the FSI problem:**

  F. Auricchio, A. Lefieux, A. Reali, and A. Veneziani. A locally anisotropic fluid-structure interaction remeshing strategy for thin structures with application to a hinged rigid leaflet.
  To be submitted, 2015.

Contributions

- **In mathematics:**
  F. Auricchio, D. Boffi, L. Gastaldi, A. Lefieux, and A. Reali. On a fictitious domain method with distributed lagrange multiplier for interface problems.
  Applied Numerical Mathematics, 2014.
- **In computational hemodynamics:**
  F. Auricchio, M. Conti, A. Lefieux, S. Morganti, A. Reali, F. Sardanelli, F. Secchi, S. Trimarchi, and A. Veneziani. Patient-specific analysis of post-operative aortic hemodynamics: a focus on thoracic endovascular repair (tevar)
  Computational Mechanics, 2014.

  G. van Bogerijen, F. Auricchio, M. Conti, A. Lefieux, A. Reali, A. Veneziani, J. Tolenaar, F. Moll, V. Rampoldi, and S. Trimarchi. Aortic hemodynamics after thoracic endovascular aortic repair, with particular attention to the bird-beak configuration.
  Journal of Endovascular Therapy, 21(6):791–802, 2014.

The iCardioCloud project

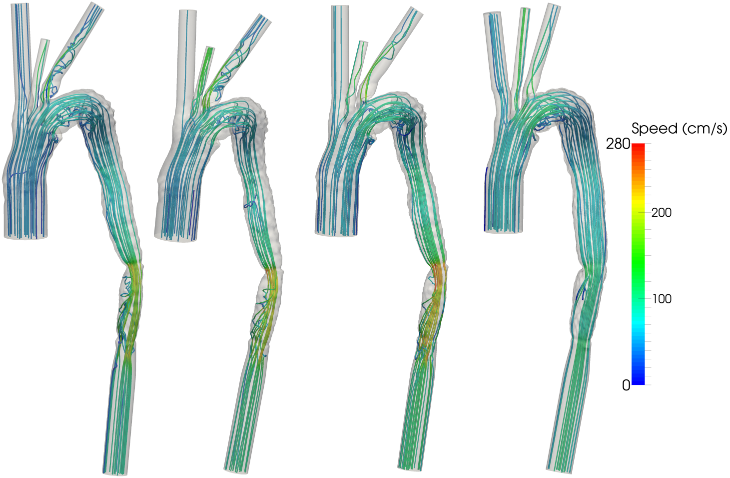
- **Pre-processing:**
  - **Patient specific geometry**: reconstructed from Computed Tomography
  - **Patient specific inflow**: from Magnetic Resonance Imaging MRI
  - Reconstructed with VMTK/Tetgen

The iCardioCloud project

- **Pre-processing:**
  - **Patient specific geometry**: reconstructed from Computed Tomography
  - **Patient specific inflow**: from Magnetic Resonance Imaging MRI
  - Reconstructed with VMTK/Tetgen
- **Simulations:**
  - **LifeV:** a collaborative parallel (Trilinos based) finite element library: half million lines of C++ code
  - Mesh: Around 3 millions tetrahedra: $\approx$ 160 Gigas in RAM
  - Navier-Stokes with MINI ($\mathbf{P}_1^+/P_1$) and inf-sup stable
  - Second order in time Backward-Euler scheme
  - In house HPC cluster (& more with FERMI (IBM BlueGene/Q)): design and installation
- **Post-processing:**
  - Paraview: streamlines, WSS, OSI
  - VMTK: Particule tracking, Pathlines,

Last but not least: a confrontation MRI - CFD

Last but not least: post vs followup operative



Thank you for your attention!

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