An introduction to the Immersed boundary method and its finite element approximation

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History

- Introduced by Peskin
  - Flow patterns around heart valves: a digital computer method for solving the equations of motion
    <PhD Thesis, 1972>
  - *Numerical analysis of blood flow in the heart*
    <J. Comput. Phys., 1977>
- Extended by Peskin and McQueen since '83 to simulate the blood flow in a three dimensional model of heart and great vessels
- Review article by Peskin
  The immersed boundary method
  <Acta Numerica, 2002>
- Several applications in biology, when a fluid interacts with a flexible structure.

Main features

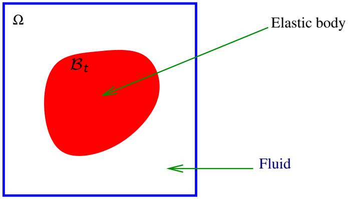
- The structure is a part of the fluid with additional forces and mass.
- The Navier-Stokes equations are solved everywhere.
- The interaction with the structure is obtained by means of a singular force term defined by a Dirac delta function.
- The immersed material is modeled as a collection of fibers.
- Discretization based on finite differences.

FE discretization of IBM

<Boffi–G., C&S 2003>
<Boffi–G.–Heltai 2004, M3AS 2007>
<Boffi–G.–Heltai–Peskin CMAME 2008>
<Boffi–Cavallini–G. 2010>

- Variational formulation of the FSI force
- No approximation of the Dirac delta function
- Better interface approximation (less diffusion, sharp pressure jump)
- Effective mixed finite elements for the approximation of the fluid equations.
- Submerged elastic solid occupying volume may also be considered.

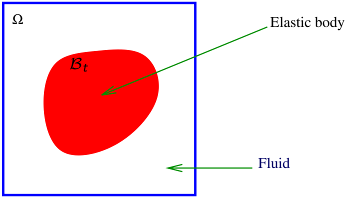
Immersed elastic bodies



Immersed body of codimension 0

the fluid domain and the immersed body have the same dimension

Immersed elastic bodies



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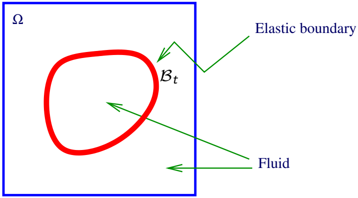
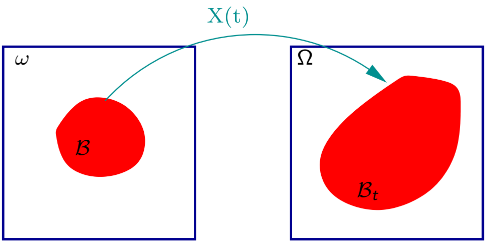


Immersed body of codimension 1

the immersed body is either a curve in 2D or a surface in 3D

Notation



Ω fluid + solid	$\mathcal{B}_t$ deformable structure domain
$\Omega \subset \mathbb{R}^d$, $d = 2, 3$	$\mathcal{B}_t \subset \mathbb{R}^m$, $m = d, d-1$

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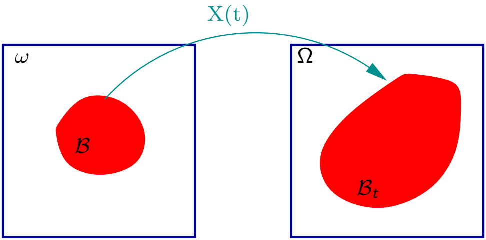
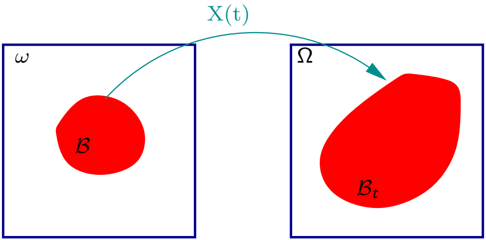


Ω fluid + solid
$\Omega \subset \mathbb{R}^d$, $d = 2, 3$
$\mathbf{x}$ Euler. var. in Ω

$\mathcal{B}_t$ deformable structure domain
$\mathcal{B}_t \subset \mathbb{R}^m$, $m = d, d - 1$
s Lagrangian var. in $\mathcal{B}$
$\mathcal{B}$ reference domain

Notation



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$\mathbf{x}$ Euler. var. in Ω	s Lagrangian var. in $\mathcal{B}$
	$\mathcal{B}$ reference domain
$\mathbf{u}(\mathbf{x}, t)$ fluid velocity	$\mathbf{X}(\cdot, t) : \mathcal{B} \to \mathcal{B}_t$ position of the solid
$p(\mathbf{x}, t)$ fluid pressure	$\mathbb{F} = \dfrac{\partial \mathbf{X}}{\partial s}$ deformation grad. ($\det \mathbb{F} > 0$)

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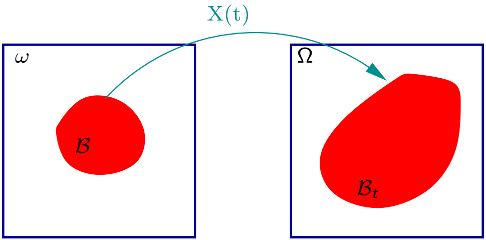

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$$\mathbf{u}(\mathbf{x}, t) = \frac{\partial \mathbf{X}}{\partial t}(s, t) \text{ where } \mathbf{x} = \mathbf{X}(s, t)$$

Case: $m = d$

From conservation of momenta, in absence of external forces, it holds

$$\rho\dot{\mathbf{u}} = \rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\,\mathbf{u}\right) = \nabla\cdot\boldsymbol{\sigma} \quad \text{in } \Omega$$

In our case the Cauchy stress tensor has the following form

$$\boldsymbol{\sigma} = \begin{cases} \boldsymbol{\sigma}_f & \text{in } \Omega\setminus\mathcal{B}_t \\ \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s & \text{in } \mathcal{B}_t \end{cases}$$

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- Incompressible fluid: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_f = -p\mathbb{I} + \mu(\nabla\,\mathbf{u} + (\nabla\,\mathbf{u})^T)$
- Visco-elastic material: $\boldsymbol{\sigma} = \boldsymbol{\sigma}_f + \boldsymbol{\sigma}_s$ with $\boldsymbol{\sigma}_s$ elastic part of the stress

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Moreover, if the structural material has a density ρ_s different from the fluid density ρ_f, we have

$$\rho = \begin{cases} \rho_f & \text{in } \Omega\setminus\mathcal{B}_t \\ \rho_s & \text{in } \mathcal{B}_t \end{cases}$$

Virtual work principle

$$\rho_f \int_\Omega \dot{\mathbf{u}} \cdot \mathbf{v} d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \nabla \mathbf{v} d\mathbf{x} - \int_{\partial\Omega} \boldsymbol{\sigma}_f \mathbf{n} \cdot \mathbf{v} da$$
$$= -(\rho_s - \rho_f) \int_{\mathcal{B}_t} \dot{\mathbf{u}} \cdot \mathbf{v} d\mathbf{x} - \int_{\mathcal{B}_t} \boldsymbol{\sigma}_s : \nabla \mathbf{v} d\mathbf{x}, \quad \forall \mathbf{v}$$

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The elastic stress $\boldsymbol{\sigma}_s$ can be expressed in Lagrangian variables by means of the Piola-Kirchhoff stress tensor by:

$$\tilde{\mathbb{P}}(s,t) = |\mathbb{F}(s,t)| \boldsymbol{\sigma}_s(\mathbf{X}(s,t),t) \mathbb{F}^{-T}(s,t), \quad s \in \mathcal{B}$$

so that

$$\rho_f \int_\Omega \dot{\mathbf{u}} \cdot \mathbf{v} d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \nabla \mathbf{v} d\mathbf{x} - \int_{\partial\Omega} \boldsymbol{\sigma}_f \mathbf{n} \cdot \mathbf{v} da$$
$$= -(\rho_s - \rho_f) \int_{\mathcal{B}_t} \dot{\mathbf{u}} \cdot \mathbf{v} d\mathbf{x} - \int_{\mathcal{B}} \tilde{\mathbb{P}} : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\, ds \quad \forall \mathbf{v}$$

Using the Lagrangian description in the solid domain, the material derivative is the same as the time derivative, hence $\dot{\mathbf{u}} = \partial^2\mathbf{X}/\partial t^2$.

$$
\begin{aligned}
&\rho_f \int_\Omega \dot{\mathbf{u}}\mathbf{v}\,d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \nabla\,\mathbf{v}\,d\mathbf{x} = \\
&\quad -(\rho_s - \rho_f)\int_{\mathcal{B}} \frac{\partial^2\mathbf{X}}{\partial t^2}\cdot\mathbf{v}(\mathbf{X}(s,t))ds - \int_{\mathcal{B}} \tilde{\mathbb{P}} : \nabla_s\,\mathbf{v}(\mathbf{X}(s,t))\,ds
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\end{aligned}
$$

At the end, after integration by parts,

$$
\begin{aligned}
\int_\Omega (\rho\dot{\mathbf{u}} - \nabla\cdot\boldsymbol{\sigma}_f)\mathbf{v}\,d\mathbf{x} &= -(\rho_s-\rho_f)\int_\mathcal{B} \frac{\partial^2\mathbf{X}}{\partial t^2}\cdot\mathbf{v}(\mathbf{X}(s,t)ds \\
&\quad + \int_\mathcal{B} (\nabla_s\cdot\tilde{\mathbb{P}})\cdot\mathbf{v}(\mathbf{X}(s,t))\,ds - \int_{\partial\mathcal{B}} \tilde{\mathbb{P}}\mathbf{N}\cdot\mathbf{v}(\mathbf{X}(s,t))\,dA
\end{aligned}
$$

From Lagrangian to Eulerian variables

Implicit change of variables using the Dirac delta function

$$\mathbf{v}(\mathbf{X}(s,t)) = \int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}$$

From Lagrangian to Eulerian variables

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$$+\int_\mathcal{B}(\nabla_s\cdot\tilde{\mathbb{P}})\cdot\mathbf{v}(\mathbf{X}(s,t))\,ds - \int_{\partial\mathcal{B}}\tilde{\mathbb{P}}\mathbf{N}\cdot\mathbf{v}(\mathbf{X}(s,t))\,dA$$

$\Downarrow$ substitute $\mathbf{v}(\mathbf{X}(s,t))$

$$= -(\rho_s-\rho_f)\int_\mathcal{B}\frac{\partial^2\mathbf{X}}{\partial t^2}\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)ds$$
$$+\int_\mathcal{B}(\nabla_s\cdot\tilde{\mathbb{P}})\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)ds$$
$$-\int_{\partial\mathcal{B}}\tilde{\mathbb{P}}\mathbf{N}\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)dA$$

$$
\begin{aligned}
\int_\Omega (\rho\dot{\mathbf{u}} - \nabla\cdot\boldsymbol{\sigma}_f)\cdot\mathbf{v}\,d\mathbf{x}
&= -(\rho_s-\rho_f)\int_{\mathcal{B}} \frac{\partial^2\mathbf{X}}{\partial t^2}\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)ds\\
&\quad+\int_{\mathcal{B}}(\nabla_s\cdot\tilde{\mathbb{P}})\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)ds\\
&\quad-\int_{\partial\mathcal{B}}\tilde{\mathbb{P}}\mathbf{N}\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)dA
\end{aligned}
$$

$$\int_\Omega (\rho\dot{\mathbf{u}} - \nabla\cdot\boldsymbol{\sigma}_f)\cdot\mathbf{v}\,d\mathbf{x}$$

$$= -(\rho_s-\rho_f)\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)ds$$

$$+\int_{\mathcal{B}}(\nabla_s\cdot\tilde{\mathbb{P}})\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)ds$$

$$-\int_{\partial\mathcal{B}}\tilde{\mathbb{P}}\mathbf{N}\cdot\left(\int_\Omega \mathbf{v}(\mathbf{x})\delta(\mathbf{x}-\mathbf{X}(s,t))\,d\mathbf{x}\right)dA$$

$\Downarrow$ and change the order of the integrals

$$= -(\rho_s-\rho_f)\int_\Omega\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds\cdot\mathbf{v}\,d\mathbf{x}$$

$$+\int_\Omega\int_{\mathcal{B}}(\nabla_s\cdot\tilde{\mathbb{P}})\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds\cdot\mathbf{v}\,d\mathbf{x}$$

$$-\int_\Omega\int_{\partial\mathcal{B}}\tilde{\mathbb{P}}\mathbf{N}\delta(\mathbf{x}-\mathbf{X}(s,t))\,dA\cdot\mathbf{v}\,d\mathbf{x}$$

Since $\mathbf{v}$ is arbitrary, we get

$$
\begin{aligned}
\rho\dot{\mathbf{u}} - \nabla\cdot\boldsymbol{\sigma}_f &= -(\rho_s-\rho_f)\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds \\
&\quad + \int_{\mathcal{B}}\nabla_s\cdot\tilde{\mathbb{P}}\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds - \int_{\partial\mathcal{B}}\tilde{\mathbb{P}}\mathbf{N}\delta(\mathbf{x}-\mathbf{X}(s,t))\,dA
\end{aligned}
$$

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The source term can be split into three contributions: excess Lagrangian mass density

$$\mathbf{d}(\mathbf{x},t) = -(\rho_s-\rho_f)\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds$$

inner force density

$$\mathbf{g}(\mathbf{x},t) = \int_{\mathcal{B}}\nabla_s\cdot\tilde{\mathbb{P}}(s,t)\delta(\mathbf{X}(s,t)-\mathbf{x})\,ds$$

and transmission force density

$$\mathbf{t}(\mathbf{x},t) = -\int_{\partial\mathcal{B}}\tilde{\mathbb{P}}(s,t)\mathbf{N}(s)\delta(\mathbf{X}(s,t)-\mathbf{x})\,ds$$

Strong formulation

Navier–Stokes

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\,\mathbf{u}\right)-\mu\Delta\mathbf{u}+\nabla\,p=\mathbf{d}+\mathbf{g}+\mathbf{t}\quad \text{in } \Omega\times\,]0,T[$$

$$\operatorname{div}\mathbf{u}=0\qquad \text{in } \Omega\times\,]0,T[$$

Excess of Lagrangian mass and force densities in $\Omega\times]0,T[$

$$\mathbf{d}(\mathbf{x},t)=-(\rho_s-\rho_f)\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds$$

$$\mathbf{g}(\mathbf{x},t)=\int_{\mathcal{B}}\nabla_s\cdot\mathbb{P}(s,t)\delta(\mathbf{x}-\mathbf{X}(s,t))ds$$

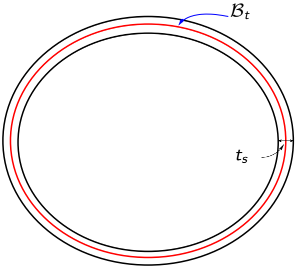
$$\mathbf{t}(\mathbf{x},t)=-\int_{\partial\mathcal{B}}\mathbb{P}(s,t)\mathbf{N}(s)\delta(\mathbf{x}-\mathbf{X}(s,t))ds$$

Immersed structure motion

$$\frac{\partial\mathbf{X}}{\partial t}(s,t)=\mathbf{u}(\mathbf{X}(s,t),t)\quad \text{in } \mathcal{B}\times\,]0,T[$$

Initial and boundary condition

Case: $m = d - 1$



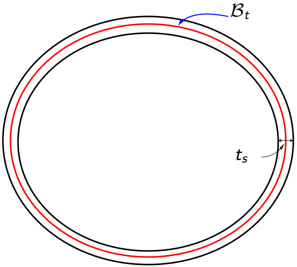
The domain occupied by the structure is $\mathcal{B}_t \times]-t_s/2, t_s/2[$.

The physical quantities depend only on the variables which represent a middle section and are constant in the direction orthogonal to it.

$$\int_\Omega \rho_f \dot{\mathbf{u}} \cdot \mathbf{v} d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \nabla \mathbf{v} d\mathbf{x} - \int_{\partial\Omega} \boldsymbol{\sigma}_f \mathbf{n} \cdot \mathbf{v} da$$

$$= - (\rho_s - \rho_f) \int_{-t_s/2}^{t_s/2} \int_{\mathcal{B}_t} \dot{\mathbf{u}} \cdot \mathbf{v} d\tilde{\mathbf{x}} \mathrm{d}\tau - \int_{-t_s/2}^{t_s/2} \int_{\mathcal{B}_t} \boldsymbol{\sigma}_s : \nabla \mathbf{v} d\tilde{\mathbf{x}} \mathrm{d}\tau$$

Case: $m = d - 1$



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$$
\begin{aligned}
&\int_\Omega \rho_f \dot{\mathbf{u}} \cdot \mathbf{v} d\mathbf{x} + \int_\Omega \boldsymbol{\sigma}_f : \nabla \mathbf{v} d\mathbf{x} - \int_{\partial\Omega} \boldsymbol{\sigma}_f \mathbf{n} \cdot \mathbf{v} da \\
&= -(\rho_s - \rho_f) \int_{-t_s/2}^{t_s/2} \int_{\mathcal{B}_t} \dot{\mathbf{u}} \cdot \mathbf{v} d\tilde{\mathbf{x}} \mathrm{d}\tau - \int_{-t_s/2}^{t_s/2} \int_{\mathcal{B}_t} \boldsymbol{\sigma}_s : \nabla \mathbf{v} d\tilde{\mathbf{x}} \mathrm{d}\tau \\
&= -(\rho_s - \rho_f) t_s \int_{\mathcal{B}_t} \dot{\mathbf{u}} \cdot \mathbf{v} d\tilde{\mathbf{x}} - t_s \int_{\mathcal{B}_t} \boldsymbol{\sigma}_s : \nabla \mathbf{v} d\tilde{\mathbf{x}},
\end{aligned}
$$

Working as before, we arrive at

$$
\begin{aligned}
\rho_f \dot{\mathbf{u}} - \nabla \cdot \boldsymbol{\sigma}_f = &-(\rho_s - \rho_f) t_s \int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}}{\partial t^2} \delta(\mathbf{x} - \mathbf{X}(\boldsymbol{\sigma}, t)) ds \\
&+ t_s \int_{\mathcal{B}} \nabla_s \cdot \tilde{\mathbb{P}} \delta(\mathbf{x} - \mathbf{X}(\boldsymbol{\sigma}, t)) ds \\
&- t_s \int_{\partial\mathcal{B}} \tilde{\mathbb{P}} \mathbf{N} \delta(\mathbf{x} - \mathbf{X}(\boldsymbol{\sigma}, t)) dA.
\end{aligned}
$$

Working as before, we arrive at

$$\begin{aligned}\rho_f \dot{\mathbf{u}} - \nabla\cdot\boldsymbol{\sigma}_f &= -(\rho_s-\rho_f)t_s\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\delta(\mathbf{x}-\mathbf{X}(\boldsymbol{\sigma},t))ds\\ &\quad + t_s\int_{\mathcal{B}}\nabla_s\cdot\tilde{\mathbb{P}}\delta(\mathbf{x}-\mathbf{X}(\boldsymbol{\sigma},t))ds\\ &\quad - t_s\int_{\partial\mathcal{B}}\tilde{\mathbb{P}}\mathbf{N}\delta(\mathbf{x}-\mathbf{X}(\boldsymbol{\sigma},t))dA.\end{aligned}$$

Excess of Lagrangian mass and force densities in $\Omega\times]0,T[$

$$\begin{aligned}\mathbf{d}(\mathbf{x},t) &= -(\rho_s-\rho_f)t_s\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds\\ \mathbf{g}(\mathbf{x},t) &= t_s\int_{\mathcal{B}}\nabla_s\cdot\tilde{\mathbb{P}}(s,t)\delta(\mathbf{x}-\mathbf{X}(s,t))ds\\ \mathbf{t}(\mathbf{x},t) &= -t_s\int_{\partial\mathcal{B}}\tilde{\mathbb{P}}(s,t)\mathbf{N}(s)\delta(\mathbf{x}-\mathbf{X}(s,t))dA\end{aligned}$$

NB: if $\mathcal{B}$ is a closed curve in 2D or a closed surface in 3D then $\partial\mathcal{B}$ is empty and the transmission force density $\boldsymbol{\tau}$ vanishes.

Strong formulation

Navier–Stokes

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}\right)-\mu\Delta\mathbf{u}+\nabla p=\mathbf{d}+\mathbf{g}+\mathbf{t}\quad \text{in } \Omega\times]0,T[$$

$$\operatorname{div}\mathbf{u}=0\quad \text{in } \Omega\times]0,T[$$

Excess of Lagrangian mass and force densities in $\Omega\times]0,T[$

$$\mathbf{d}(\mathbf{x},t)=-\delta\rho\int_{\mathcal{B}}\frac{\partial^2\mathbf{X}}{\partial t^2}\delta(\mathbf{x}-\mathbf{X}(s,t))\,ds$$

$$\mathbf{g}(\mathbf{x},t)=\int_{\mathcal{B}}\nabla_s\cdot\mathbb{P}(s,t)\delta(\mathbf{x}-\mathbf{X}(s,t))ds$$

$$\mathbf{t}(\mathbf{x},t)=-\int_{\partial\mathcal{B}}\mathbb{P}(s,t)\mathbf{N}(s)\delta(\mathbf{x}-\mathbf{X}(s,t))dA$$

$$\delta\rho=\begin{cases}\rho_s-\rho_f & \text{if } m=d\\ (\rho_s-\rho_f)t_s & \text{if } m=d-1\end{cases}\qquad \mathbb{P}=\begin{cases}\tilde{\mathbb{P}} & \text{if } m=d\\ t_s\tilde{\mathbb{P}} & \text{if } m=d-1.\end{cases}$$

Immersed structure motion

$$\frac{\partial\mathbf{X}}{\partial t}(s,t)=\mathbf{u}(\mathbf{X}(s,t),t)\quad \text{in } \mathcal{B}\times]0,T[$$

Variational formulation

- Navier–Stokes

$$\rho_f \frac{d}{dt}(\mathbf{u}(t),\mathbf{v}) + a(\mathbf{u}(t),\mathbf{v}) + b(\mathbf{u}(t),\mathbf{u}(t),\mathbf{v}) - (\operatorname{div}\mathbf{v}, p(t))$$
$$= -\delta\rho \int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}}{\partial t^2} \mathbf{v}(\mathbf{X}(s,t))ds + < \mathbf{F}(t), \mathbf{v} > \qquad \forall \mathbf{v} \in H^1_0(\Omega)^d$$
$$(\operatorname{div}\mathbf{u}(t), q) = 0 \qquad \forall q \in L^2_0(\Omega)$$

- $\langle \mathbf{F}(t), \mathbf{v} \rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\, ds \;\forall \mathbf{v} \in H^1_0(\Omega)^d$
- $\dfrac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$
- $\mathbf{u}(\mathbf{x},0) = \mathbf{u}_0(\mathbf{x}) \;\forall \mathbf{x} \in \Omega, \quad \mathbf{X}(s,0) = \mathbf{X}_0(s) \;\forall s \in \mathcal{B}.$

$$a(\mathbf{u},\mathbf{v}) = \mu(\nabla \mathbf{u}, \nabla \mathbf{v})$$

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \frac{\rho_f}{2}\left((\mathbf{u}\cdot\nabla \mathbf{v}, \mathbf{w}) - (\mathbf{u}\cdot\nabla \mathbf{w}, \mathbf{v})\right)$$

Equivalence with standard formulation

$m = d$

The definition of $\mathbf{g}$ and $\mathbf{t}$ implies that

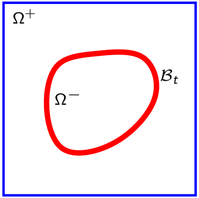
$$\mathbf{g}(\mathbf{x}, t) = 0 \quad \text{for } \mathbf{x} \neq \mathcal{B}_t, \qquad \mathbf{t}(\mathbf{x}, t) = 0 \quad \text{for } \mathbf{x} \neq \partial\mathcal{B}_t.$$

Recall that $\dot{\mathbf{u}}(\mathbf{X}(s,t),t) = \dfrac{\partial^2 \mathbf{X}}{\partial t^2}$. Then the given Navier-Stokes equations are equivalent to:

$$\begin{array}{ll}
\rho_f \left(\dfrac{\partial \mathbf{u}^f}{\partial t} + \mathbf{u}^f \cdot \nabla \mathbf{u}^f \right) - \boldsymbol{\sigma}^f = 0 & \text{in } (\Omega \setminus \mathcal{B}_t) \times]0, T[\\
\rho_s \left(\dfrac{\partial \mathbf{u}^s}{\partial t} + \mathbf{u}^s \cdot \nabla \mathbf{u}^s \right) - \boldsymbol{\sigma}^s_f - \boldsymbol{\sigma}^s_s = 0 & \text{in } \mathcal{B}_t \times]0, T[\\
\mathbf{u}^f = \mathbf{u}^s & \text{on } \partial\mathcal{B}_t \times]0, T[\\
\boldsymbol{\sigma}^f_f \mathbf{n}^f + \boldsymbol{\sigma}^s_f \mathbf{n}^s = \mathbf{t} \ (= |\mathbb{F}|^{-1} \tilde{\mathbb{P}} \mathbf{N}) & \text{on } \partial\mathcal{B}_t \times]0, T[
\end{array}$$

Equivalence with standard formulation

$$m = d - 1$$



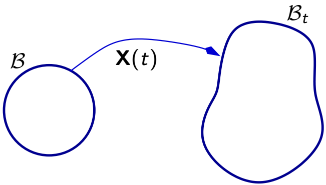
$$\rho_f\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\,\mathbf{u}\right)-\mu\Delta\mathbf{u}+\nabla\,p=0 \quad \text{in } (\Omega^+\cup\Omega^-)\times]0,T[$$

$$\nabla\cdot\mathbf{u}=0 \quad \text{in } (\Omega^+\cup\Omega^-)\times]0,T[.$$

$$u=\frac{\partial \mathbf{X}}{\partial t}\Rightarrow u_{|\Omega^+}=u_{|\Omega^-} \quad \text{on } \mathcal{B}_t$$

$$\left(\mu\left[\left[\frac{\partial u}{\partial \mathbf{n}}\right]\right]+[\![p]\!]\mathbf{n}\right)|\mathbb{F}|=-(\rho_s-\rho_f)t_s\frac{\partial^2\mathbf{X}}{\partial t^2}+t_s\nabla_s\,\mathbb{P} \quad \text{on } \mathcal{B}.$$

Incompressible and viscous hyper-elastic materials



Trajectory of a material point
$\mathbf{X} : \mathcal{B} \times [0, T] \to \mathcal{B}_t$
Deformation gradient
$$\mathbb{F}(s,t) = \frac{\partial \mathbf{X}}{\partial s}(s,t)$$

Wide class of elastic materials are characterized by:

- **potential energy density** $W(\mathbb{F}(s,t))$
- **total elastic potential energy**
$$E(\mathbf{X}(t)) = \int_{\mathcal{B}} W(\mathbb{F}(s,t))ds$$
- **Piola-Kirchoff stress tensor** $\tilde{\mathbb{P}}(s,t) = \dfrac{\partial W}{\partial \mathbb{F}}(s,t)$
- **inner force density** $\nabla_s \cdot \tilde{\mathbb{P}}(s,t) = \nabla_s \cdot \left(\dfrac{\partial W}{\partial \mathbb{F}}(\mathbb{F}(s,t))\right)$

Stability

<Boffi–Cavallini–G. '10>

Recalling that

$$\frac{\partial \mathbf{X}}{\partial t}(s,t) = \mathbf{u}(\mathbf{X}(s,t),t) \quad \forall s \in \mathcal{B}$$

it holds

$$\frac{\rho_f}{2}\frac{d}{dt}||\mathbf{u}(t)||_0^2 + \mu||\nabla \mathbf{u}(t)||_0^2 + \frac{d}{dt}E(\mathbf{X}(t)) + \frac{1}{2}\delta\rho\frac{d}{dt}\left\|\frac{\partial \mathbf{X}}{\partial t}\right\|_B^2 = 0$$

where E is the total elastic potential energy

$$E\left(\mathbf{X}(t)\right) = \int_{\mathcal{B}} W(\mathbb{F}(s,t))\,ds$$

Fluid - Structure coupling

Let $\Omega_t = \Omega_t^f \cup \Omega_t^s$ be a time dependent region, with Ω_t^f fluid region, Ω_t^s solid region

in Ω_t^f:
incompressible Navier-Stokes equations with Eulerian or ALE formulation
unknowns: velocity $\mathbf{u}$ and pressure p

in Ω_t^s:
elastodynamics equation with Lagrangian formulation
unknown: structure displacement η

coupled through transmission condition on $\Sigma_t = \partial\Omega_t^s \cap \partial\Omega_t^f$:
continuity of velocities and stresses between fluid and structure

Numerical strategies for coupled problems

<Matthies-Niekamp '03>
<Matthies-Niekamp-Steindorf '06>

Two approaches:

- Strongly coupled or monolithical
  Direct solution of the coupled problem.
  Stable but requires the solution of a *big* nonlinear problem.
- Weakly coupled or partitioned
  Separate solution of fluid and solid equations iteratively.
  Considerable reduction of computational cost, but undesirable instability effects.

Added-mass and partitioned procedure

In coupled FSI problems, the fluid acts over the structure as an added-mass at the interface.

When $\rho_s \gg \rho_f$ (ρ_s and ρ_f solid and fluid density) the added-mass effect is negligible and standard partitioned procedures converge in few iterations.

The method fails to converge when $\rho_s/\rho_f \approx 1$.

<Nobile '01, Causin-Gerbeau-Nobile '05>

Different strategies have been proposed based on fully coupled implicit algorithms solved via iterative solvers using domain decomposition or algebraic splitting:

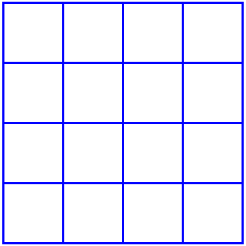
<Le Tallec-Mouro '01, Deparis-Fernández-Formaggia '03>
<Badia-Nobile-Vergara '08, Giorda-Nobile-Vergara '09>
<Badia-Quaini-Quarteroni '08-'09>

or based on time-discretization via operator splitting:

<Guidoboni-Glowinski-Cavallini-Canic '09>

Finite element approximation

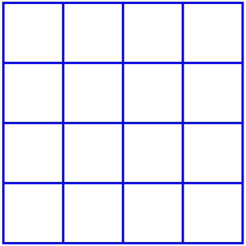
- Uniform background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)



Finite element approximation

- Uniform background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- Inf-sup stable finite element pair

$$
\begin{aligned}
V_h &= \{\mathbf{v} \in H^1_0(\Omega)^d : \mathbf{v} \in Q2\} \\
Q_h &= \{q \in L^2_0(\Omega) : q \in P1\}
\end{aligned}
$$



Finite element approximation

- Uniform background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- Inf-sup stable finite element pair

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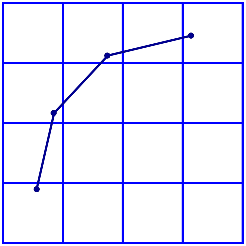
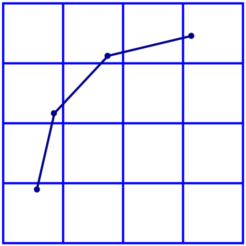
$$Q_h = \{q \in L^2_0(\Omega) : q \in P1\}$$

- Grid $\mathcal{S}_h$ for $\mathcal{B}$ (meshsize h_s)



Finite element approximation

- Uniform background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- Inf-sup stable finite element pair

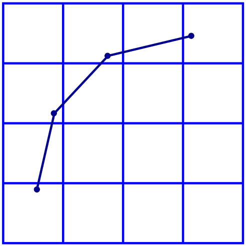


$$V_h = \{\mathbf{v} \in H^1_0(\Omega)^d : \mathbf{v} \in Q2\}$$
$$Q_h = \{q \in L^2_0(\Omega) : q \in P1\}$$

- Grid $\mathcal{S}_h$ for $\mathcal{B}$ (meshsize h_s)
- Piecewise linear finite element space for $\mathbf{X}$
  $S_h = \{\mathbf{Y} \in C^0(\mathcal{B}; \Omega) : \mathbf{Y} \in P1\}$

Finite element approximation

- Uniform background grid $\mathcal{T}_h$ for the domain Ω (meshsize h_x)
- Inf-sup stable finite element pair



$$V_h = \{\mathbf{v} \in H^1_0(\Omega)^d : \mathbf{v} \in Q2\}$$
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- Grid $\mathcal{S}_h$ for $\mathcal{B}$ (meshsize h_s)
- Piecewise linear finite element space for $\mathbf{X}$
  $S_h = \{\mathbf{Y} \in C^0(\mathcal{B};\Omega) : \mathbf{Y} \in P1\}$

Notation

- T_k, $k = 1, \dots, M_e$ elements of $\mathcal{S}_h$
- $\mathbf{s}_j$, $j = 1, \dots, M$ vertices of $\mathcal{S}_h$
- $\mathcal{E}_h$ set of the edges e of $\mathcal{S}_h$

Discrete source term

Source term:

$$\langle \mathbf{F}(t), \mathbf{v}\rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}_h(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}_h(s,t))\, ds \quad \forall \mathbf{v} \in V_h$$

Discrete source term

Source term:

$$\langle \mathbf{F}(t), \mathbf{v} \rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}_h(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}_h(s,t))\, ds \quad \forall \mathbf{v} \in V_h$$

$\mathbf{X}_h$ p.w. linear $\Rightarrow \mathbb{F}_h$, $\mathbb{P}_h$ p.w. constant

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$\mathbf{X}_h$ p.w. linear $\Rightarrow \mathbb{F}_h$, $\mathbb{P}_h$ p.w. constant

By integration by parts

$$\begin{aligned} \langle \mathbf{F}_h(t), \mathbf{v} \rangle_h &= -\sum_{k=1}^{M_e} \int_{T_k} \mathbb{P}_h : \nabla_s \mathbf{v}(\mathbf{X}(s,t))\, ds \\ &= -\sum_{k=1}^{M_e} \int_{\partial T_k} \mathbb{P}_h \mathbf{N} \mathbf{v}(\mathbf{X}(s,t))\, dA \end{aligned}$$

Discrete source term

Source term:

$$\langle \mathbf{F}(t), \mathbf{v} \rangle = -\int_{\mathcal{B}} \mathbb{P}(\mathbb{F}_h(s,t)) : \nabla_s \mathbf{v}(\mathbf{X}_h(s,t))\, ds \quad \forall \mathbf{v} \in V_h$$

$\mathbf{X}_h$ p.w. linear $\Rightarrow \mathbb{F}_h$, $\mathbb{P}_h$ p.w. constant

By integration by parts

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that is

$$\langle \mathbf{F}_h(t), \mathbf{v} \rangle_h = -\sum_{e \in \mathcal{E}_h} \int_e [\![\mathbb{P}_h]\!] \cdot \mathbf{v}(\mathbf{X}(s,t))\, dA$$

$[\![\mathbb{P}]\!] = \mathbb{P}^+\mathbf{N}^+ + \mathbb{P}^-\mathbf{N}^-$ jump of $\mathbb{P}$ across e for internal edges

$[\![\mathbb{P}]\!] = \mathbb{P}\mathbf{N}$ *jump* when $e \subset \partial\mathcal{B}$

The *semidiscrete* problem becomes: find $(\mathbf{u}_h, p_h) :]0,T[\to V_h \times Q_h$ and $\mathbf{X}_h : [0,T] \to S_h$ such that

$$\begin{cases} \rho_f \dfrac{d}{dt}(\mathbf{u}_h(t),\mathbf{v}) + a(\mathbf{u}_h(t),\mathbf{v}) + b(\mathbf{u}_h(t),\mathbf{u}_h(t),\mathbf{v}) \\ \qquad -(\operatorname{div}\mathbf{v}, p_h(t)) = -\delta\rho \displaystyle\int_{\mathcal{B}} \frac{\partial^2 \mathbf{X}_h}{\partial t^2}\mathbf{v}(\mathbf{X}_h(s,t))ds \\ \qquad\qquad - \displaystyle\sum_{e\in\mathcal{E}_h}\int_e [\![\mathbb{P}_h]\!]\cdot\mathbf{v}(\mathbf{X}_h(s,t))dA \qquad \forall \mathbf{v}\in V_h \\ (\operatorname{div}\mathbf{u}_h(t), q) = 0 \qquad \forall q \in Q_h \end{cases}$$

$$\frac{d\mathbf{X}_{hi}}{dt}(t) = \mathbf{u}_h(\mathbf{X}_{hi}(t),t) \quad \forall i = 1,\dots,M$$

$$\mathbf{u}_h(0) = \mathbf{u}_{0h} \text{ in } \Omega$$

$$\mathbf{X}_{hi}(0) = \mathbf{X}_0(s_i) \quad \forall i = 1,\dots,M$$

Semidiscrete stability

<Boffi–Cavallini–G. '10>

$$\frac{\rho}{2}\frac{d}{dt}||\mathbf{u}_h(t)||_0^2+\mu||\nabla\mathbf{u}_h(t)||_0^2+\frac{d}{dt}E(\mathbf{X}_h(t))\\+\frac{1}{2}\delta\rho\frac{d}{dt}\left\|\frac{\partial\mathbf{X}_h}{\partial t}\right\|_B^2=0.$$

Fully discrete problem
Backward Euler – **BE**

Find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ e $\mathbf{X}_h^{n+1} \in S_h$ such that

$$\langle \mathbf{F}_h^{n+1}, \mathbf{v}\rangle_h = -\sum_{e\in\mathcal{E}_h}\int_e [\![\mathbb{P}_h]\!]^{n+1} \cdot \mathbf{v}(\mathbf{X}_h^{n+1}(s))dA \qquad \forall \mathbf{v} \in V_h$$

$$\textbf{NS}\begin{cases} \rho_f\left(\dfrac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t},\mathbf{v}\right) + a(\mathbf{u}_h^{n+1},\mathbf{v}) + b(\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1},\mathbf{v}) \\ \quad -(\operatorname{div}\mathbf{v}, p_h^{n+1}) = \\ \quad -\delta\rho\displaystyle\int_{\mathcal{B}}\frac{\mathbf{X}_h^{n+1}-2\mathbf{X}_h^n+\mathbf{X}_h^{n-1}}{\Delta t^2}\cdot\mathbf{v}(\mathbf{X}_h^{n+1}(s))ds \\ \qquad + < \mathbf{F}_h^{n+1}, \mathbf{v} >_h \qquad\qquad \forall \mathbf{v}\in V_h \\ (\operatorname{div}\mathbf{u}_h^{n+1}, q) = 0 \qquad \forall q \in Q_h; \end{cases}$$

$$\frac{\mathbf{X}_{hi}^{n+1}-\mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^{n+1}) \quad \forall i = 1,\dots,M.$$

Fully discrete problem

Modified bckward Euler – **MBE**

Step 1. $\langle \mathbf{F}_h^n, \mathbf{v}\rangle_h = -\sum_{e\in\mathcal{E}_h}\int_e [\![\mathbb{P}_h]\!]^n \cdot \mathbf{v}(\mathbf{X}_h^n(s,t))\,dA \qquad \forall \mathbf{v}\in V_h$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h \times Q_h$ such that

$$\text{NS}\begin{cases} \rho_f\left(\dfrac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t},\mathbf{v}\right) + a(\mathbf{u}_h^{n+1},\mathbf{v}) + b(\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1},\mathbf{v}) \\ \quad -(\operatorname{div}\mathbf{v}, p_h^{n+1}) = \\ \quad -\delta\rho\displaystyle\int_{\mathcal{B}} \frac{\mathbf{X}_h^{n+1}-2\mathbf{X}_h^n+\mathbf{X}_h^{n-1}}{\Delta t^2}\cdot\mathbf{v}(\mathbf{X}_h^n(s))ds \\ \qquad +\langle \mathbf{F}_h^n,\mathbf{v}\rangle_h & \forall \mathbf{v}\in V_h \\ (\operatorname{div}\mathbf{u}_h^{n+1}, q) = 0 & \forall q\in Q_h; \end{cases}$$

Step 3. $\dfrac{\mathbf{X}_{hi}^{n+1}-\mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i=1,\dots,M.$

Using **Step 3** in **Step 2** we arrive at:

Step 1. $\langle \mathbf{F}_h^n, \mathbf{v}\rangle_h = -\sum_{e\in\mathcal{E}_h}\int_e [\![\mathbb{P}_h]\!]^n \cdot \mathbf{v}(\mathbf{X}_h^n(s,t))\,dA \qquad \forall \mathbf{v}\in V_h$

Step 2. find $(\mathbf{u}_h^{n+1}, p_h^{n+1}) \in V_h\times Q_h$ such that

$$
\textbf{NS}\begin{cases}
\rho_f\left(\dfrac{\mathbf{u}_h^{n+1}-\mathbf{u}_h^n}{\Delta t},\mathbf{v}\right) + a(\mathbf{u}_h^{n+1},\mathbf{v}) + b(\mathbf{u}_h^{n+1},\mathbf{u}_h^{n+1},\mathbf{v}) \\
\quad -(\operatorname{div}\mathbf{v}, p_h^{n+1}) = \\
\quad -\delta\rho\displaystyle\int_{\mathcal{B}}\frac{\mathbf{u}_h^{n+1}(\mathbf{X}_h^n(s)) - \mathbf{u}_h^n(\mathbf{X}_h^{n-1}(s))}{\Delta t}\cdot\mathbf{v}(\mathbf{X}_h^n(s))ds \\
\qquad +\langle \mathbf{F}_h^n,\mathbf{v}\rangle_h \qquad \forall\mathbf{v}\in V_h \\
(\operatorname{div}\mathbf{u}_h^{n+1}, q) = 0 \qquad \forall q\in Q_h;
\end{cases}
$$

Step 3. $\dfrac{\mathbf{X}_{hi}^{n+1}-\mathbf{X}_{hi}^n}{\Delta t} = \mathbf{u}_h^{n+1}(\mathbf{X}_{hi}^n) \quad \forall i = 1,\dots,M.$

Discrete Energy Estimate

<Boffi–Cavallini–G. '10>

Assumption Set

$$\mathbb{H}_{i\alpha j\beta} = \frac{\partial^2 W}{\partial \mathbb{F}_{i\alpha} \partial \mathbb{F}_{j\beta}}(\mathbb{F})$$

There exist $\kappa_{min} > 0$ and $\kappa_{max} > 0$ s.t. for all tensors $\mathbb{E}$

$$\kappa_{min}\mathbb{E}^2 \le \mathbb{E} : \mathbb{H} : \mathbb{E} \le \kappa_{max}\mathbb{E}^2$$

Artificial Viscosity Theorem

Let $\mathbf{u}_h^n$, p_h^n and $\mathbf{X}_h^n$ be a solution to the FE-IBM.
Let Σ_n be the sum of the kinetic and elastic energy:

$$\Sigma_n = \frac{\rho_f}{2} \|\mathbf{u}^n\|^2_{0,\Omega} + \frac{\delta\rho}{2} \left\| \frac{\mathbf{X}^n - \mathbf{X}^{n-1}}{\Delta t} \right\|^2_{0,\mathcal{B}} + E\left[\mathbf{X}_h^n\right].$$

Then $$\Sigma_{n+1} - \Sigma_n + (\mu + \mu_a)\|\nabla \mathbf{u}_h^{n+1}\|_0^2 \le 0$$

CFL Condition: $\mu + \mu_a > 0$

CFL condition

BE is unconditionally stable, while MBE requires the term μ_a to be not too large

$$\mu_a = -\kappa_{max} C \frac{h_s^{(m-2)} \Delta t}{h_x^{(d-1)}} L^n$$

space dim.	solid dim.	CFL condition
2	1	$L^n \Delta t \leq C h_x h_s$
2	2	$L^n \Delta t \leq C h_x$
3	2	$L^n \Delta t \leq C h_x^2$
3	3	$L^n \Delta t \leq C h_x^2 / h_s$

Numerical experiments

- $\Omega =]0,1[\times]0,1[$, $\mathcal{B}=[0,L]$
- $\rho_s = \rho_f$
- $\mathbb{P} = \kappa \dfrac{\partial \mathbf{X}}{\partial s}$

Numerical parameters

Ω partitioned into 16 by 16 subsquares
Immersed boundary discretized by 618 uniformly spaced nodes
$\Delta t = .01$

V_h continuous piecewise biquadratics $\Rightarrow Q_2$
Q_h discontinuous piecewise linears $\Rightarrow P_1$
S_h continuous piecewise linears

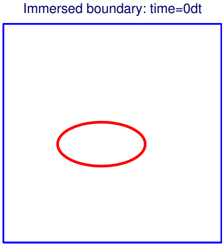
Programs written in **C++** with the support of `deal.II` libraries.
Pictures and movies obtained with Matlab and IBM Opendx.

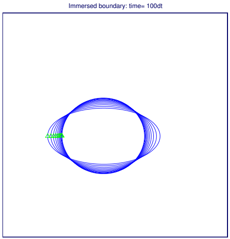
Ellipse immersed in a static fluid

Aim: to examine the influence of the elastic force of the immersed boundary on the whole system.

Fluid initially at rest: $\mathbf{u}_{0h} = 0$

$$\mathbf{X}_0(s) = \begin{pmatrix} 0.2\cos(2\pi s) + 0.45 \\ 0.1\sin(2\pi s) + 0.45 \end{pmatrix} \quad s \in [0,1],$$





Percentage of area loss at t=2.

M	16	32	64	128	256	512
$N = 4$	$\boxed{36.468}$	35.944	37.166	37.988	38.413	38.635
$N = 8$	$\boxed{15.951}$	$\boxed{14.090}$	13.057	12.809	12.809	12.843
$N = 16$	20.182	$\boxed{9.015}$	$\boxed{7.255}$	7.011	7.105	7.191
$N = 32$	45.293	9.763	$\boxed{2.788}$	$\boxed{2.308}$	2.303	2.325

Inflated baloon in a fluid at rest

Structure model:

$$\langle F(t), v \rangle = \kappa \int_0^L \frac{\partial^2 X(s,t)}{\partial s^2} v(X(s,t))\, ds, \quad \kappa = 1$$

Initial condition:

$$X_0(s) = \begin{pmatrix} R\cos(s/R) + 0.5 \\ R\sin(s/R) + 0.5 \end{pmatrix}, \quad s \in [0, 2\pi R] \qquad R = 0.4.$$

Analytical solution:

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{0} \quad \forall \mathbf{x} \in \Omega, \quad \forall t \in]0, T[$$

$$p(\mathbf{x}, t) = \begin{cases} \kappa(1/R - \pi R), & |\mathbf{x}| \le R \\ -\kappa \pi R, & |\mathbf{x}| > R \end{cases} \quad \forall t \in]0, T[.$$

Q2/P1^d solution

Uniform grid: 64×64 squares, $\Delta t = 0.005$, $T = 3s$

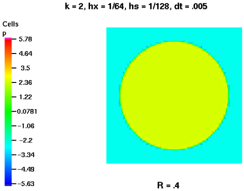


Figure: Pressure

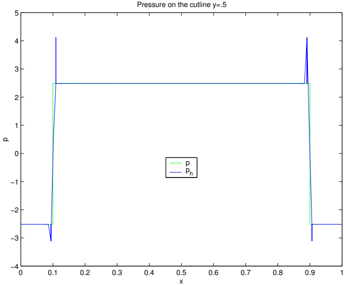
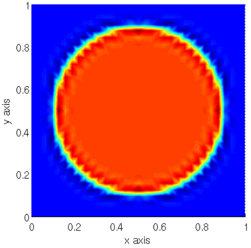


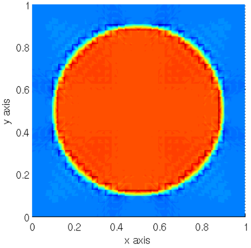
Figure: Pressure cutline

P1isoP2/P1^c and P1isoP2/P1^c+P0 solutions

Uniform grid: 64×64 squares squares divided into two triangles
$\Delta t = 0.0001, \; T = 0.1s$



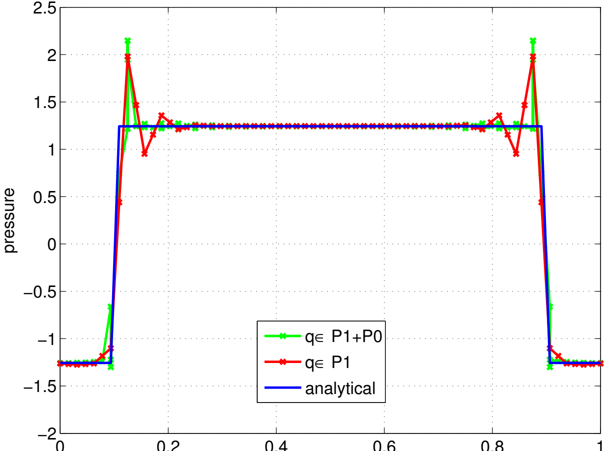
Pressure P1isoP2/P1^c



Pressure P1isoP2/P1^c+P0

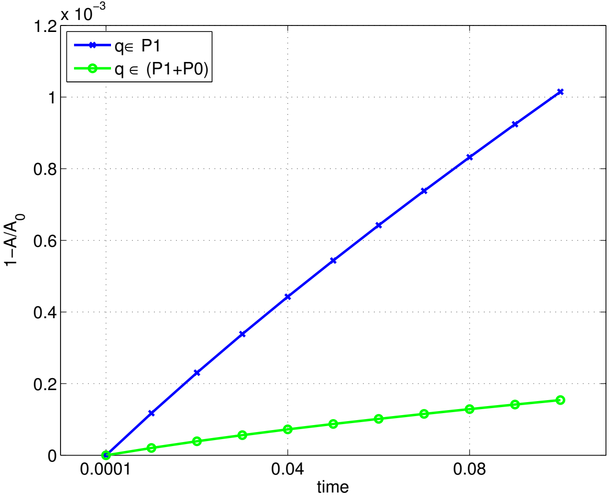
P1isoP2/P1^c versus P1isoP2/P1^c+P0 cutline

Uniform grid: 64×64 squares divided into two triangles

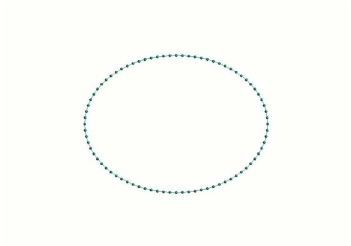


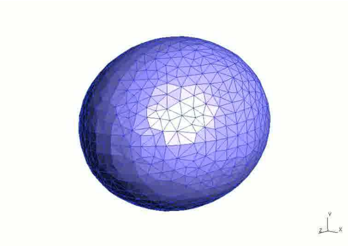
Area loss

Uniform grid: 64×64 squares divided into two triangles



Examples of instability





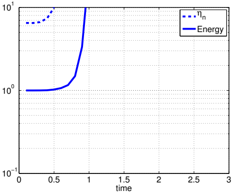
Stability analysis

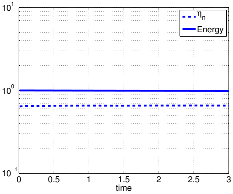
Plot: $-\mu_a$ VS Σ_n for different μ and Δt

$$\boxed{\mu_a = -\kappa C \frac{\Delta t}{h_s h_x} L^n}$$

$\rho_f = 1$, $\rho_s = 2$, $\kappa = 1$, `N=64`, `M=1024`,

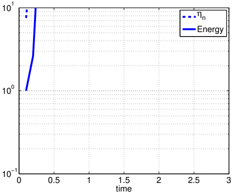
$\mu = 1$

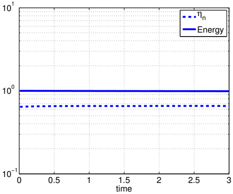




$\Delta t = 10^{-1}$ $\Delta t = 10^{-2}$

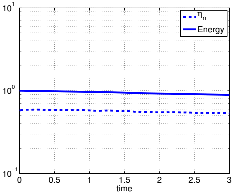
$\mu = 0.1$



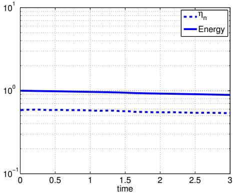


Stability analysis for different ρ_s

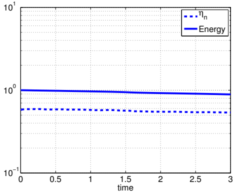
$\rho_f = 1$, $\mu = 0.1$, $\kappa = 1$, `N=64`, `M=1024`,



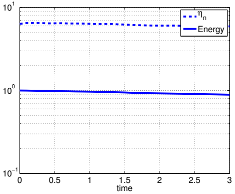
$\rho_s = 11$



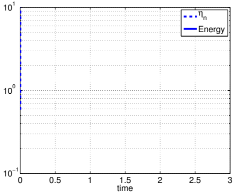
$\rho_s = 1.1$



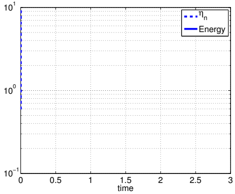
$\rho_s = 0.9$



$\rho_s = 0.8$



$\rho_s = 0.4$



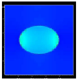
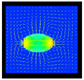
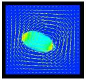
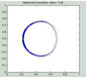
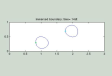
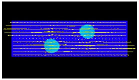
$\rho_s = 0.2$

Some numerical experiments
Equal densities

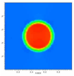
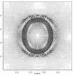
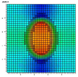
Original 2D code in Fortran 77, ported to DEAL.II (c++) (`www.dealii.org`) by L. Heltai

Rectangular mesh, $Q2/P1^d$

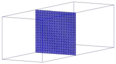
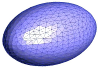
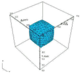
2D - Codimension 1

2D - Codimension 0

3D - Codimension 1

Some numerical experiments

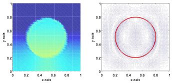
Different densities

Original 2D code in Fortran 90 by N. Cavallini

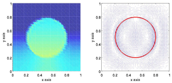
Triangular mesh P1isoP2/P1^c

Inflated balloon falling down in a liquid

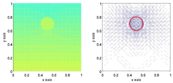
Densities: $\rho_s = 21$ and $\rho_f = 1$



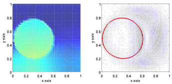
$\kappa = 1$



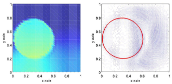
$\kappa = 0.1$



$\kappa = 0.1$



$\kappa = 1$



$\kappa = 0.1$

Conclusions

- The Immersed Boundary Method is extended to the treatment of thick materials modeled by viscous-hyper-elastic constitutive laws.
- The case of fluid and structure with different densities has been also considered.
- The finite element approach is efficient and can easily handle the case of thick materials.
- Stability analysis of the partitioned space-time discretization is provided.
- The CFL condition does not depend on the values of ρ_f and of ρ_s.