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Master Degree in Earthquake Engineering

Geometrically exact three-dimensional beam theory: modeling and FEM implementation for statics and dynamics analysis

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Abstract

This thesis presents a geometrically exact theory for elastic beams and its finite element formulation and implementation. Since there is a growing interest in the treatment of structural nonlinearities, geometrically exact analysis has increased its appealing in engineering community basically for its innate capability to fully reproduce large displacements in three-dimensional space, without the approximations typical of co-rotational approaches.

In the present study, the original beam formulation based on the Reissner-Simo geometrically exact approach is investigated and re-examined to provide an organic and unitary treatment, suitable for FEM implementation. Since the examination of large rotations is of paramount importance to understand the kinematic of continuum body in 3D space, the rotation manifold is thoroughly investigated, and the formulation is developed within the mathematical framework of differential geometry on manifolds. The beam cross-section is assumed to remain rigid, but it can undergoes any general three-dimensional movement, without conserving normality with beam axis (i.e. shear deformations are properly taken into account). The relevant engineering strain measures at any material point on the beam cross-section are obtained through the deformation gradient tensor, whereas the stress resultants and couples are defined in the classical sense. The governing equations of motion are derived from linear and angular momentum balance.

This work derives also a weak formulation of the virtual work principle, suitable to be discretized in space by means of the standard Galerkin finite element approach. The classical Newmark scheme is included for the step-by-step integration in time.

The study concludes with a suite of numerical simulations, performed with the aim to illustrate the good accuracy and effectiveness of the FEM code relying on the presented formulation. Examples include elastic finite deformation responses of simple structures in both static and dynamic regime.

Keywords: geometrically exact beam theory; nonlinear dynamics; finite rotations; finite element method; elastic material

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Chapter 1

Introduction

Load-displacement behaviors of all mechanical systems existing in nature are not linear both in geometry and material: linearity is only a convenient artifice created by lazy.

A. Mastropasqua

1.1 From linear to nonlinear analysis.

The ability to predict the structural behavior under assigned external loading is a traditional need in engineering. To this end, a sound theoretical framework, accurate modeling and suitable analysis strategies need to be available. Great efforts have been made in this direction, particularly in the last two centuries, leading to the classical Cauchy theory of elastic three-dimensional bodies, the Saint Venant beam theory, the plates and shells theories and, more recently, due to the great improvements in computational tools, such as the finite element method.

The *small displacements* assumption, allowing reference to relatively simple formulation, has played an important role in this evolution. Many of the results obtained by the research, in particular the derivation of structures models such as beams and plates from the three-dimensional continuum, and their description in terms of finite elements, are strictly related to this assumption. Unfortunately, the linear approximation is not always applicable for a realistic prediction of the structural response. This leads the desire to extend the approach to the more general set of nonlinear problems.

The three-dimensional geometrical nonlinear analysis of beams has captured the interest of many researchers during the past decades, and it still constitutes an active branch of research. In the field of earthquake engineering, second-order effects, such as the so-called $P-\Delta$, which occurs due to the changes of configuration of the structure during the earthquake, frequently affects the seismic design of high-rise buildings or bridge piers.

Considering that a great part of the elements are prismatic with one dimension grater than the other two, reduced or one-dimensional (1D) formulations appears as solution combining both numerical precision and reasonable computational costs when compared with fully three-dimensional (3D) descriptions. In fact, even though fully 3D numerical models are more accurate, on the other hand the required computing time makes their applications unpractical.

In addition, in the past structural analysis for practical purposes was restricted to small deformations response and elastic case. However, in several areas of engineering the inelastic nonlinear response of the structures is often strictly required, as in the case of earthquake engineering, where the behavior of structures subjected to ground motion is significantly characterized by the damage to structural elements. In this contest, structural damage results in inelastic deformations occurring during the earthquake, which in turns lead to large displacements and large rotations in the general three-dimensional space. Traditional analysis methods based on the linear elastic behavior of the structure can predict such kind of behavior only implicitly, while nonlinear methods are naturally well suited to provide the direct prediction of the expected inelastic displacements and deformations. Even though the latter approach is recognized to be precise and complete than the former, nonlinear modeling requires significant knowledge from the analyst in order to account properly both the material and geometrical nonlinearities involves in the practical problems.

1.2 State-of-art of geometrically exact beam theory.

Since Euler [19], a one-dimensional continuum called *beam* was used as an adequate representation for the class of three-dimensional bodies having one of the three dimensions significantly grater that two others. Nowadays, beam models have found applications in civil, mechanical and aero-space engineering. Exact and efficient nonlinear analysis of structures, build up from beam components, using robust numerical methods, e.g. finite element methods, should be based on proper nonlinear beam theories.

Since Saint Venant and Kirchhoff, the derivation of a beam model from the three-dimensional theory has been based on some simplifying kinematic hypotheses. At the beginning, such an approach allowed to derive effectively the simplest one-dimensional models. Later, the application of kinematic hypothesis became just the natural element of the derivation. It took the form of an automatic adoption of a more or less advanced imagination about the deflected shape. But a real deformation process is usually very complicated indeed, and cannot be a priori prescribed. As consequence, the use of any kinematic hypothesis introduces inevitably some approximations into such a model already from the very beginning, imposing artificial constraints on the continuum motion model.

It's a fact the model which was long considered the stone of a structural engineer (the Euler-Bernoulli beam theory) is even nowadays introduced within the framework of geometrically linear theory, limited to small or rather infinitesimal displacements, rotations and deformations. As mentioned by Makinen [36], it seems it is long forgotten in part of the mechanics community, that the original developments of beam model of "Euler elastica" were indeed presented in a geometrically nonlinear setting.

It was only with work of Reissner (see [52]), on beam theory capable of dealing with arbitrary large displacements and deformations and moderate rotations, that interest was spurred again in truly geometrically nonlinear models. Geometrically exact beam theory is sometimes referred as the *Reissner's beam theory*, but strictly speaking, the latter is geometrically exact only in 2D (see [50]). Treatment of rotations in 3D becomes nontrivial primarily because of the nonlinear character of 3D rotations in space. For this reason Reissner [52] proposed a simplification of the rotation matrix, which enabled the derivation of the required strain-configuration relationship, but unfortunately, also spoiled the geometric exactness of the theory. However, Reissner's finite strain beam theory is one of the most important ones, subsequentially extended and used by many other authors for 2-D and 3-D cases for both static and dynamic problems.

In a modern contest, the research on geometrically exact beam theory with finite element implementation is initiated by Simo and Vu-Quoc, and has been mainly developed by Simo and Vu-Quoc, Cardona and Gèradin, Ibrahimbegovic *et al.*

In the early and pioneering work of Simo [56], the author gave a dynamic formulation for Reissner's beam. In that paper, a spin rotation vector is used as a variable, and the beam placement is updated with the aid of a rotation tensor and an exponential mapping. The main drawbacks of this formulation are that: (i) the consistent stiffness tensor is an unsymmetrical tensor away from equilibrium, (ii) the need for secondary storage variables (quaternions) and their manipulations, (iii) the solution has a path-dependent property even when a conservative loading is applied. Later, Simo and Vu-Quoc [59] implemented the numerical integration of the beam equations of motion in the context of the finite element method, both for static and dynamic cases. It is also in this work that Simo first introduced the still used terminology *geometrically exact beam* to indicate that Reissner's theory was recasted in a form which is valid for any magnitude of displacements and rotations [1].

In an important paper, Cardona and Geradin [16] gave another finite element implementation for Reissner beam element with a different updating procedure, based on an updated Lagrangian formulation with the rotation vector as a dependent variable. This formulation can bypass the singularity problem of the total Lagrangian formulation, which takes place when the rotation angle approaches the angle 2π, and its multiples. The updated Lagrangian formulation has additional benefits, such as a fully symmetrical stiffness tensor when applying a conservative loading. On the other hand, it requires some secondary storage variables for the curvature and rotation vector at every spatial integration point.

Ibrahimbegovic et al. [31]proposed a total Lagrangian formulation in a static cases, with the stiffness tensor consistently derived. The consistent stiffness tensor, which is a symmetric tensor, has the same form in the total and updated Lagrangian formulation, and is considerably more complicated than the consistent stiffness tensor in an Eulerian formulation, which leads to an unsymmetrical stiffness tensor away from the equilibrium.

[1]Strictly speaking, geometrically exact beam theory doesn't account large strains as soon as a linear elastic constitutive law for stress resultants and couples is used.

Also Makinen [37] provided a total Lagrangian geometrically exact finite element formulation, where the singularity problem at the rotation angle 2π is bypassed by adopting the a wise change of parametrization of the rotation manifold.

Even though the three-dimensional nonlinear analysis of beam structures captured the interest of many researchers during the past decades, as cited by Mata et al. [41] many contributions have been focused on the formulation of geometrically consistent models of beams undergoing large displacements and rotations, but considering material linear elastic behavior and, therefore, employing simplified linear constitutive relations in terms of cross-sectional forces and moments. Only few works have been carried out using fully geometrical and material nonlinear formulations for beams, but they have mainly focused on plasticity (see e.g. Mata et al. [41]).

More recently, Mata et al. [41] included the general nonlinear constitutive behavior in the static version of the geometrically exact formulation for beams proposed by Simo [56], considering an intermediate curved reference configuration between the straight reference beam and the current configuration.

An important effort has been devoted to develop time-stepping schemes for the integration of the nonlinear dynamic equations of motion involving finite rotations. The main difficulty arises in the fact that the deformation map takes values in the differentiable manifold $SO(3) \times \mathbb{R}^3$ and not in a linear space, as it is the case in classical dynamics. An implicit time-stepping algorithm is developed in Simo et al. [59] extending the classical Newmark's scheme to $SO(3)$, and obtaining a formulation similar to that of the linear case. A comparison among implicit time stepping schemes according to different choices of rotational parameters can be reviewed in Ibrahimbegovic et al. [34].

Even when Newmark's scheme has been widely applied to the study of the dynamic response of structures, Makinen [35] stated that it only constitutes an approximated version of the corrected algorithm, which are given in his work for the spatial and material description.

Recently, Mata et all. [42] developed a fully geometric and constitutive nonlinear model for the description of the dynamic behavior of beam element. The proposed formulation is based on the geometrically exact formulation for beams due to Simo, but an intermediate reference configuration is considered. A complete treatment of the constitutive nonlinearity in the context of fiber-like approaches including the corresponding cross-sectional analysis is adopted. Viscosity is included at the constitutive level by means of a thermodynamically consistent visco-damage model, developed in terms of the material description of the first Piola-Kirchhoff stress vector. The motion in time is managed using an appropriated version of Newmark's scheme in updating the kinematic variables.

From a more engineering viewpoint, Mata et al. [43] tries to combine simplicity and sophistication by coupling a reduced models for prismatic elements with full 3D models for the connecting joints. Basically they adopted a two-scale approach developed for obtaining the nonlinear dynamic response of RC buildings with local non-prismatic parts. In particular, at global scale level, elements are prismatic rods based on the geometrically exact formulation due to Simo and Reissner, appropriately extended to include material nonlinearities. A fully 3D model is introduced to describe the local scale level, along with a surface-interfaces kinematic hypothesis assumed to manage the coupling between two scales. Special attention is also paid to the use of damage indices able of estimating the remaining load carrying capacity of structures after a seismic action. According to authors, such an approach appears as a convenient formulation for studying the dynamic nonlinear behavior of realistic RC structures, which response is dominated by local irregularities such as in the case of precast structures.

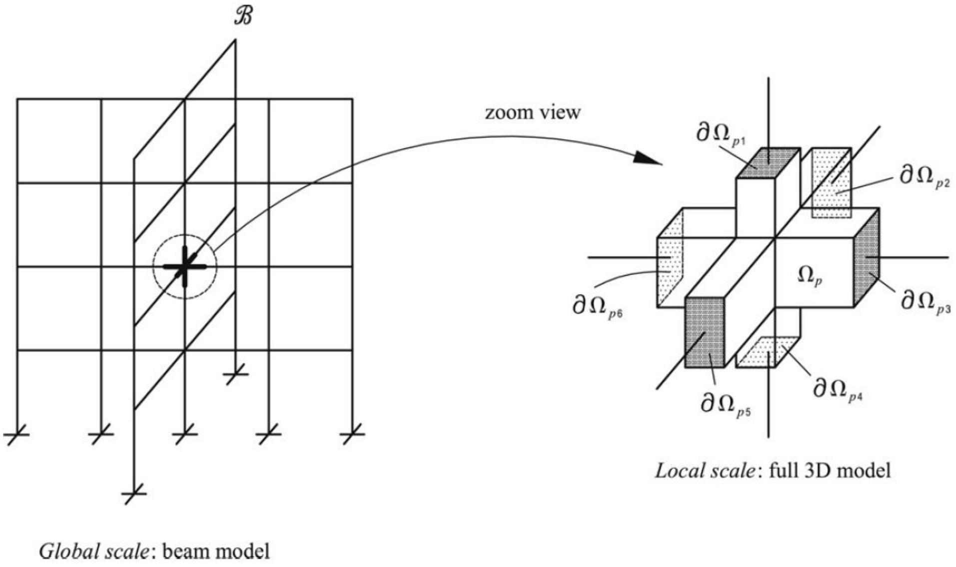


Figure 1.1: Schematic representation of the two-scale model (Mata et al. [43]).

1.3 Overview

1.3.1 Problem statement

As mentioned in Bozorgna & Bertero [9], a modern numerical approach to the structural analysis and design of three-dimensional engineering structures (especially in the field of seismic analysis) should take into account the two major sources of nonlinear behavior, namely:

(i) *Geometric non-linearity* due to the change in the configuration experienced by flexible structures during loading conditions, and the consequent inclusion of large displacements and large rotations in the compatibility and equilibrium equations.

(ii) *Constitutive non-linearity* resulting from the material nonlinear relationship between force and deformations.

In general, the engineering community agrees with the fact that, although models which consider both two nonlinearities are more expensive, in terms of computational cost, they allow to estimate more precisely the response of RC and other kind of structures.

Focusing on the second type of nonlinearity, the following section presents the objectives of the present work, which tries to be a contribution to a unified treatment of geometrical nonlinearities for beam elements, in the rigorous framework of the principles of continuum mechanics.

1.3.2 Scope and objectives

The main purpose of this thesis consists in re-examining, in a unified framework, the fully 3D geometrically exact beam theory, and its finite element implementation aimed to the development of a formulation able to reproduce geometric nonlinearity in both static and dynamic range. In this sense, the following list of objectives can be defined:

- **(I)** *Theoretical objectives*
  - **(I.1)** To carry on a comprehensive study on the mathematical framework in which finite rotations are naturally collocated.
  - **(I.2)** To investigate the nature of rotation group, its main properties (e.g. non-commutativity) and the parametrization of finite rotations, as well as the formalization of tangent space of rotation manifold and the fundamental relations between rotation tensor, rotation vector, total rotation vector, and their spatial and time derivative.
  - **(I.3)** To present in a unitary and consistent treatment of the basic kinematics governing the geometrically exact beam theory, under the Reissner-Simo hypothesis.
  - **(I.4)** To deduce explicit expressions for the objective strain measures and the corresponding energetically conjugated stress measures acting on each material point of the beam cross-section.
  - **(I.5)** To carry out the consistent linearization of the most important kinematical quantities necessary for expressing the principle of virtual work.
  - **(I.6)** To derive the strong and weak form of the balance equations for linear and angular momentum, and to deduce the principle of virtual work in spatial and material form.
- **(II)** *Numerical objectives*
  - **(II.1)** To perform the discretization in space of the mechanical problem using Galerkin finite element interpolation of the displacement variables.
  - **(II.2)** To perform the time discretization according to the Newmark's method of the mechanical problem.
  - **(II.3)** To implement (computationally) a load and displacement control algorithm for solving the nonlinear algebraic system of equations coming out from the finite element discretization.
  - **(II.4)** To test the described FEM formulation through a set of linear elastic numerical examples in both static and dynamic cases, and compare the results with those provided in existing literature.

1.3.3 Outline

The present study starts with an introduction of the finite rotation mathematical framework, proceeds with describing the three-dimensional beam theory and ends up with the finite element implementation and its application to some test cases. The organization of the present document is as follows.

In Chapter 2, for reader's convenience, a thorough review of relevant mathematical aspects regarding differential geometry is made, with the purpose to give a solid introduction to the treatment of finite rotations. Special care is reserved to the definitions of differential rotation manifold, special orthogonal group and Lie algebra, as well as the description of the properties of rotation tensor and its parameterizations. Discussion is also focused on rotation vector, total rotation vector and its relations with the spatial and time derivatives of rotation tensor.

Chapter 3 is devoted to the presentation of a geometrically exact formulation for beams capable of undergoing finite deformations based in that originally proposed by Reissner and Simo. Firstly a section is dedicated to a detailed description of the kinematic of the model, with special attention paid on the formal definition of the configuration and placement manifolds as well as their tangent spaces. Subsequently, after calculating the deformation gradient tensor, the strain and strain rate measures at both, material point and dimensionally reduced levels, are deduced.

The beam's equations of motion in both spatial and material description are obtained in Chapter 4, starting from the local form of the linear and angular momentum balance conditions.

An appropriated (weak) form for numerical implementation is discussed in Chapter 5 for the nonlinear functional corresponding to the virtual work principle. Next, the hyperelastic cross-sectional constitutive law is discussed in the same Chapter.

Chapter 6 describes the spatial discretization based on the Galerkin isoparametric finite element approximation of the variational equation of virtual work. The applied procedure yields to a system of nonlinear algebraic equations well suited to be solved by numerical iterative method. An entire section is devoted to the derivation of the so-called force residual equation, and some remarks on the numerical procedures commonly adopted dot its solution are given.

In Chapter 7 highly geometrical nonlinear plane and spatial problems are investigated. Results obtained from numerical simulations are in agreement with those available in literature, showing the ability of the implemented formulations in simulating the full geometric nonlinear response of beam-like structure.

Finally, in Chapter 8 conclusions about the work developed are presented. A detailed survey is given in section 8.1, and an additional section is included for considering further developments which may take advantage from the present work.

The thesis is complemented with a set of Appendices. They include: (i) some mathematical recalls useful for the comprehension of the critical derivations of main equations, (ii) the introduction of a useful family of scalar quantities involving trigonometric functions, which yields to a more rational presentation of some nasty equation regarding rotation tensor, (iii) an alternative expression of compound rotation about fixed axis by using Cayley-Rodrigues parametrization, (iv) the complete derivation of relation between spin-like vectors and total rotation vector and (v) the determination of the balance equation of angular momentum with respect to a spatial pole and its time derivative.

1.4 Notation

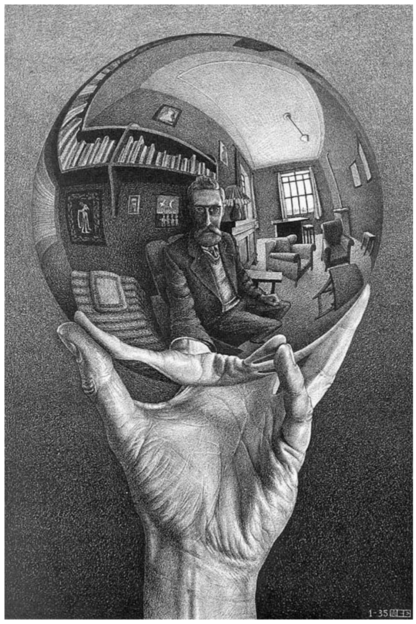
Along the chapters, intensive use of vector and tensor calculus is made. Through the document, scalar quantities are denoted using lightfaced letters with italic or calligraphic style, or lightfaces mathematical symbols, e.g. $(a, b, \ldots)$. Tensors are written in boldfaced letters, for instance $(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \ldots)$. As a special case, the second order skew-symmetric tensors are identified in boldface and equipped with an over-head tilde $(\tilde{\cdot})$ such as $(\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, \ldots, \tilde{\boldsymbol{\Omega}}, \tilde{\boldsymbol{\Psi}}, \ldots, \tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\psi}})$, whereas $(\boldsymbol{A}, \boldsymbol{B}, \ldots, \boldsymbol{\Omega}, \boldsymbol{\Psi}, \ldots, \boldsymbol{\omega}, \boldsymbol{\psi})$ indicate their associated axial vectors. If not otherwise specified, upper or lower case letters are used for scalars, vectors (first order tensors) or tensors, but subjected to the previously defined convention. A special notation is adopted to distinguish *material* and *spatial* quantities: upper-case letters are used for material vectors and tensors, whereas lower-case letters are used to denote spatial vectors and tensors. The superscript $(\cdot)^T$ is used to denote the transpose of a given quantity. The superscript $(\cdot)^r$ is sometimes used for denoting the material (reference) description. Other sub and superscripts are employed in several quantities through the text, but they are defined the first time they are used. Summation index convention applies through the text. Latin indices, such as i, j range over the values: $\{1, 2, 3\}$, and Greek indices, such as α, β range over the values $\{1, 2\}$. If it is not the case, specific ranges are given in the text. The dot (inner), cross, and tensorial products are denoted by means of the symbols $(\cdot)$, $(\times)$ and $(\otimes)$, respectively. The overhead dot is used to denote the time derivative, i.e. $(\dot{\cdot})$, whereas prime symbol marks the spatial derivative, i.e. $(\cdot)'$.

Part I

Geometrically Exact Beam Theory

Chapter 2

Introduction to finite rotations



Quelli che s'innamoran di pratica senza scienza son come 'l nocchiere ch'entra in naviglio senza timone o bussola, che mai ha la certezza di dove vada.

L. Da Vinci

The main aim of the present chapter is to pave the way for the work in the next chapters concerning to the development of a geometrically exact three-dimensional beam theory involving finite displacements, where large rotations are coupled with large translations. In particular, the results here presented impacts on the accurate description of the rotational motion. The term *large* or *finite* rotations is normally employed in continuous mechanics as opposite to *small* of *infinitesimal* rotations.

The chapter opens with a short introduction to manifolds, differentiable manifolds and Lie groups, with special care to special orthogonal group. Then, a formal presentation of the main properties of rotation group is done, revealing a rich mathematical structure which corresponds to the Lie group isomorphic to the special orthogonal group of rotation tensors. With the help of an example, we explain also the non-commutative nature of large rotations and their non additivity in vectorial sense. Subsequently, a rigorous definition of the tangent space to the rotational manifold is presented in terms of the Lie algebra associated to the rotation group. Then, a rather detailed discussion about possible parametrization of the rotation manifold is presented, addressing the practical advantages and limitations of using in particular the the vectorial representation. Since rotations behave differently in observer transformation and in objective derivatives, there exist the need to distinguish the spaces from which a rotations are described. To this end a configurational approach for describing large rotations in three-dimensional space is given, and subsequently the so called spatial and material updating procedure for compound rotations is explained. Finally, an intense paragraph is dedicated to introduce Lie derivative, as well as their physical meaning in the contest of rotations.

As these are complex mathematical topics quite unfamiliar to engineering literature, only the concepts and formalism strictly necessary for this work will be reviewed. However, more extensive and detailed works about the mathematical theory of finite rotations can be found in [1], [2], [4], and on application to beam, shell and flexible mechanics theories in [33], [31].

2.1 Manifolds

Definition 2.1 (Manifold)**.** *Given a n-dimensional Euclidean space* [1] $\mathbb{E}^n$*, a set* $\mathcal{M} \in \mathbb{E}^n$ *is a manifold with dimension d, if there exists a bijection* [2] $\varphi_i : \mathcal{U}_i \to \mathbb{E}^n$ *from an open domain* $\mathcal{U}_i \subset \mathbb{E}^d$ *subset of a* d*-dimensional Euclidean parameter space, onto some open set in the manifold,* $\varphi_i(\mathcal{U}_i) \equiv \mathcal{V} \subset \mathcal{M}$*, such that every point* $P \in \mathcal{M}$ *is an image under a mapping. A pair* $(\mathcal{U}_i, \varphi_i)$ *is called a chart or a parametrization chart.*

Definition 2.2 (Differentiable manifold)**.** *A manifold* $\mathcal{M}$ *is called differentiable if for every point* $P \in \mathcal{M}$ *there exist images* $\varphi_1(\mathcal{U}_1)$ *and* $\varphi_2(\mathcal{U}_2)$ *where the point* $P \in \mathcal{M}$ *belongs to, such that the composite mapping* $\varphi_2^{-1} \circ \varphi_1$ *is a diffeomorphism* [3]*.*

The basic idea of differentiable manifold is symbolically depicted in Figure 2.1. A differentiable manifold can be mapped from a chart in a parameter space into a chart of manifold in an embedded space. The composite map is called *change of parametrization*, and is differentiable for differentiable manifolds.

From a geometrical point of view, a differentiable manifold $\mathcal{M}$ can be imagined as a generalization of a surface in the n-dimensional space, as depicted for instance in Figure 2.2 for the Calabi-Yau manifolds.

Let $\mathbf{\Lambda}(t)$ be any differentiable curve on the manifold $\mathcal{M}$ parametrized in terms of the real parameter $t \in \mathbb{R}$, that passes through the base point [4] $\mathbf{\Lambda}_0 \in \mathcal{M}$ such that

[1] The Euclidean space is a real , finite-dimensional, linear, inner-product space with an Euclidean metric.

[2] A mapping is a bijection if it is injective and surjective, i.e. one-to-one mapping.

[3] A diffeomorphism is a bijection with continuously differentiable mapping and its inverse mapping.

[4] A base point is a point of the manifold on which a tangent space is induced.

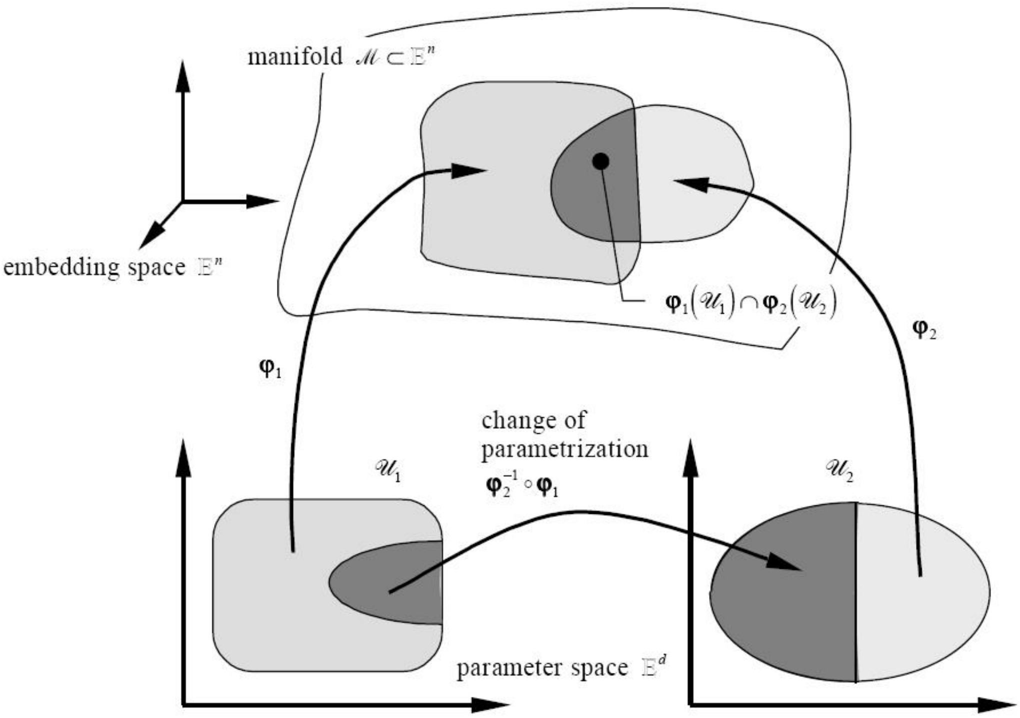


Figure 2.1: A geometric interpretation for a parametrization of a manifold when $n = 3$ and $d = 2$.

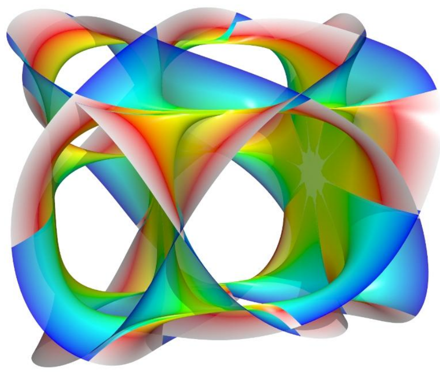


Figure 2.2: The Calabi-Yau manifolds is a complex manifold with important applications in superstring theory.

$\mathbf{\Lambda}|_{t=0} = \mathbf{\Lambda}_0$. Then the derivative with respect to t

$$\frac{\mathrm{d}\mathbf{\Lambda}}{\mathrm{d}t} = \lim_{t\to 0} \frac{\mathbf{\Lambda}(t) - \mathbf{\Lambda}(0)}{t} \qquad \text{where} \qquad \mathbf{\Lambda}(0) = \mathbf{\Lambda}_0, \quad \mathbf{\Lambda}(t) \in \mathcal{M} \tag{2.1}$$

is said to be the tangent vector to $\mathcal{M}$ at $\mathbf{\Lambda}_0$. The set of all tangent vectors at $\mathbf{\Lambda}_0$, denoted by $\mathcal{T}_{\mathbf{\Lambda}_0}\mathcal{M}$, forms a vector space called *tangent space* to $\mathcal{M}$ at $\mathbf{\Lambda}_0$. More formally we have the following definition.

Definition 2.3 (Tangent space on manifold)**.** *Let $\mathcal{M} \in \mathbb{R}^n$ be an open set (manifold), and let $P \in \mathcal{M}$. The tangent space to $\mathcal{M}$ at P is simply the vector space $\mathbb{R}^n$ emanating from P. This tangent space is denoted as $\mathcal{T}_P\mathcal{M}$.*

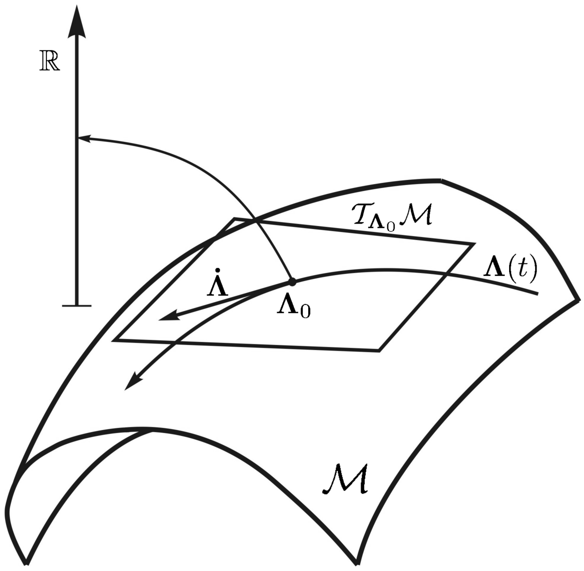


Figure 2.3: A geometric representation of the parametrized curve $\mathbf{\Lambda}(t)\colon \mathbb{R}^1 \to \mathcal{M}$ and the tangent vector $\dot{\mathbf{\Lambda}}$ and its tangent space $\mathcal{T}_{\mathbf{\Lambda}_0}\mathcal{M}$ on the manifold $\mathcal{M}$ at the point $\mathbf{\Lambda}_0$.

With the help of Figure 2.3, where a differentiable manifold is depicted in a symbolic manner, a curve on the manifold is a map of an interval of $\mathbb{R}^1$ into a curve on the manifold, and consequently a tangent vector to a curve on the manifold is, for instance, the velocity vector of an object which moves along the curve, where the velocity vector has the usual meaning of time derivative displacement parameter.

Linearization process. Let consider a manifold $\mathcal{M}$ with a generic element $\boldsymbol{x}$, its tangent space $\mathcal{TM}$ with a generic element $\boldsymbol{u}$, and a function $F = F(\boldsymbol{x}) \,|\, F\colon \mathcal{M} \to \mathcal{C}$, where $\mathcal{C}$ is a generic set. We indicate the *tangent operator* or *linearization* of F by the notation $\boldsymbol{\delta}_{\boldsymbol{u}}F(\boldsymbol{x})$, and we call $\boldsymbol{x}$ the point of linearization and $\boldsymbol{u}$ the direction of linearization. We also state that this operation must be linear in the direction of linearization and we define the following equivalent expressions:

$$\boldsymbol{\delta}_{\boldsymbol{u}}F(\boldsymbol{x}) = \boldsymbol{\delta}F \cdot \boldsymbol{u} = \lim_{\varepsilon\to 0} \frac{F(\boldsymbol{x}_\varepsilon) - F(\boldsymbol{x})}{\varepsilon} = \left.\frac{\mathrm{d}F(\boldsymbol{x}_\varepsilon)}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \operatorname{grad} F \cdot \boldsymbol{u} \tag{2.2}$$

where $\boldsymbol{x}_\varepsilon = \boldsymbol{x} + \varepsilon\boldsymbol{u}$ represents a configuration infinitesimally near to $\boldsymbol{x}$, obtained perturbing $\boldsymbol{x}$ in the direction of the tangent element $\boldsymbol{u}$ by the quantity $\varepsilon > 0 \in \mathbb{R}$. The *perturbed* configuration $\boldsymbol{x}_\varepsilon$ is an admissible variation from an admissible configuration which satisfies the conditions $\boldsymbol{x}_\varepsilon \in \mathcal{M}$ and $\lim_{\varepsilon\to 0} \boldsymbol{x}_\varepsilon = \boldsymbol{x}$.

2.2 Special orthogonal group

Definition 2.4 (Group)**.** *A group $\mathcal{G}$ is a set equipped with an internal operation " $\times$ " that combines any two elements $\boldsymbol{A}$ and $\boldsymbol{B}$ to form another element denoted as $\boldsymbol{A} \times \boldsymbol{B} \in \mathcal{G}$. To qualify as a group, the set and operation $(\mathcal{G}, \times)$ must satisfy four requirements known as the group axioms*

- Closure: *a set is closed if the result of the operation $\boldsymbol{A}\times\boldsymbol{B}$ is also in $\mathcal{G}$, $\forall \boldsymbol{A}, \boldsymbol{B} \in \mathcal{G}$;*
- Associativity: *an internal operation is associative if the relation $\boldsymbol{A}(\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{A}\boldsymbol{B})\boldsymbol{C}$ holds $\forall \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C} \in \mathcal{G}$;*
- Identity element: *exists a unique element $\boldsymbol{I} \in \mathcal{G}$ called identity such that $\boldsymbol{A}\boldsymbol{I} = \boldsymbol{I}\boldsymbol{A} = \boldsymbol{A}$ $\forall \boldsymbol{A} \in \mathcal{G}$;*
- Inverse element: *For each $\boldsymbol{A} \in \mathcal{G}$ there exists a unique element of $\mathcal{G}$ called the inverse of $\boldsymbol{A}$ such that $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$ holds.*

The group is called an Abelian group, or a *commutative* group, if $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{B}\boldsymbol{A}, \forall \boldsymbol{A}, \boldsymbol{B} \in \mathcal{G}$. Conversely the group is called *non-commutative* if this property does not hold.

Definition 2.5 (Lie group)**.** *A Lie group $\mathcal{L}$ is a differentiable n-dimensional manifold $\mathcal{M}^n$ endowed with the following two smooth mappings:*

1. *An internal operation*

$$\begin{aligned}&\mathscr{F}_\alpha \colon \mathcal{L} \times \mathcal{L} \to \mathcal{L}\\ &(\boldsymbol{A}, \boldsymbol{B}) \mapsto \mathscr{F}_\alpha(\boldsymbol{A}, \boldsymbol{B}) = \boldsymbol{A} \odot \boldsymbol{B} \quad \forall \boldsymbol{A}, \boldsymbol{B} \in \mathcal{L}\end{aligned}$$

   where $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{L}$, the symbol $\times$ is used to denote pairing between elements, and the symbol $\odot$ is used to indicate an abstract operation (multiplication) between elements of the group $\mathcal{L}$.

2. *A smooth mapping which defines the inverse element*

$$\begin{aligned}&\mathscr{F}_\mathcal{V} \colon \mathcal{L} \to \mathcal{L}\\ &\boldsymbol{A} \mapsto \mathscr{F}_\mathcal{V}(\boldsymbol{A}) = (\boldsymbol{A})^{-1}\end{aligned}$$

Definition 2.6 (Orthogonal group)**.** *The orthogonal group of degree n over a field* [5] *K is written as $O(n, K)$, and represents the group of n-by-n orthogonal matrices $\boldsymbol{Q}$*

[5] In abstract algebra, a field is an algebraic structure with notions of addition, subtraction, multiplication, and division, satisfying certain axioms.

with entries from K. This is a subgroup of the general linear group [6] $\mathcal{GL}(n, K)$, and it is given by

$$\boxed{O(n, K) := \left\{ \boldsymbol{Q} \in \mathcal{GL}(n, K) \;\mid\; \boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I} \right\}} \tag{2.3}$$

where $\boldsymbol{Q}^T$ is the transpose of $\boldsymbol{Q}$, and $\boldsymbol{Q}^T\boldsymbol{Q} = \boldsymbol{I}$ or $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$ defines the orthogonality conditions.

More in general, over the field $\mathbb{R}$ of real numbers, the orthogonal group $O(n, \mathbb{R})$ is often simply denoted by $O(n)$ if no confusion is matter. The group of all 3-by-3 orthogonal real matrices is then denoted as $O(3)$, and it consists of all proper and improper rotations in 3-dimensional space.
Every orthogonal matrix has determinant either +1 or -1. The orthogonal n-by-n matrices with determinant +1 form a *proper* subgroup of $O(n, K)$ known as the special orthogonal group $SO(n, K)$. Improper matrices correspond to orthogonal matrices with $\det \boldsymbol{Q} = -1$, and they do not form a group, because the product of two improper matrices is a proper matrix.

Definition 2.7 (Special orthogonal group)**.** *The special non-commutative Lie group of proper orthogonal linear transformations in the real space $\mathbb{R}$ with differentiable structure, is defined as the set of orthogonal n-by-n matrices $\boldsymbol{\Lambda}$ such that*

$$\boxed{SO(n, \mathbb{R}) := \left\{ \boldsymbol{\Lambda} \colon \mathbb{R} \to \mathbb{R} \;\mid\; \boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^T = \boldsymbol{I} \;,\; \det \boldsymbol{\Lambda} = +1 \right\}} \tag{2.4}$$

Over the field $\mathbb{R}$ of real numbers, the special orthogonal group $SO(n, \mathbb{R})$ is often simply denoted by $SO(n)$ if no confusion is possible. Similarly, in the Euclidean space $\mathbb{R}^3$ the previous becomes

$$\boxed{\mathrm{SO}(3) := \left\{ \boldsymbol{\Lambda} \colon \mathbb{R}^3 \to \mathbb{R}^3 \;\mid\; \boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^T = \boldsymbol{I} \;,\; \det \boldsymbol{\Lambda} = +1 \right\}} \tag{2.5}$$

2.3 Rotation group

Let introduce a rotation operator $\boldsymbol{\Lambda}$ which transforms linearly and isometrically an orthonormal basis of $\mathbb{R}^3$ to another orthonormal basis in the Euclidean space. Like any linear transformation of finite-dimensional vector spaces, a rotation can always be represented by a tensor (say $\boldsymbol{\Lambda}$) , or alternatively by a rotation vector (say $\boldsymbol{\Psi} \in \mathbb{R}^3$), which contains information on the axis of rotation and on the entity (with sign) of the rotation itself. In geometry, the group of all rotations about the origin of three-dimensional Euclidean space $\mathbb{R}^3$ is called *rotation group.*

By definition, a *proper rotation* about the origin is a linear transformation that preserves length (isometry), volume (with sign), and the angle between pairs of vectors (orientation). Formally, given three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^3$

1. A rotation tensor $\boldsymbol{\Lambda}$ preserves the length of vectors if it satisfies the conditions $||\boldsymbol{u}|| = ||\boldsymbol{\Lambda u}|| \quad \forall \boldsymbol{u} \in \mathbb{R}^3$;

[6] The general linear group is the group of all real matrices with nonzero determinant.

2. A rotation tensor $\boldsymbol{\Lambda}$ preserves volume if it satisfies the condition $\boldsymbol{u} \cdot \boldsymbol{v} \times \boldsymbol{w} = \boldsymbol{\Lambda u} \cdot \boldsymbol{\Lambda v} \times \boldsymbol{\Lambda w}$;
3. A rotation tensor $\boldsymbol{\Lambda}$ preserves the inner product if it satisfies the condition $\boldsymbol{\Lambda}\mathbf{u} \cdot \boldsymbol{\Lambda}\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$, i.e. whether the angle between $\mathbf{u}$ and $\mathbf{v}$ is preserved.

Because of these three properties, a proper rotation is termed *rigid transformation*. Conversely, a length-preserving transformation which reverses orientation is called an *improper rotation*. Physically, every improper rotation of three-dimensional Euclidean space is a *reflection* in a plane through the origin.

In particular, an orthogonal tensor preserves or reverses orientation according to whether the determinant of the tensor is unit positive or negative, respectively. In fact, assuming $\boldsymbol{\Lambda}$ being an orthogonal tensor, from definition (2.3) it follows that

$$\det(\boldsymbol{\Lambda}^T\boldsymbol{\Lambda}) = \det \boldsymbol{I} = 1 \tag{2.6}$$

and from determinant properties

$$\det(\boldsymbol{\Lambda}^T\boldsymbol{\Lambda}) = \det(\boldsymbol{\Lambda}^T)\det\boldsymbol{\Lambda} = \det(\boldsymbol{\Lambda})^2 \tag{2.7}$$

which implies $\det\boldsymbol{\Lambda} = \pm 1$. It is possible to prove that every proper rotation can be expressed uniquely by an orthogonal matrix $\boldsymbol{\Lambda}$ with unit positive determinant, such that $\boldsymbol{\Lambda}$ belongs to the set of all orthogonal matrices as follow (see also definition (2.5))

$$\mathrm{SO}(3) := \left\{ \boldsymbol{\Lambda} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^T = \mathbf{1} \, , \, \det\boldsymbol{\Lambda} = +1 \right\} \tag{2.8}$$

As consequence of that, all the group axioms are satisfied for a set of arbitrary rotations, namely:

1. Composition of two rotations results in another rotation, i.e. $\boldsymbol{\Lambda}_c = \boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2$ $\forall\, \boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2 \in SO(3)$ with $\boldsymbol{\Lambda}_c \in SO(3)$;
2. The product of three rotations is associative, i.e. $(\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2)\boldsymbol{\Lambda}_3 = \boldsymbol{\Lambda}_1(\boldsymbol{\Lambda}_2\boldsymbol{\Lambda}_3)$ $\forall\, \boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2, \boldsymbol{\Lambda}_3 \in SO(3)$;
3. There exists a neutral element, the identity matrix, such that $\mathbf{I}\,\boldsymbol{\Lambda}\mathbf{I} = \mathbf{I}\,\boldsymbol{\Lambda} = \boldsymbol{\Lambda} \quad \forall\, \boldsymbol{\Lambda} \in SO(3)$;
4. Every rotation has a unique inverse rotation such that $\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}^{-1}\boldsymbol{\Lambda} = \mathbf{I} \quad \forall\, \boldsymbol{\Lambda} \in SO(3)$.

In addition to these properties, it is of fundamental importance to point out other two features of rotation group:

A. The rotation product is *not commutative* in general, i.e. $\boldsymbol{\Lambda}_1\boldsymbol{\Lambda}_2 \neq \boldsymbol{\Lambda}_2\boldsymbol{\Lambda}_1$ with $\boldsymbol{\Lambda}_1, \boldsymbol{\Lambda}_2 \in SO(3)$;

B. Rotations are *not additive* in general, since the set of rotations is not a linear space.

These special characteristics of rotations are found only in the three-dimensional case, while in two spatial dimensions rotations can be added and even commuted, leading a simpler analysis.

Focusing on property A., with the help of Figure 2.4 it is easy to see that the result of applying a set of successive large rotations on a body, depends on the order in which they are performed. In this example three rotations of magnitude $\pi/2$ are arranged as triplet $\{\phi_{xx}, \phi_{yy}, \phi_{zz}\}$, and they are applied to a rigid box in two different orders. The final configuration of the box in general will be different for each one of the options. Therefore, one may conclude that rotations do not commute and consequently the order of application of rotations is crucial for representing uniquely a set of spatial rotations.

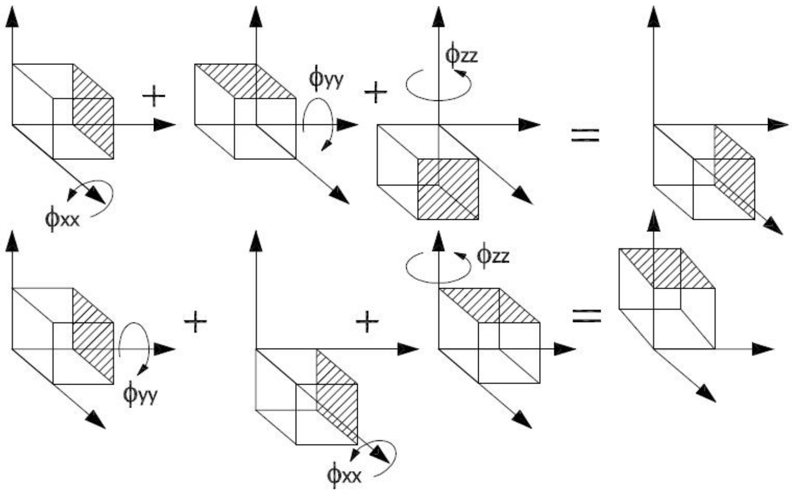


Figure 2.4: Noncommutativity of successive rotations. The symbol "+" is used to indicate an abstract operation of composition of rotations.

According to property B., the composition of successive rotations can not be expressed by simply applying the parallelogram rule, adding the corresponding rotation vectors, unless they take place around the same axis.
To show this property, let consider a rotation denoted by a rotation vector $\boldsymbol{\theta}_A = \theta_A \boldsymbol{e}_A$ and the corresponding rotation tensor $\boldsymbol{\Lambda}_A$, that brings triad $\mathcal{I} = (\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3)$ into triad $\mathcal{J} = (\boldsymbol{j}_1, \boldsymbol{j}_2, \boldsymbol{j}_3)$. This first rotation is followed by a second rotation, characterized by rotation vector $\boldsymbol{\theta}_B = \theta_B \boldsymbol{e}_B$ and rotation tensor $\boldsymbol{\Lambda}_B$, that brings $\mathcal{J}$ into $\mathcal{K} = (\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$. We have

$$\boldsymbol{j}_k = \boldsymbol{\Lambda}_A \boldsymbol{i}_k, \qquad \boldsymbol{k}_k = \boldsymbol{\Lambda}_B \boldsymbol{j}_k, \qquad k = 1, 2, 3 \tag{2.9}$$

and hence, eliminating the intermediate configuration $\mathcal{J}$

$$\boldsymbol{k}_k = \boldsymbol{\Lambda}_B \boldsymbol{\Lambda}_A \boldsymbol{i}_k, \qquad k = 1, 2, 3 \tag{2.10}$$

The total rotation from $\mathcal{I}$ to $\mathcal{K}$ is given by

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}_B \boldsymbol{\Lambda}_A \tag{2.11}$$

This product of rotation tensors is called *compound* of rotations, and equation (2.11) can be used recursively to compose as many successive rotations as necessary. The word "compound" used when combining rotations, stresses the fact that successive

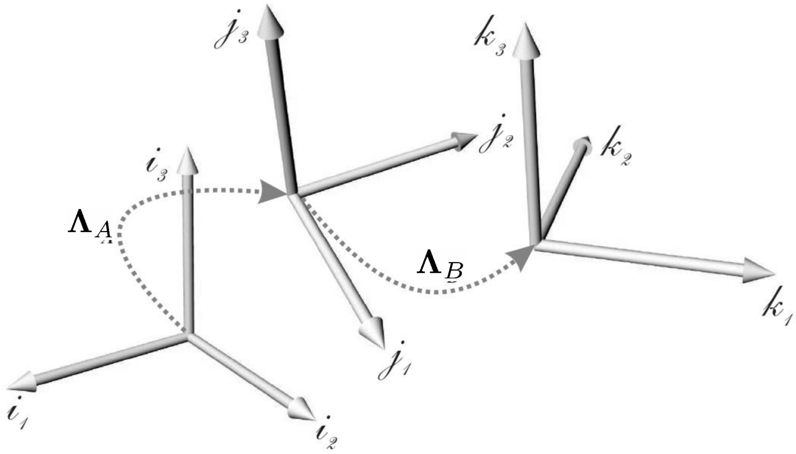


Figure 2.5: Rotation of triad $\mathcal{I}$ into triad $\mathcal{J}$, followed by rotation of $\mathcal{J}$ into $\mathcal{K}$.

rotations cannot be obtained by simply adding their corresponding rotation vectors as in linear spaces. In fact, if $\boldsymbol{\theta} = \theta\, \boldsymbol{e}$ is the rotation vector corresponding to the composed rotation $\boldsymbol{\Lambda}$, then

$$\boldsymbol{\theta} \neq \boldsymbol{\theta}_A + \boldsymbol{\theta}_B \tag{2.12}$$

This fact can be shown by a self-evident geometric example, as the one depicted in Figure 2.6), which considers two successive rotations with orthogonal axes.

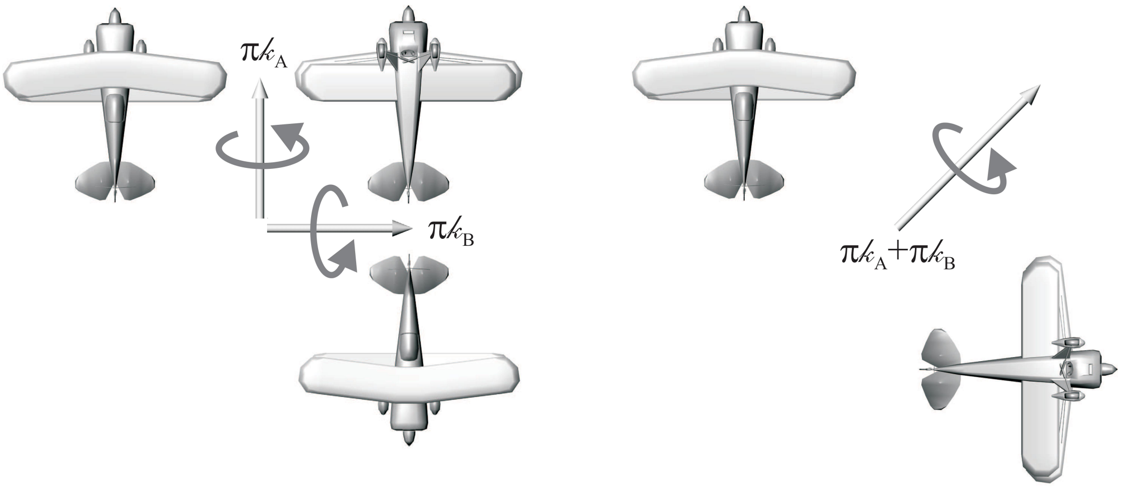


Figure 2.6: At left, rotation $\pi \boldsymbol{k}_A$, followed by rotation $\pi \boldsymbol{k}_B$, with $\boldsymbol{k}_B$ orthogonal to $\boldsymbol{k}_A$. At right, rotation about $\pi(\boldsymbol{k}_A + \boldsymbol{k}_B)$.

2.4 Lie Algebra

Definition 2.8 (Lie algebra). *A Lie algebra $\mathscr{L}$ of the Lie group $\mathcal{L}$ is a tangent vector space at the identity $\mathcal{T}_{\boldsymbol{I}}\mathcal{L}$, equipped with a bilinear, skew-symmetric brackets operator* $[\cdot,\cdot]$ *such that* $[\boldsymbol{a},\boldsymbol{b}] = -[\boldsymbol{b},\boldsymbol{a}] \quad \forall \boldsymbol{a},\boldsymbol{b} \in \mathscr{L}$ *satisfying Jacobi's identity* $[\boldsymbol{a},[\boldsymbol{b},\boldsymbol{c}]] + [\boldsymbol{b},[\boldsymbol{c},\boldsymbol{a}]] + [\boldsymbol{c},[\boldsymbol{a},\boldsymbol{b}]] = \boldsymbol{0} \quad \forall \boldsymbol{a},\boldsymbol{b},\boldsymbol{c} \in \mathscr{L}$.

Recalling Definition 2.3, in the case of rotational manifold the tangent space at the identity $\boldsymbol{\Lambda}_0 = \boldsymbol{I}$ is given a special name, the *Lie algebra* [7] of $SO(3)$ and is denoted by $so(3)$. It defines the fundamental isomorphism between $so(3)$ and $\mathbb{R}^3$, particularly important for linearization process.

Let now focus on a t-parametrized operator $\boldsymbol{\Lambda}(t) \in SO(3)$ belonging to the special orthogonal group. Suppose that it is given by formula [8] $\boldsymbol{\Lambda}(t) = \exp(t\tilde{\boldsymbol{\Psi}})$, with $\tilde{\boldsymbol{\Psi}}$ a convenient skew-symmetric tensor. Differentiating this expression with respect to the parameter t at $t = 0$, one obtains the tangent vector space at the identity $\boldsymbol{I} \in SO(3)$, i.e.

$$\left.\frac{\mathrm{d}\boldsymbol{\Lambda}}{\mathrm{d}t}\right|_{t=0} = \left.\frac{\mathrm{d}\,\exp(t\tilde{\boldsymbol{\Psi}})}{\mathrm{d}t}\right|_{t=0} = \tilde{\boldsymbol{\Psi}} \tag{2.13}$$

Thus, the skew-symmetric tensor $\tilde{\boldsymbol{\Psi}}$ belongs to the tangent space of the rotation manifold $SO(3)$, denoted by $\mathcal{T}_{\boldsymbol{I}}SO(3)$, where the identity $\boldsymbol{I} \in SO(3)$ represents a base point of the rotation manifold. It is clear that the base point is the identity $\boldsymbol{I} \in SO(3)$ since $\boldsymbol{\Lambda} = \exp(t\tilde{\boldsymbol{\Psi}})$ at $t = 0$ is equal to the identity $\boldsymbol{I}$. It could be proven that the skew-symmetric tensor $\tilde{\boldsymbol{\Psi}}$ is also an element of Lie algebra $so(3)$ for the corresponding Lie group $SO(3)$. Thereby, we could mark $so(3) = \mathcal{T}_{\boldsymbol{I}}SO(3)$, i.e. Lie algebra is canonical isomorphic to the tangent space of the rotation manifold at the identity. In other words, it exists a one-to-one correspondence between elements of $so(3)$ and elements of $\mathcal{T}_{\boldsymbol{I}}SO(3) \in \mathbb{R}^3$. The correspondence can be identified through the vector product $\times$ on $\mathbb{R}^3$ by the formula (also referred as *axial vector relation*)

$$\boxed{\tilde{\boldsymbol{\Psi}}\boldsymbol{h} = \boldsymbol{\Psi} \times \boldsymbol{h} \qquad \forall \boldsymbol{h} \in \mathbb{R}^3} \tag{2.14}$$

with

$$\tilde{\boldsymbol{\Psi}} = \mathrm{skew}[\boldsymbol{\Psi}] = \begin{bmatrix} 0 & -\Psi_3 & \Psi_2 \\ \Psi_3 & 0 & -\Psi_1 \\ -\Psi_2 & \Psi_1 & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Psi} = \mathrm{axial}[\tilde{\boldsymbol{\Psi}}] = \begin{bmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \end{bmatrix} \tag{2.15}$$

where the vector $\boldsymbol{\Psi} \in \mathbb{R}^3$ is called *axial vector* of the skew-symmetric tensor $\tilde{\boldsymbol{\Psi}} \in so(3)$. Sometimes in literature is used the notation $[\boldsymbol{\Psi}\times]$ to indicate the skew tensor $\tilde{\boldsymbol{\Psi}}$ in order to emphasize its axial vector.

Having established that the skew-symmetric tensor $\tilde{\boldsymbol{\Psi}}$ belongs to the Lie algebra, the set $so(3)$ may be defined as the set of all skew-symmetric tensors, i.e.

$$\boxed{so(3) = \left\{ \tilde{\boldsymbol{\Psi}} : \mathbb{R}^3 \to \mathbb{R}^3 \;\middle|\; \tilde{\boldsymbol{\Psi}}^T = -\tilde{\boldsymbol{\Psi}} \right\}} \tag{2.16}$$

where $\tilde{\boldsymbol{\Psi}}^T = -\tilde{\boldsymbol{\Psi}}$ represents the skewness condition. It is worth to remark that, up to this stage, no assumptions are made on the magnitude of the elements of Lie-algebra $so(3)$, which therefore are not required to be infinitesimal.

[7] In mathematics, an algebraic structure consists of one or more sets closed under one or more operations, satisfying some axioms.

[8] We anticipate here the form of the so called exponential mapping whose detailed formalism will be presented in the next section.

We report below some important relations between the skew-symmetric tensor $\tilde{\boldsymbol{\Psi}}$ and its associated rotation vector $\boldsymbol{\Psi}$, which are frequently found in the development of geometrically exact formulation for beams. Given two skew-symmetric tensors $\tilde{\boldsymbol{\Psi}}$ and $\tilde{\boldsymbol{W}}$ and their axial vectors, respectively $\boldsymbol{\Psi}$ and $\boldsymbol{W}$, the following identities holds

$$\tilde{\boldsymbol{\Psi}}\boldsymbol{W} = \boldsymbol{\Psi} \times \boldsymbol{W} = -\boldsymbol{W} \times \boldsymbol{\Psi} = -\tilde{\boldsymbol{W}}\boldsymbol{\Psi} = \tilde{\boldsymbol{W}}^T\boldsymbol{\Psi} = \boldsymbol{\Psi}^T\tilde{\boldsymbol{W}} \tag{2.17}$$

$$\tilde{\boldsymbol{\Psi}}\boldsymbol{\Psi} = \boldsymbol{\Psi}^T\tilde{\boldsymbol{\Psi}} = \boldsymbol{\Psi} \times \boldsymbol{\Psi} = \boldsymbol{0} \tag{2.18}$$

$$\boldsymbol{\Psi}^T\tilde{\boldsymbol{W}} = -\tilde{\boldsymbol{W}}\boldsymbol{\Psi} = -\boldsymbol{W} \times \boldsymbol{\Psi} = \boldsymbol{\Psi} \times \boldsymbol{W} = \tilde{\boldsymbol{\Psi}}\boldsymbol{W} = \boldsymbol{W}^T\tilde{\boldsymbol{\Psi}}^T = -\boldsymbol{W}^T\tilde{\boldsymbol{\Psi}} \tag{2.19}$$

$$[\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{W}} - \tilde{\boldsymbol{W}}\tilde{\boldsymbol{\Psi}}]\boldsymbol{h} = (\boldsymbol{\Psi} \times \boldsymbol{W}) \times \boldsymbol{h} \qquad \forall\, \boldsymbol{h} \in \mathbb{R}^3 \tag{2.20}$$

$$[\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{W}}]\boldsymbol{h} = [\boldsymbol{W} \otimes \boldsymbol{\Psi} - \boldsymbol{\Psi} \cdot \boldsymbol{W}\boldsymbol{I}]\boldsymbol{h} \qquad \forall\, \boldsymbol{h} \in \mathbb{R}^3 \tag{2.21}$$

$$\tilde{\boldsymbol{\Psi}}^2\boldsymbol{h} = \tilde{\boldsymbol{\Psi}}(\tilde{\boldsymbol{\Psi}}\boldsymbol{h}) = [\boldsymbol{\Psi} \otimes \boldsymbol{\Psi} - \Psi^2\boldsymbol{I}]\boldsymbol{h} \qquad \forall\, \boldsymbol{h} \in \mathbb{R}^3 \tag{2.22}$$

where $\Psi = ||\boldsymbol{\Psi}|| = \sqrt{\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}}$. Note, in equation (2.20) the term in squared brackets denotes the *Lie brackets*.

2.5 Parametrization of finite rotations

Any form describing a three-dimensional rotation in terms of suitable degrees of freedom, is called *parametrization* of rotation matrix, or more properly, rotation parametrization[9].

Over the years, several forms have been proposed in literature to cope with the description of rotation manifold (an extensive investigation on the rotational vector representation can be found in [1], [21] and [39]). Basically, they can be subdivided in two main families: *vector-like parametrization*, of which Euler-Rodrigues and Cayley-Rodrigues parametrizations are the most notably representative, and *non vector-like parametrization*, such as classical Euler's angles representation, Cardan's angles and quaternions.

In general, some representations involve three parameters, some others require more (say up to nine parameters employed in the matrix complete representation). Although, the minimum set of degree of freedom needed to describe a finite rotation is three (in fact the nine components of the rotation matrix $\boldsymbol{\Lambda}$ are related each other by the six orthonormality conditions which characterize a rotation tensor in $SO(3)$, i.e. $\boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \boldsymbol{I}$, and hence the nine components can be expressed by only three independent parameters), it has long been established that a unique global representation based on only three parameters does not exist. It has also been established (e.g. see [69]) that for a unique global representation of finite rotations one needs a minimum of five parameters, in order to establish differentiable 1-1 map (bijection) with the differentiable inverse for representation of $\boldsymbol{\Lambda} \in SO(3)$. Early work of Argyris [1] has also shown that a 4-parameter representation of finite rotations, based on the so-called *quaternions*, is also potentially useful for practical purposes, regardless of the fact that it is not strictly 1-1 but rather a 2-1 representation.

[9]Mathematically, a parametrization is a mapping from an open set of Euclidean space into some open set of the manifold.

Usually, both theoretical and computational issues can play a meaningful role in the choice of the rotation parametrization, which is also influenced by the possible specific requirements of its application.

In the following, an intuitive geometrical deduction of an explicit expression for the rotation tensor in terms of the Cartesian components of a rotation vector through the well known Euler's theorem is presented, and then its properties as well as other possible parameterization are discussed. The subject is extensively treated in [1], [21] and [39].

2.5.1 Vector-like parametrization

The vector-like parametrizations feature a set of three parameters that defines the Cartesian components of a rotation vector. The rotation vector is a vector with direction along the physical axis of rotation, versus defined by the right-hand-rule in dependence of the clockwise or counter-clockwise sense of rotation, and norm equal to the angle of rotation. The general form of such a vectors is

$$\boldsymbol{p} = f\,\boldsymbol{e} \tag{2.23}$$

where $\boldsymbol{p}$ is termed the *generalized rotation vector* (see [8]), while $f = f(\Psi)$ is some yet unspecified function of the angle of rotation Ψ, termed *generating function*, and the unit vector $\boldsymbol{e}$ identifies the axis of rotation. Other than obvious requirement of regularity, the generating function f must obey certain conditions. First we must have

$$f(0) = 0 \tag{2.24}$$

which means that the magnitude of the generalized rotation vector must be null for a null rotation angle Ψ. This ensures that, if $\boldsymbol{b} = \boldsymbol{\Lambda}\boldsymbol{a}$ and $\Psi = 0$, then $\boldsymbol{b} = \boldsymbol{a}$. Taking now the time derivative of $\boldsymbol{p}$ at the origin

$$\dot{\boldsymbol{p}}(0) = \left.\frac{\partial f}{\partial \Psi}\right|_{\Psi=0} \frac{\mathrm{d}\Psi}{\mathrm{d}t}\boldsymbol{e} + f(0)\frac{\mathrm{d}\boldsymbol{e}}{\mathrm{d}t} \tag{2.25}$$

$$= \left.\frac{\partial f}{\partial \Psi}\right|_{\Psi=0} \dot{\Psi}\,\boldsymbol{e} + f(0)\,\dot{\boldsymbol{e}} \tag{2.26}$$

one recognizes that the generating function f must be such that

$$\left.\frac{\partial f}{\partial \Psi}\right|_{\Psi=0} \neq 0 \tag{2.27}$$

ensuring $\dot{\boldsymbol{p}}(0)$ to be different from zero when $\dot{\Psi}$ is different from zero.

This paramentrization has the following features:

i. It has a simple geometric meaning (see Figure 2.7) and for a certain extent it permits a treatment of rotations similar to what is commonly done with translations;

ii. It requires a minimal set of parameters, which for the special orthogonal group in three-dimensional space $SO(3)$ is three;

Table 2.1: Commonly used vector-like parametrizations of three-dimensional rotations, with their correspondent generating function and range of validity.

PARAMETRIZATION TYPE	$f(\Psi)$	Range
Exponential map	Ψ	$-2\pi < \Psi < 2\pi$
Linear parameter	$\sin\Psi$	$-\pi < \Psi < \pi$
Euler-Rodrigues parameter	$2\sin(\Psi/2)$	$-\pi < \Psi < \pi$
Cayley-Gibbs-Rodrigues parameter	$2\tan(\Psi/2)$	$-\pi < \Psi < \pi$
Wiener-Milenkovich parameter	$4\sin(\Psi/4)$	$-2\pi < \Psi < 2\pi$

iii. It has differentiability holes for $\Psi = 2k\pi$, however the drawbacks can be considered mild if the values of Ψ to consider are always far from the first singularity value (say 2π), i.e. in the range of moderate rotations;

iv. It provides a 1-1 corresponding to the orthogonal tensor holding up to moderate rotations within the range of validity of the parametrization itself (see for detail Table 2.5.1);

v. It requires using trigonometric functions;

2.5.2 Euler-Rodrigues parametrization

The expression of the rotation tensor in terms of the rotation vector can be derived at least in two alternative ways: using purely geometric arguments, or using a differential approach. While both ways of looking at the problem clearly lead to the same results, both have interesting peculiarities and highlight different important properties of rotations. Here, for sake of simplicity, we tackle only the geometric approach, while for a more complete treatment of this subject the reader should refer to [8] and references therein.

The Euler's theorem.

The geometric derivation of the rotation tensor can be based on the fundamental Euler's theorem. It states that any arbitrary displacement of a rigid body that leaves one point fixed is a rotation about the unit vector $\boldsymbol{e} \in \mathbb{R}^3$ of the axis of rotation, with magnitude $\Psi = (\boldsymbol{\Psi} \cdot \boldsymbol{\Psi})^{1/2} \in [0, 2\pi]$. A schematic representation of the theorem is depicted in Figure 2.7. From a physical point of view, any three-dimensional rotation can be interpreted as a two-dimensional rotation (measured by the angle Ψ) that takes place in a plane orthogonal to a suitably chosen direction (axis of rotation RR) identified by the unit rotation vector $\boldsymbol{e}$. The two quantities $(\boldsymbol{e}, \Psi)$ are sometimes labeled as the *principal axis* of rotation and the *principal angle* pf rotation, respectively. They completely define the rotational displacement represented by the rotation tensor $\boldsymbol{\Lambda}$. By this way, the notion of a *rotation vector* $\boldsymbol{\Psi} = \Psi\boldsymbol{e}$ introduced in equation (2.23) for describing rotations, is recovered [10].

[10] The vector $\boldsymbol{\Psi}$ is sometime called *pseudo-vector*

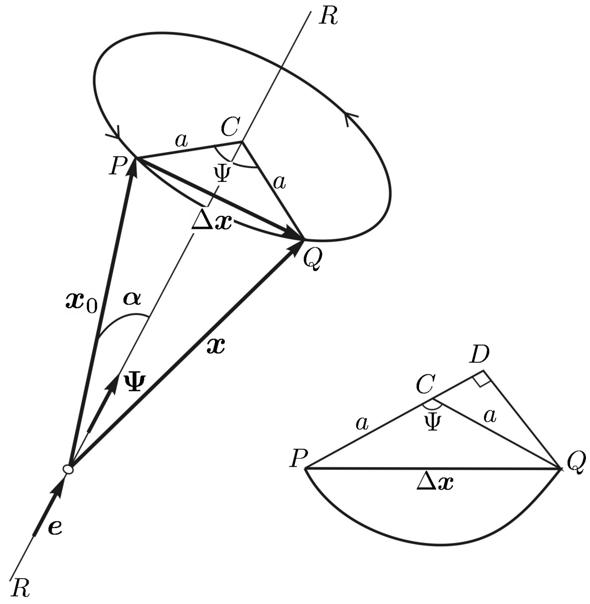


Figure 2.7: Construction of rotation matrix $\boldsymbol{\Lambda}$ for an arbitrary rotation vector $\boldsymbol{\Psi}$.

Euler-Rodrigues formula: geometric approach.

With reference to the *rotation vector*[11]

$$\boldsymbol{\Psi} = \Psi \boldsymbol{e} \tag{2.28}$$

by definition, $\boldsymbol{\Psi}$ has norm equal to Ψ and is directed as the unit rotation axis vector $\boldsymbol{e}$ about which the rotation occurs. Its orientation is such that they form a right-handed screw. Indicating with ϕ, θ, ψ the components [12] of $\boldsymbol{\Psi}$ in a Cartesian reference system $\{O, X, Y, Z\}$, the rotation vector may be expressed as

$$\boldsymbol{\Psi} = \begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \phi \boldsymbol{e}_1 + \theta \boldsymbol{e}_2 + \psi \boldsymbol{e}_3 = \Psi \boldsymbol{e} \tag{2.29}$$

and its norm $\Psi = ||\boldsymbol{\Psi}||$ as

$$\Psi = \sqrt{\phi^2 + \theta^2 + \psi^2} = \sqrt{\boldsymbol{\Psi}^T \boldsymbol{\Psi}} \tag{2.30}$$

Now, consider a vector $\boldsymbol{x_0}$ which is mapped into $\boldsymbol{x}$ by a rotation of an angle Ψ around the axis in direction of $\boldsymbol{e}$, as depicted in Figure 2.7. This mapping takes the form

$$\boldsymbol{x} = \boldsymbol{\Lambda} \boldsymbol{x_0} \tag{2.31}$$

where $\boldsymbol{\Lambda}\colon \mathbb{E}^3 \to \mathbb{E}^3$ is a rotation tensor, nonlinear function of $\boldsymbol{\Psi}$, which transforms linearly and isometrically a vector into another vector in a rotational motion that

[11] It will also regarded as the *cross-section rotational degree of freedom* in the forthcoming chapters.
[12] It's worth to note that, in contrast with small rotations, finite rotations ϕ, θ, ψ cannot be interpreted as component rotations about the Cartesian axis X, Y, Z.

is represented with the aid of the rotation vector. Here we want to find an explicit expression of $\boldsymbol{\Lambda}$ as a function of $\boldsymbol{\Psi}$, i.e. to establish the transformation (see [1] for more details)

$$\boldsymbol{x} = \boldsymbol{\Lambda}(\boldsymbol{\Psi})\, \boldsymbol{x_0} \tag{2.32}$$

which relates the rotated vector $\boldsymbol{x}$ to the original vector $\boldsymbol{x_0}$. On the other hand, looking at the Figure 2.7, the position vector $\boldsymbol{x_0}$ moves to its final position $\boldsymbol{x}$, and the relation between these two vectors is

$$\boldsymbol{x} = \boldsymbol{x_0} + \Delta\boldsymbol{x} \tag{2.33}$$

Since both equations (2.33) and (2.31) hold for the same rotational movement, we are here interested to find the expression of Λ such that the right-hand side of (2.33) equals the matrix product (2.32).
To this end, from Figure 2.7 we first introduce the following notation

$$\Delta\boldsymbol{x} = \overline{PD} + \overline{DQ} \tag{2.34}$$

where $\overline{DQ}$ is drawn normal to $\overline{PC}$. We also note that the vector $\overline{DQ}$ stands perpendicular to the plane OPC, and points hence in the direction $(\boldsymbol{e} \times \boldsymbol{x})$. To find its magnitude we observe that

$$DQ = a \sin\Psi \tag{2.35}$$

On the other hand, we observe that the magnitude of $(\boldsymbol{e} \times \boldsymbol{x})$ is

$$||\boldsymbol{e} \times \boldsymbol{x}|| = 1 \cdot x \sin\alpha = x\frac{a}{x} = a \tag{2.36}$$

It follows in conjunction with (2.35), (2.36) and (2.28) that

$$\overline{DQ} = (\boldsymbol{e} \times \boldsymbol{x}) \sin\Psi = \frac{\sin\Psi}{\Psi}(\boldsymbol{\Psi} \times \boldsymbol{x}) \tag{2.37}$$

We next proceed to the determination of the vector $\overline{PD}$. Figure 2.7 demonstrates immediately that it is not only perpendicular to $(\boldsymbol{e} \times \boldsymbol{x})$, but also to $\boldsymbol{e}$, since it lies in the plane PCQ normal to $\boldsymbol{e}$. Hence it may be assigned the direction of $\boldsymbol{e} \times (\boldsymbol{e} \times \boldsymbol{x})$. Now the absolute value of the last vector is clearly again a since $\boldsymbol{e}$ is a unit vector and is also normal to $(\boldsymbol{e} \times \boldsymbol{x})$, i.e.

$$||\boldsymbol{e} \times (\boldsymbol{e} \times \boldsymbol{x})|| = ||\boldsymbol{e} \times \boldsymbol{x}|| = a \tag{2.38}$$

At the same time Figure 2.7 yields to [13]

$$PD = a - a\cos\Psi = (1 - a\cos\Psi)a = 2\sin^2\frac{\Psi}{2}a \tag{2.39}$$

[13] We used the half-angle trigonometric formula

$$\sin^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2}$$

Hence, using (2.39) in conjunction with the direction of $\overline{PD}$ and (2.28), we deduce

$$\overline{PD} = 2\sin^2\frac{\Psi}{2}(\boldsymbol{e}\times(\boldsymbol{e}\times\boldsymbol{x})) = \frac{1}{2}\frac{\sin^2(\Psi/2)}{(\Psi/2)^2}(\boldsymbol{\Psi}\times(\boldsymbol{\Psi}\times\boldsymbol{x})) \tag{2.40}$$

Applying (2.37) and (2.40) in (2.33) and substituting into (2.34), the vector $\boldsymbol{x}$ takes the form

$$\begin{aligned}\boldsymbol{x} &= \boldsymbol{x}_0 + \boldsymbol{\Delta x}\\ &= \boldsymbol{x}_0 + \overline{PD} + \overline{DQ}\\ &= \boldsymbol{x}_0 + \frac{\sin\Psi}{\Psi}(\boldsymbol{\Psi}\times\boldsymbol{x}) + \frac{1}{2}\frac{\sin^2(\Psi/2)}{(\Psi/2)^2}(\boldsymbol{\Psi}\times(\boldsymbol{\Psi}\times\boldsymbol{x}))\end{aligned} \tag{2.41}$$

Now we can rewrite equation (2.41) in matrix form

$$\boldsymbol{x} = \boldsymbol{x}_0 + \frac{\sin\Psi}{\Psi}\tilde{\boldsymbol{\Psi}}\boldsymbol{x}_0 + \frac{1}{2}\frac{\sin^2(\Psi/2)}{(\Psi/2)^2}\tilde{\boldsymbol{\Psi}}^2\boldsymbol{x}_0 \tag{2.42}$$

where we have

$$\tilde{\boldsymbol{\Psi}} = \begin{bmatrix} 0 & -\psi & \theta \\ \psi & 0 & -\phi \\ -\theta & \phi & 0 \end{bmatrix} \qquad \text{and} \tag{2.43}$$

$$\tilde{\boldsymbol{\Psi}}^2 = \tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Psi}} = \begin{bmatrix} -(\theta^2+\psi^2) & \theta\phi & \phi\psi \\ \theta\phi & -(\psi^2+\phi^2) & \theta\psi \\ \phi\psi & \theta\psi & -(\phi^2+\theta^2) \end{bmatrix} \tag{2.44}$$

The reader will observe that the form of $\tilde{\boldsymbol{\Psi}}$ is identical to the well-known antisymmetrical matrix representing infinitely small rotations. However, in the present (more general) context, $\tilde{\boldsymbol{\Psi}}$ contains the Cartesian components of a finite rotational axial vector $\boldsymbol{\Psi}$. The equality between equations (2.41) and (2.42) is easily shown recalling that by definition, the cross product $\boldsymbol{\Psi}\times\boldsymbol{x}$ and $\boldsymbol{\Psi}\times(\boldsymbol{\Psi}\times\boldsymbol{x})$ can be given in the matrix form

$$\boldsymbol{\Psi}\times\boldsymbol{x} = \tilde{\boldsymbol{\Psi}}\boldsymbol{x} \qquad \text{and} \qquad \boldsymbol{\Psi}\times(\boldsymbol{\Psi}\times\boldsymbol{x}) = \tilde{\boldsymbol{\Psi}}^2\boldsymbol{x} \tag{2.45}$$

Equation (2.42) is the transformation we were looking for, i.e.

$$\boldsymbol{x} = \boldsymbol{\Lambda}(\boldsymbol{\Psi})\,\boldsymbol{x_0} \quad \text{with} \quad \boxed{\boldsymbol{\Lambda}(\boldsymbol{\Psi}) = \boldsymbol{I} + \frac{\sin\Psi}{\Psi}\tilde{\boldsymbol{\Psi}} + \frac{1}{2}\frac{\sin^2(\Psi/2)}{(\Psi/2)^2}\tilde{\boldsymbol{\Psi}}^2} \tag{2.46}$$

Substituting the trigonometric identity $\sin^2\alpha = \frac{1-cos(2\alpha)}{2}$ into the argument of $\tilde{\boldsymbol{\Psi}}^2$ we obtain an equivalent form of the transformation $\boldsymbol{\Lambda}(\boldsymbol{\Psi})$

$$\boxed{\boldsymbol{\Lambda}(\boldsymbol{\Psi}) = \boldsymbol{I} + \frac{\sin\Psi}{\Psi}\tilde{\boldsymbol{\Psi}} + \frac{1-\cos\Psi}{\Psi^2}\tilde{\boldsymbol{\Psi}}^2} \tag{2.47}$$

These two equivalent formula establish the relation between the total rotation vector $\boldsymbol{\Psi}$ and the rotation tensor $\boldsymbol{\Lambda}$, and represent the Euler rotation vector parametrization of rotation tensor $\mathbf{\Lambda}(\mathbf{\Psi})$. They are known in literature as Euler-Rodrigues formula. It is worth to note that the two trigonometric functions which compares into (2.46) are real and continuous.

Alternative form of Euler-Rodrigues formula.

The equation establish in (2.47) can be recast in alternative, but equivalent forms, bringing out the unit rotation vector $\boldsymbol{e} = \mathbf{\Psi}/\Psi$. Writing (2.46) using the vector notation for skew tensor $\tilde{\mathbf{\Psi}} = [\mathbf{\Psi}\times]$, we obtain

$$\mathbf{\Lambda}(\mathbf{\Psi}) = \boldsymbol{I} + \frac{\sin\Psi}{\Psi}[\mathbf{\Psi}\times] + \frac{1-\cos\Psi}{\Psi^2}[\mathbf{\Psi}\times[\mathbf{\Psi}\times]] \tag{2.48}$$

which, pointing out $\boldsymbol{e}$, becomes

$$\boxed{\mathbf{\Lambda}(\mathbf{\Psi}) = \boldsymbol{I} + \sin\Psi[\boldsymbol{e}\times] + (1-\cos\Psi)[\boldsymbol{e}\times[\boldsymbol{e}\times]]} \tag{2.49}$$

Alternatively, introducing in (2.47) the identity (2.22) which we recall below

$$\tilde{\mathbf{\Psi}}^2\boldsymbol{b} = \tilde{\mathbf{\Psi}}(\tilde{\mathbf{\Psi}}\boldsymbol{b}) = (\mathbf{\Psi}\otimes\mathbf{\Psi} - \Psi^2\boldsymbol{I})\boldsymbol{b} \qquad \forall\,\boldsymbol{b}\in\mathbb{R}^3 \tag{2.50}$$

we obtain

$$\begin{aligned}\mathbf{\Lambda}(\mathbf{\Psi}) &= \boldsymbol{I} + \frac{\sin\Psi}{\Psi}\tilde{\mathbf{\Psi}} + \frac{1-\cos\Psi}{\Psi^2}(\mathbf{\Psi}\otimes\mathbf{\Psi} - \Psi^2\boldsymbol{I}) \\ &= \cos\Psi\boldsymbol{I} + \frac{\sin\Psi}{\Psi}\tilde{\mathbf{\Psi}} + \frac{1-\cos\Psi}{\Psi^2}\mathbf{\Psi}\otimes\mathbf{\Psi}\end{aligned} \tag{2.51}$$

Recognizing again $\boldsymbol{e}$, by substitution into the previous equation, we get another form of Euler-Rodrigues formula

$$\boxed{\mathbf{\Lambda}(\mathbf{\Psi}) = \cos\Psi\boldsymbol{I} + \sin\Psi[\boldsymbol{e}\times] + (1-\cos\Psi)(\boldsymbol{e}\otimes\boldsymbol{e})} \tag{2.52}$$

This relation makes comprehensible how the rotation operator does not depend on the multiplies of the rotation revolution counts, i.e. $\mathbf{\Lambda}(\mathbf{\Psi}) = \mathbf{\Lambda}(\mathbf{\Psi}+2i\pi\boldsymbol{e})$ with $i = 1, 2, \ldots$.

Orthogonality conditions.

As previously stated, the rotation tensor is characterized by some important properties, among which the orthogonality condition is one of the most fundamentals. In order to show how equation (2.47) satisfies this property, one can note that, since $\tilde{\mathbf{\Psi}}$ is a skew-tensor such that $\tilde{\mathbf{\Psi}}^T = -\tilde{\mathbf{\Psi}}$, the transpose $\mathbf{\Lambda}^T(\mathbf{\Psi})$ takes form

$$\mathbf{\Lambda}^T(\mathbf{\Psi}) = \boldsymbol{I} - \frac{\sin\Psi}{\Psi}\tilde{\mathbf{\Psi}} + \frac{1-\cos\Psi}{\Psi^2}\tilde{\mathbf{\Psi}}^2 \tag{2.53}$$

Evaluating equation (2.47) for $\mathbf{\Psi} = -\mathbf{\Psi}$, recalling that $\sin(-\Psi) = -\sin(\Psi)$, and resuming the cross product anticommutativity property $(-\mathbf{\Psi}\times\boldsymbol{x}) = -(\mathbf{\Psi}\times\boldsymbol{x})$, and also $-\mathbf{\Psi}\times(-\mathbf{\Psi}\times\boldsymbol{x}) = \mathbf{\Psi}\times(\mathbf{\Psi}\times\boldsymbol{x})$, we can state that

$$\mathbf{\Lambda}^T(\mathbf{\Psi}) = \mathbf{\Lambda}(-\mathbf{\Psi}) \tag{2.54}$$

and finally the inverse relation to (2.46) reads

$$\boldsymbol{x_0} = \boldsymbol{\Lambda}(-\boldsymbol{\Psi})\,\boldsymbol{x} = \boldsymbol{\Lambda}^T(\boldsymbol{\Psi})\,\boldsymbol{x} \tag{2.55}$$

where $\boldsymbol{\Lambda}(-\boldsymbol{\Psi})$ clearly maps the rotated vector $\boldsymbol{x}$ back into its original position $\boldsymbol{x_0}$, since $\boldsymbol{\Lambda}(-\boldsymbol{\Psi})$ is the rotation around the inverted rotation vector. Therefore $\boldsymbol{\Lambda}(-\boldsymbol{\Psi}) = \boldsymbol{\Lambda}^{-1}(\boldsymbol{\Psi})$ and accordingly we have

$$\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}(-\boldsymbol{\Psi}) = \boldsymbol{\Lambda}^T(\boldsymbol{\Psi}) \tag{2.56}$$

which finally confirms that $\boldsymbol{\Lambda}(\boldsymbol{\Psi})$ is orthogonal.

2.5.3 Exponential mapping.

Expression (2.46) could be transformed into a theoretically more convenient function of $\tilde{\boldsymbol{\Psi}}$. It will be show that this representation has remarkable advantage to simplify differentiation of rotation $\boldsymbol{\Lambda}$. Because of this favorable property, the exponential map has become a favorite of implementation where large angles may occur in a three-dimensional motion.

Theorem 2.9 (Exponential function)**.** *Consider the series expansion of* $\exp[\tilde{\boldsymbol{\Psi}}]$

$$\exp[\tilde{\boldsymbol{\Psi}}] = \boldsymbol{I} + \tilde{\boldsymbol{\Psi}} + \frac{1}{2!}\tilde{\boldsymbol{\Psi}}^2 + \frac{1}{3!}\tilde{\boldsymbol{\Psi}}^3 + \ldots + \frac{1}{n!}\tilde{\boldsymbol{\Psi}}^n + \ldots \tag{2.57}$$

which is by definition the exponential function *of the skew-tensor* $\tilde{\boldsymbol{\Psi}}$*. It can be proved (see [27] page 228) that the exponential function of a generic skew-symmetric tensor is a rotation tensor, i.e. the exponential is the chart which maps a skew-symmetric tensor in a proper rotation tensor, i.e.*

$$\boxed{\boldsymbol{\Lambda} = \exp[\tilde{\boldsymbol{\Psi}}]} \tag{2.58}$$

Proof. Here we proof [14] that the expression of $\boldsymbol{\Lambda}$ (2.46) yields to the exponential map of $\tilde{\boldsymbol{\Psi}}$. We deduce it in two steps: in the first the trigonometric functions (2.46) are expanded in series in Ψ; subsequentually in the second step, by a judicious consideration of the powers in $\tilde{\boldsymbol{\Psi}}$, we transform the series finally into one in $\tilde{\boldsymbol{\Psi}}$. Lastly, according to [23] (see [23] page 287) a simple proof of orthogonality conditions and $\det \boldsymbol{\Lambda} = +1$ is reported.

We start expanding in series with respect to $\boldsymbol{\Psi}$ the trigonometric functions [15] in

[14]The surprisingly concise and elegant result of (2.58) may also be deduced by arguments based on Lie's theory of groups as is done in quantum mechanics [1].

[15]The Taylor's series expansion for sine and cosine function holds

$$\sin\Psi = \Psi - \frac{\Psi^3}{3!} + \frac{\Psi^5}{5!} - \ldots$$
$$\cos\Psi = 1 - \frac{\Psi^2}{2!} + \frac{\Psi^4}{4!} - \ldots$$

(2.46), which yields to

$$\boldsymbol{\Lambda}(\tilde{\boldsymbol{\Psi}}) = \boldsymbol{I} + \left[1 - \frac{\Psi^2}{3!} + \frac{\Psi^4}{5!} + \ldots + (-1)\frac{\Psi^{2n}}{(2n+1)!} \pm \ldots\right] \tilde{\boldsymbol{\Psi}} + \left[\frac{1}{2!} - \frac{\Psi^2}{4!} + \frac{\Psi^4}{6!} + \ldots + (-1)\frac{\Psi^{2n}}{(2n+2)!} \pm \ldots\right] \tilde{\boldsymbol{\Psi}}^2 \tag{2.59}$$

Next we consider the skew-symmetric tensor $\tilde{\boldsymbol{\Psi}}$ with axial vector $\boldsymbol{\Psi}$ and relative norm $\Psi = ||\boldsymbol{\Psi}||$. By explicit computation, we can observe the interesting relations

$$\tilde{\boldsymbol{\Psi}}^3 = -\Psi^2\tilde{\boldsymbol{\Psi}}, \qquad \tilde{\boldsymbol{\Psi}}^5 = +\Psi^4\tilde{\boldsymbol{\Psi}}$$
$$\tilde{\boldsymbol{\Psi}}^4 = -\Psi^2\tilde{\boldsymbol{\Psi}}^2, \qquad \tilde{\boldsymbol{\Psi}}^6 = +\Psi^4\tilde{\boldsymbol{\Psi}}^2 \tag{2.60}$$

which leads to the recurrence formula

$$\tilde{\boldsymbol{\Psi}}^{2n-1} = (-1)^{n-1}\Psi^{2(n-1)}\tilde{\boldsymbol{\Psi}} \qquad \tilde{\boldsymbol{\Psi}}^{2n} = (-1)^{n-1}\Psi^{2(n-1)}\tilde{\boldsymbol{\Psi}}^2 \tag{2.61}$$

Developing the multiplications in equation (2.59), and then substituting into it the right-hand side of (2.61), we deduce $\boldsymbol{\Lambda}$ as a series expansion of $\tilde{\boldsymbol{\Psi}}$

$$\boldsymbol{\Lambda}(\tilde{\boldsymbol{\Psi}}) = \boldsymbol{I} + \tilde{\boldsymbol{\Psi}} + \frac{1}{2!}\tilde{\boldsymbol{\Psi}}^2 + \frac{1}{3!}\tilde{\boldsymbol{\Psi}}^3 + \ldots + \frac{1}{n!}\tilde{\boldsymbol{\Psi}}^n + \ldots \tag{2.62}$$

which proves, in fact, the equality (2.58)

$$\boldsymbol{\Lambda} = \exp[\tilde{\boldsymbol{\Psi}}] \tag{2.63}$$

In terms of the rotational vector $\boldsymbol{\Psi}$, equations (2.57) and (2.58) give the exact value of the current rotation matrix. Using truncated MacLaurin's series of various order in equation (2.57), approximated values of the rotation matrix are obtained and corresponding simplified theories can be derived. For example, a so called first order theory is obtained if small rotations are assumed, so that the quadratic and higher order terms in (2.57) may be neglected. However, in this work no simplification are addressed in order to maintain the geometric exactness in the kinematic description of body motion.

Orthogonality condition is quickly obtained by noting from (2.58) $\boldsymbol{\Lambda}^T = \exp[\tilde{\boldsymbol{\Psi}}^T] = \exp[-\tilde{\boldsymbol{\Psi}}^T] = \boldsymbol{\Lambda}^{-1}$.

We recall that for $\det\boldsymbol{\Lambda} = +1$ we have a proper rotation, while for $\det\boldsymbol{\Lambda} = -1$ we get a combination of a rotation and a reflection in a coordinate plane. Now, if the eigenvalues of $\tilde{\boldsymbol{\Psi}}$ are μ_i, hence by definition $\mathrm{Tr}[\tilde{\boldsymbol{\Psi}}] = \sum_{i=1}^{n}\mu_i = 0$. From (2.58), the eigenvalues of $\boldsymbol{\Lambda}e^{\tilde{\boldsymbol{\Psi}}}$ are $\lambda_1 = e^{\mu_1}$, $\lambda_2 = e^{\mu_2}$, ..., $\lambda_n = e^{\mu_n}$, i.e. $\lambda_i = e^{\mu_i}$, thus we have

$$\begin{aligned}\det\boldsymbol{\Lambda} &= \prod_{i=1}^{n}\lambda_i \\ &= e^{\mu_1}e^{\mu_2}\ldots e^{\mu_n} \\ &= e^{\sum_{i=1}^{n}\mu_i} \\ &= e^0 = 1\end{aligned} \tag{2.64}$$

□

2.5.4 Cayley-Rodrigues parametrization.

Because of trigonometric functions involved, Euler-Rodrigues parametrization could show an high computational cost. According to [21] it can be possible to reformulate expression (2.46) in slightly different forms by means of the so-called *Rodrigues rotation parameters*. This parametrization, also known as Cayley-Gibbs-Rodrigues parametrization, was primarily proposed by Rodrigues.

Instead of using the rotation vector $\boldsymbol{\Psi}$ (2.29), we can establish a pseudo-vector $\boldsymbol{\varpi}$ as

$$\boxed{\boldsymbol{\varpi} = \varpi\, \boldsymbol{e} \triangleq 2\tan\frac{\Psi}{2}\,\boldsymbol{e} = \frac{\tan(\Psi/2)}{\Psi/2}\,\boldsymbol{\Psi}} \tag{2.65}$$

with Cartesian components $\{\varpi_x, \varpi_y, \varpi_z\}$, with

$$\varpi = 2\tan\frac{\Psi}{2} = \sqrt{\varpi_x^2 + \varpi_y^2 + \varpi_z^2} \tag{2.66}$$

Sometimes these parameters are defined without the factor 2 in equation (2.65). However, definition as in (2.65) has the advantage to lead $\varpi = \Psi$ up to second order.

While the Euler-Rodrigues representation $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(\boldsymbol{\Psi})$ is valid for any value of Ψ, the use of Rodrigues rotation vectors implies to have an existence domain $0 \leq \Psi \leq \pi$ because definition of (2.65). In fact equation (2.65) collapses as Ψ nears π, since $\tan\frac{1}{2}\Psi \to \pm\infty$ as $\Psi \to \pi$, and singularity hold for $\Psi = \pi + 2k\pi$, $k = 1, 2, 3, \ldots$.

In order to determine the form of the rotation matrix according to the new parametrization, we start recalling equation (2.41), and substituting the last of (2.65), the following trigonometric identities hold

i.

$$\begin{aligned}
\frac{\sin\Psi}{\Psi}\cdot\frac{\Psi}{2\tan(\Psi/2)} &= \frac{\sin\Psi}{\Psi}\cdot\frac{\Psi}{2}\cdot\frac{1+\cos\Psi}{\sin\Psi}\\
&= \frac{1+\cos\Psi}{2}\\
&= \cos^2\frac{\Psi}{2}
\end{aligned}$$

ii.

$$\begin{aligned}
\frac{\Psi}{2\tan(\Psi/2)}\cdot\frac{\Psi}{2\tan(\Psi/2)}\cdot\frac{1}{2}\left(\frac{\sin(\Psi/2)}{\Psi/2}\right)^2 &= \frac{\Psi}{2\tan(\Psi/2)}\cdot\frac{\Psi}{4}\cdot\frac{1+\cos\Psi}{\sin\Psi}\cdot\frac{\sin^2(\Psi/2)}{(\Psi/2)^2}\\
&= \frac{\Psi}{2\tan(\Psi/2)}\cdot\frac{1}{\Psi}\cdot\frac{1+\cos\Psi}{\sin\Psi}\cdot\frac{1-\cos\Psi}{2}\\
&= \frac{\Psi}{2\tan(\Psi/2)}\cdot\frac{\sin\Psi}{2\Psi}\\
&= \frac{1+\cos\Psi}{2\sin\Psi}\cdot\frac{\sin\Psi}{2}\\
&= \frac{1}{2}\cdot\cos^2\frac{\Psi}{2}
\end{aligned}$$

consequentially equation (2.41) becomes

$$\boldsymbol{x} = \boldsymbol{x}_0 + \cos^2\frac{\Psi}{2}(\boldsymbol{\varpi}\times\boldsymbol{x}) + \frac{1}{2}\cos^2\frac{\Psi}{2}(\boldsymbol{\varpi}\times(\boldsymbol{\varpi}\times\boldsymbol{x})) \tag{2.67}$$

Using the trigonometric identity $\cos^2\alpha = (1+\tan^2\alpha)$ and the scalar product $\boldsymbol{\varpi}^T\cdot\boldsymbol{\varpi} = 4\tan^4\frac{\Psi}{2}$, we note from (2.65) that

$$\cos^2\frac{\Psi}{2} = \frac{1}{1+\tan^2(\Psi/2)} = \frac{1}{1+\frac{1}{4}\boldsymbol{\varpi}^T\cdot\boldsymbol{\varpi}} \tag{2.68}$$

and finally equation (2.67) turns into

$$\boldsymbol{x} = \boldsymbol{x}_0 + \frac{1}{1+\frac{1}{4}\boldsymbol{\varpi}^T\cdot\boldsymbol{\varpi}}\left[(\boldsymbol{\varpi}\times\boldsymbol{x}) + \frac{1}{2}(\boldsymbol{\varpi}\times(\boldsymbol{\varpi}\times\boldsymbol{x}))\right] \tag{2.69}$$

We introduce now the auxiliary matrix $\tilde{\boldsymbol{R}} = \text{skew}(\boldsymbol{\varpi})$ which takes the place of $\tilde{\boldsymbol{\Psi}}$ in (2.43), and write

$$\tilde{\boldsymbol{R}} = \begin{bmatrix} 0 & -\varpi_z & \varpi_y \\ \varpi_z & 0 & -\varpi_x \\ -\varpi_y & \varpi_x & 0 \end{bmatrix} = \frac{\tan(\Psi/2)}{\Psi/2}\begin{bmatrix} 0 & -\psi & \theta \\ \theta & 0 & -\phi \\ -\theta & \phi & 0 \end{bmatrix} = \frac{\tan(\Psi/2)}{\Psi/2}\tilde{\boldsymbol{\Psi}} \tag{2.70}$$

In analogy to (2.45) we note the vector and matrix rules

$$\boldsymbol{\varpi}\times\boldsymbol{x} = \tilde{\boldsymbol{R}}\boldsymbol{x} \qquad \text{and} \qquad \boldsymbol{\varpi}\times(\boldsymbol{\varpi}\times\boldsymbol{x}) = \tilde{\boldsymbol{R}}^2\boldsymbol{x} \tag{2.71}$$

which yield in pursuance of (2.69) the matrix expression

$$\boldsymbol{x} = \boldsymbol{x}_0 + \frac{1}{1+\frac{1}{4}\boldsymbol{\varpi}^T\cdot\boldsymbol{\varpi}}\left[\tilde{\boldsymbol{R}} + \frac{1}{2}\tilde{\boldsymbol{R}}^2\right]\boldsymbol{x} = \boldsymbol{\Lambda}(\boldsymbol{\varpi})\,\boldsymbol{x} \tag{2.72}$$

with the rotation matrix

$$\boxed{\boldsymbol{\Lambda}(\boldsymbol{\varpi}) = \boldsymbol{I} + \frac{1}{1+\frac{1}{4}\boldsymbol{\varpi}^T\cdot\boldsymbol{\varpi}}\left[\tilde{\boldsymbol{R}} + \frac{1}{2}\tilde{\boldsymbol{R}}^2\right]} \tag{2.73}$$

which is formally the same assumed in [48], i.e.

$$\boldsymbol{\Lambda}(\boldsymbol{\varpi}) = \boldsymbol{I} + \frac{4}{4+\boldsymbol{\varpi}^2}\left[\tilde{\boldsymbol{R}} + \frac{1}{2}\tilde{\boldsymbol{R}}^2\right] \tag{2.74}$$

and with inverse transformation matrix $\boldsymbol{\Lambda}(-\boldsymbol{\varpi})$, being the rotation matrix orthogonal

$$\boldsymbol{\Lambda}(-\boldsymbol{\varpi}) = \boldsymbol{I} + \frac{4}{4+\boldsymbol{\varpi}^2}\left[-\tilde{\boldsymbol{R}} + \frac{1}{2}\tilde{\boldsymbol{R}}^2\right] = \boldsymbol{\Lambda}^T(\boldsymbol{\varpi}) \tag{2.75}$$

2.6 Configurational description of compound rotations

2.6.1 Spatial and material description of compound rotations

As it has been introduced, rotations can be defined by means of rotation operators. It is straightforward to see that the components of a given rotation operator depend on the reference frame adopted. In fact they could be directly expressed in terms of a fixed reference frame, usually called *material*, or alternatively in terms of a movable reference frame rigidly attached to the rotating body, commonly called *spatial*.

To explain better the nature of the two descriptions, let suppose a sequence of N consecutive rotations $\mathbf{\Lambda}_1, \ldots, \mathbf{\Lambda}_i, \ldots, \mathbf{\Lambda}_N$, which are composed together to generate another rotation $\mathbf{\Lambda}_c$ applied to an arbitrary vector $\boldsymbol{h} \in \mathbb{R}^3$. The resulting compound rotation can be defined by two different, through completely equivalent (dual), ways:

1. **Spatial description**: In this case the vector $\boldsymbol{h}_c \in \mathbb{R}^3$ obtained by the application of the sequence of rotations on vector $\boldsymbol{h}$ can be seen as the result of the application of a compound rotation $\mathbf{\Lambda}_c \in SO(3)$ obtained by the consecutive application of the rotation tensor $\mathbf{\Lambda}_i \in SO(3)$ $(i = 1, \ldots, N)$ on the previous rotated vector, i.e.

$$\boldsymbol{h}_c = \mathbf{\Lambda}_N(\ldots(\mathbf{\Lambda}_i(\ldots(\mathbf{\Lambda}_1(\boldsymbol{h}))))) = \mathbf{\Lambda}_N \ldots \mathbf{\Lambda}_i \ldots \mathbf{\Lambda}_1(\boldsymbol{h}) = \mathbf{\Lambda}_c(\boldsymbol{h}) \tag{2.76}$$

   Therefore, the inverse multiplicative rule for rotation tensors is valid for the composition of rotations.

2. **Material description**: In this case the direct multiplicative rule is valid for the composition of rotations, i.e.

$$\boldsymbol{h}_c = \mathbf{\Lambda}_1(\ldots(\mathbf{\Lambda}_i(\ldots(\mathbf{\Lambda}_N(\boldsymbol{h}))))) = \mathbf{\Lambda}_1 \ldots \mathbf{\Lambda}_i \ldots \mathbf{\Lambda}_N(\boldsymbol{h}) = \mathbf{\Lambda}_c(\boldsymbol{h}) \tag{2.77}$$

   where the components of the generic rotation tensor $\mathbf{\Lambda}_{i+1}$ representing the $(i+1)$-th rotation are expressed in the new rotated, or updated, reference system, affected by rotation $\mathbf{\Lambda}_i$ and previous ones.

It is important to note that in both cases, the resulting configuration is the same. Therefore, the composition of two or more rotations, defined in terms of a spatially fixed reference frame, is the same as these obtained applying the same sequence of rotations referred to a rotating frame, but inverting the order of the composition. Thus, the order of the matrix products that appear in the composition of successive rotations is crucial, and depends on the bases in which the components of the various rotation tensors are measured.

2.6.2 Compound rotation

In this section we perform a more formal description of compound rotations in terms of configurational description of the rotational motion. To this end, let $\{\boldsymbol{E}_i\}$ and $\{\boldsymbol{t}_i\}$ be two spatially fixed (inertial) reference coordinate systems, identified as the material and spatial coordinate system, respectively. Given two rotation vectors $\boldsymbol{\Psi} = \Psi_i \boldsymbol{E}_i$ for material frame and the respective $\boldsymbol{\psi} = \psi_i \boldsymbol{t}_i$ for spatial frame, it is possible to obtain $\boldsymbol{\Lambda} \in SO(3)$ by means of applying the exponential mapping as

$$\boldsymbol{\Lambda} = \exp[\tilde{\boldsymbol{\Psi}}] = \exp[\tilde{\boldsymbol{\psi}}] \tag{2.78}$$

with $\tilde{\boldsymbol{\Psi}}$ and $\tilde{\boldsymbol{\psi}}$ being the skew-symmetric tensor obtained from $\boldsymbol{\Psi}$ and $\boldsymbol{\psi}$, respectively. In this manner, the rotation tensor $\boldsymbol{\Lambda}$ is parametrized in the material or spatial description, although the rotation tensor itself can be regarded as a two point operator. If a rotation increment is applied, it is possible to obtain the new compound rotation according to equation (2.76) and (2.77), and using equation (2.58) it is possible to define the material and spatial descriptions of the compound rotations as described below.

Material description of compound rotation. Given a material incremental rotation vector $\tilde{\boldsymbol{\Theta}} = \Theta_i \boldsymbol{E}_i$, the new compound rotation tensor $\boldsymbol{\Lambda}_c$ is described by the left translation mapping, defined as an operator with base point in $\boldsymbol{\Lambda} \in SO(3)$ and described by

$$\begin{aligned} \mathrm{left}_{\boldsymbol{\Lambda}}(\cdot)\colon SO(3) &\longrightarrow SO(3) \\ \exp[\tilde{\boldsymbol{\Theta}}] &\longmapsto \boldsymbol{\Lambda}_c = \boldsymbol{\Lambda}\exp[\tilde{\boldsymbol{\Theta}}] = \boldsymbol{\Lambda}\boldsymbol{\Lambda}^{mat} \end{aligned} \tag{2.79}$$

where $\boldsymbol{\Lambda}^{mat} \in SO(3)$ is the material form of the incremental rotation operator, while $\tilde{\boldsymbol{\Theta}} \in so(3)$ is an incremental material rotation tensor. It is worth to note that the left translation map is defined as acting on an element of $so(3)$ but the final updating procedure requires the specification of a base point $\boldsymbol{\Lambda}$ on the rotational manifold $SO(3)$. This description is called material since the incremental rotation operator acts on a material tensor space.

Spatial description of compound rotation. Given a spatial incremental rotation vector $\boldsymbol{\theta} = \theta_i \boldsymbol{t}_i$, the description of the new compound rotation tensor $\boldsymbol{\Lambda}_c$ can be obtained by the right translation mapping, defined as an operator with base point in $\boldsymbol{\Lambda} \in SO(3)$ and described by

$$\begin{aligned} \mathrm{right}_{\boldsymbol{\Lambda}}(\cdot)\colon SO(3) &\longrightarrow SO(3) \\ \exp[\tilde{\boldsymbol{\theta}}] &\longmapsto \boldsymbol{\Lambda}_c = \exp[\tilde{\boldsymbol{\theta}}]\boldsymbol{\Lambda} = \boldsymbol{\Lambda}^{spat}\boldsymbol{\Lambda} \end{aligned} \tag{2.80}$$

where $\boldsymbol{\Lambda}^{spat} \in SO(3)$ is the spatial form of the incremental rotation operator, while $\tilde{\boldsymbol{\theta}} \in so(3)$ is an incremental spatial rotation tensor. The right translation map is defined as acting on an element of $so(3)$, but the final updating procedure requires the specification of a base point $\boldsymbol{\Lambda}$ on the rotational manifold $SO(3)$. This description is called spatial since the incremental rotation operator acts on a spatial tensor space. The duality appearing in the definitions (2.79) and (2.80) results from the fact that the rotation group is non commutative.

Relations between spatial and material descriptions. The spatial and material descriptions of the incremental rotation tensor, their incremental rotation vectors and the skew-symmetric tensors are related by (see [36])

1. Inner automorphism: $\mathbf{\Lambda}^{spat} = \mathbf{\Lambda}\mathbf{\Lambda}^{mat}\mathbf{\Lambda}^T$;
2. Lie algebra adjoint transformation on $so(3)$: $\tilde{\boldsymbol{\theta}} = \mathbf{\Lambda}\tilde{\mathbf{\Theta}}\mathbf{\Lambda}^T$;
3. Lie algebra adjoint transformation via cross product as Lie algebra: $\boldsymbol{\theta} = \mathbf{\Lambda}\mathbf{\Theta}$.

where the third property indicates that the rotation vector $\boldsymbol{\theta}$ is unaffected by the rotation, as in fact expected since the rotation takes place about rotation axis, that is indeed parallel (coaxial) to $\boldsymbol{\theta}$. This property is closely related to the eigen-analysis of $\mathbf{\Lambda}$. In fact, the eigenvalue problem for rotation tensor is written as

$$(\mathbf{\Lambda} - \lambda_i \boldsymbol{I})\boldsymbol{x_i} = \mathbf{0} \qquad i = 1, 2, 3 \tag{2.81}$$

where λ_i is the $i-$th eigenvalue and $\boldsymbol{x_i}$ its associated eigenvector. Rearranging Lie algebra adjoint transformation via cross product as

$$(\mathbf{\Lambda} - \boldsymbol{I})\boldsymbol{\theta} = \mathbf{0} \tag{2.82}$$

and comparing this expression with (2.81), we conclude that the axis of rotation can be regarded as the eigenvector of the rotation tensor corresponding to the eigenvalue $\lambda = 1$.

Remark 1. *As it has be shown,* $\exp[\tilde{\mathbf{\Theta}}] \in SO(3)$*, with* $\tilde{\mathbf{\Theta}}$ *being the skew-symmetric tensor obtained from* $\mathbf{\Theta} \in \mathbb{R}^3$ *that belongs to the tangential space of* $SO(3)$ *at the identity on* $SO(3)$*; i.e.* $\tilde{\mathbf{\Theta}} \in so(3) \approx \mathcal{T}_{\boldsymbol{I}}SO(3)$. □

2.6.3 Tangent spaces of rotation manifold

Material tangent space to $SO(3)$. Taking the directional (Gateaux) derivative of the compound rotation according to definition (2.77), i.e. differentiating the perturbed configuration of the material form of the compound rotation $\mathbf{\Lambda}_c = \mathbf{\Lambda}\exp[\varepsilon\tilde{\mathbf{\Theta}}]$ with respect to the scalar parameter ε and setting $\varepsilon = 0$, one obtain

$$\mathbf{\Lambda}\, exp(\varepsilon\tilde{\mathbf{\Theta}})|_{\varepsilon=0} = \mathbf{\Lambda} \tag{2.83}$$

$$\left.\frac{\mathrm{d}\mathbf{\Lambda}\, exp(\varepsilon\tilde{\mathbf{\Theta}})}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \mathbf{\Lambda}\tilde{\mathbf{\Theta}} \tag{2.84}$$

which yields the *material tangent space* to the rotation manifold $SO(3)$ at the base point $\mathbf{\Lambda} \in SO(3)$ since (2.83) holds. More formally we can define

$$\mathcal{T}^{mat}_{\mathbf{\Lambda}}SO(3) = \{\tilde{\mathbf{\Theta}}_{\mathbf{\Lambda}} := (\mathbf{\Lambda}, \tilde{\mathbf{\Theta}}) \,|\, \mathbf{\Lambda} \in SO(3), \tilde{\mathbf{\Theta}} \in so(3)\} \tag{2.85}$$

where an element of the material tangent space $\tilde{\mathbf{\Theta}}_{\mathbf{\Lambda}} \in \mathcal{T}^{mat}_{\mathbf{\Lambda}}SO(3)$ is a skew-symmetric tensor, i.e. $\tilde{\mathbf{\Theta}} \in so(3)$. The notation $(\mathbf{\Lambda}, \tilde{\mathbf{\Theta}})$ is used for indicating the pair formed by the rotation tensor $\mathbf{\Lambda}$ and the skew-symmetric tensor $\tilde{\mathbf{\Theta}}$, representing the material tangent tensor, at the base point $\mathbf{\Lambda} \in SO(3)$ (see Figure 2.8). For simplicity it is possible to omit the base point $\mathbf{\Lambda}$ by denoting $\tilde{\mathbf{\Theta}}\mathbf{\Lambda} \in \mathcal{T}^{mat}_{\mathbf{\Lambda}}SO(3)$ if there is no danger of confusion.

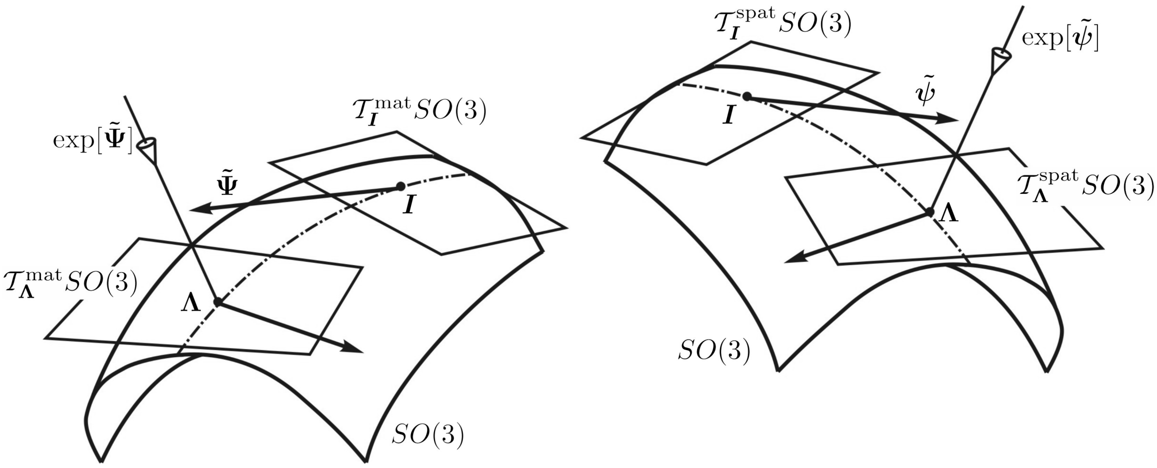


Figure 2.8: Geometric representation of the material (on the left) and spatial tangent spaces (on the right) on the rotation manifold $SO(3)$.

Spatial tangent space to $SO(3)$. Likewise above, basing on definition (2.80), we may write

$$exp(\varepsilon\tilde{\boldsymbol{\theta}})\,\boldsymbol{\Lambda}|_{\varepsilon=0} = \boldsymbol{\Lambda} \tag{2.86}$$

$$\left.\frac{\mathrm{d}exp(\varepsilon\tilde{\boldsymbol{\theta}})\,\boldsymbol{\Lambda}}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} = \tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda} \tag{2.87}$$

which define the spatial tangent space of the rotation manifold $SO(3)$ at the base point $\boldsymbol{\Lambda} \in SO(3)$ since (2.86) holds. More formally we can define

$$\mathcal{T}_{\boldsymbol{\Lambda}}^{spat}SO(3) = \{\tilde{\boldsymbol{\theta}}_{\boldsymbol{\Lambda}} := (\boldsymbol{\Lambda}, \tilde{\boldsymbol{\theta}}) \,|\, \boldsymbol{\Lambda} \in SO(3), \tilde{\boldsymbol{\theta}} \in so(3)\} \tag{2.88}$$

where, by analogy with the material case, an element of the spatial tangent space $\tilde{\boldsymbol{\theta}}_{\boldsymbol{\Lambda}} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{spat}SO(3)$ is a skew-symmetric tensor belonging to $so(3)$. Again, omitting the base point $\boldsymbol{\Lambda}$, it is possible to write $\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{spat}SO(3)$.

2.6.4 Incremental rotation vector by total rotation vector

Consider a rotation tensor $\boldsymbol{\Lambda} \in SO(3)$ which can be indistinctly parametrized (minimally) by using the spatial or material total rotation vector $\boldsymbol{\Psi} = \Psi_i \boldsymbol{E}_i$ and $\boldsymbol{\psi} = \psi_i \boldsymbol{t}_i$, respectively, i.e. we have $\boldsymbol{\Lambda} = \exp[\tilde{\boldsymbol{\Psi}}] = \exp[\tilde{\boldsymbol{\psi}}]$.

The material total rotation vector $\boldsymbol{\Psi}$ represents the rotation vector associated to a rotation with respect to the *fixed axis*, that is axes which remain fixed during the rotation sequence. Hence, according to Euler's theorem, rotation vector $\boldsymbol{\Psi}$ is not affected by rotations, i.e. remains the same when multiplied by the corresponding orthogonal tensor $\boldsymbol{\Lambda}$ such that equality $\boldsymbol{\psi} = \boldsymbol{\Lambda}\boldsymbol{\Psi} = \boldsymbol{I}\boldsymbol{\Psi}$ holds.

On the other hand, the incremental rotation vector $\boldsymbol{\Theta}$ represents the rotation vector associated to a rotation with respect to the *follower axis*, that is axis rotated by the previous rotation. Geometrically $\boldsymbol{\Theta}$ represents a superimposed infinitesimal rotation to an existing finite rotation.

Material description. A virtual [16] rotation tensor $\delta\tilde{\mathbf{\Theta}}$ is an element of the corresponding tangent space $\mathcal{T}_{\mathbf{\Lambda}}SO(3)$ for any base point $\mathbf{\Lambda} \in SO(3)$ such that it satisfies all linearized constraint equations, which naturally arise from boundary conditions. A virtual rotation vector $\delta\mathbf{\Theta}$ at the base point $\mathbf{\Lambda}$ is the associated axial vector of the virtual rotation tensor $\delta\tilde{\mathbf{\Theta}}$, via equation (2.14).

Let us consider the material description of the compound rotation vector $\mathbf{\Psi} + \delta\mathbf{\Psi}$ which parametrizes $\mathbf{\Lambda}$, with $\delta\mathbf{\Psi}$ the additive increment of the rotation vector $\mathbf{\Psi}$. In general we have

$$\exp[\tilde{\mathbf{\Psi}} + \delta\tilde{\mathbf{\Psi}}] = \exp[\tilde{\mathbf{\Psi}}]\exp[\boldsymbol{\delta}\tilde{\mathbf{\Theta}}] \neq \exp[\tilde{\mathbf{\Psi}} + \delta\tilde{\mathbf{\Theta}}] \tag{2.89}$$

where it is possible to see that $\delta\tilde{\mathbf{\Psi}}$ is the linear additive increment of $\tilde{\mathbf{\Psi}}$ because they belong to the same tangent space $\mathcal{T}_{\boldsymbol{I}}^{mat}SO(3)$, in contrast with $\delta\tilde{\mathbf{\Theta}} \in \mathcal{T}_{\mathbf{\Lambda}}^{mat}SO(3)$. The linearized relation between $\delta\mathbf{\Psi}$ and $\delta\mathbf{\Theta}$ is obtained starting to construct a perturbed configuration of $\mathbf{\Lambda}$ depending on a scalar parameter $\varepsilon \in \mathbb{R}^3$ as

$$\mathbf{\Lambda}_{\varepsilon} \triangleq \exp[\tilde{\mathbf{\Psi}} + \varepsilon\,\delta\tilde{\mathbf{\Psi}}] = \exp[\tilde{\mathbf{\Psi}}]\,\exp[\varepsilon\,\delta\tilde{\mathbf{\Theta}}] \tag{2.90}$$

For clarity, we give in Figure the geometric representation of different possibilities to construct a perturbed rotation $\mathbf{\Lambda}_{\varepsilon}$. With the help of Figure we can note that

- The virtual rotation tensor $\delta\tilde{\mathbf{\Psi}}$ belongs to the same tangent space as the rotation tensor $\tilde{\mathbf{\Psi}}$, i.e. such that $\delta\tilde{\mathbf{\Psi}}, \tilde{\mathbf{\Psi}} \in \mathcal{T}_{\boldsymbol{I}}^{mat}SO(3)$ with the identity as the base point;
- The rotation tensor $\mathbf{\Lambda} = \exp[\tilde{\mathbf{\Psi}}]$ and the virtual incremental rotation tensor $\delta\tilde{\mathbf{\Theta}}$ belong to the same material tangent space $\mathcal{T}_{\mathbf{\Lambda}}^{mat}SO(3)$;
- The skew-symmetric tangent tensors $\tilde{\mathbf{\Psi}}$ and $\delta\tilde{\mathbf{\Theta}}$ do not belong to the same tangent space of rotation as generally $\exp[\tilde{\mathbf{\Psi}}]\,\exp[\delta\tilde{\mathbf{\Theta}}] \neq \exp[\tilde{\mathbf{\Psi}} + \delta\tilde{\mathbf{\Theta}}]$ (see [38] for an organic and enlighten proof).

Considering the fact that $\exp[\tilde{\mathbf{\Psi}}]^{-1} = \exp[-\tilde{\mathbf{\Psi}}]$, equation (2.90) can be rearranged as

$$\exp[\varepsilon\,\delta\tilde{\mathbf{\Theta}}] = \exp[-\tilde{\mathbf{\Psi}}]\,\exp[\tilde{\mathbf{\Psi}} + \varepsilon\,\delta\tilde{\mathbf{\Psi}}] \tag{2.91}$$

and taking the derivative of equation (2.91) with respect to parameter ε and setting $\varepsilon = 0$, using the Rodrigues' formula, it is possible to obtain the linearized relation between the incremental rotation vector $\delta\mathbf{\Theta}$ and the linearized increment of the rotation vector $\delta\mathbf{\Psi}$ as (see e.g. [37])

$$\begin{aligned}
\delta\mathbf{\Theta} &= D[\mathbf{\Lambda}(\varepsilon, \mathbf{\Theta})] = D[\mathbf{\Lambda}_{\varepsilon}] \cdot \boldsymbol{\delta}\tilde{\mathbf{\Psi}} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\exp[\varepsilon\,\delta\tilde{\mathbf{\Theta}}]\right]\Big|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\exp[-\tilde{\mathbf{\Psi}}]\,\exp[\tilde{\mathbf{\Psi}} + \varepsilon\,\delta\tilde{\mathbf{\Psi}}]\right]\Big|_{\varepsilon=0} \\
&= \boldsymbol{T}^T\delta\mathbf{\Psi}
\end{aligned} \tag{2.92}$$

[16]Here the symbol δ is used to denote a linearized increment.

with

$$\boldsymbol{T}^T = \frac{\sin\Psi}{\Psi}\boldsymbol{I} - \frac{1-\cos\Psi}{\Psi^2}\tilde{\boldsymbol{\Psi}} + \frac{\Psi - \sin\Psi}{\Psi^3}\tilde{\boldsymbol{\Psi}}^2 \tag{2.93}$$

$$\Psi = ||\boldsymbol{\Psi}||, \quad \boldsymbol{\Lambda} = \exp(\tilde{\boldsymbol{\Psi}}), \quad \lim_{\boldsymbol{\Psi}\to\boldsymbol{0}} \boldsymbol{T}^T(\boldsymbol{\Psi}) = \boldsymbol{I} \tag{2.94}$$

where the *material tangential transformation tensor* $\boldsymbol{T}^T = \boldsymbol{T}^T(\boldsymbol{\Psi})$ is a linear mapping between the material tangent spaces $\boldsymbol{T}^T\colon\ \mathcal{T}_{\boldsymbol{I}}^{mat}SO(3) \to \mathcal{T}_{\boldsymbol{\Lambda}}^{mat}SO(3)$, i.e. the transformation $\boldsymbol{T}^T$ has an effect on the base point, changing the base point $\boldsymbol{I}$ into $\boldsymbol{\Lambda}$. As a confirmation of the fact that the virtual rotation vector $\delta\boldsymbol{\Theta}$ and the virtual total rotation vector $\delta\boldsymbol{\Psi}$ belong to different vector spaces on the manifold, one can note that the tangential transformation $\boldsymbol{T}^T$ is equal to the identity only at $\boldsymbol{\Psi} = \boldsymbol{0}$, i.e. $\boldsymbol{\Psi} \to \boldsymbol{0} \Rightarrow \boldsymbol{T}^T(\boldsymbol{\Psi}) \to \boldsymbol{I}$.

It is worth also noting that the tangential transformation $\boldsymbol{T}(\boldsymbol{\Psi})$, the corresponding rotation operator $\boldsymbol{\Lambda}(\boldsymbol{\Psi})$ and the skew-symmetric rotation tensor $\tilde{\boldsymbol{\Psi}}$ have the same eigenvectors. Hence, $\boldsymbol{T}(\boldsymbol{\Psi})$, $\boldsymbol{\Lambda}(\boldsymbol{\Psi})$ and $\tilde{\boldsymbol{\Psi}}$ are commutative (a proof is given in [31]).

Spatial description. Consider the compound spatial rotation vector $\boldsymbol{\psi} + \delta\boldsymbol{\psi}$ which parametrizes $\boldsymbol{\Lambda}$, with $\delta\boldsymbol{\psi}$ the additive increment of the rotation vector $\boldsymbol{\psi}$. In general we have

$$\exp[\tilde{\boldsymbol{\psi}} + \delta\tilde{\boldsymbol{\psi}}] = \exp[\varepsilon\delta\tilde{\boldsymbol{\theta}}]\ \exp[\delta\tilde{\boldsymbol{\psi}}] \tag{2.95}$$

It is possible to see that $\delta\tilde{\boldsymbol{\psi}}$ is the linear additive increment of $\tilde{\boldsymbol{\psi}}$ because they belong to the same tangent space $\mathcal{T}_{\boldsymbol{I}}^{spat}SO(3)$, in contrast with $\delta\tilde{\boldsymbol{\theta}} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{spat}SO(3)$. One can observe that, because of $\delta\tilde{\boldsymbol{\theta}}$ being skew-symmetric, the spatial form of the linearized increment or admissible variation of the rotation tensor $\delta\boldsymbol{\Lambda}$ is no longer orthogonal. In fact, $\delta\tilde{\boldsymbol{\theta}}$ belongs to the tangential space of the rotation tensor $\boldsymbol{\Lambda} \in SO(3)$.
The linearized relation between $\delta\boldsymbol{\psi}$ and $\delta\boldsymbol{\theta}$ is obtained likewise above as follows: construct a perturbed configuration of $\boldsymbol{\Lambda}$ depending on a scalar parameter $\varepsilon \in \mathbb{R}^3$ as

$$\boldsymbol{\Lambda}_\varepsilon \triangleq \exp[\tilde{\boldsymbol{\psi}} + \varepsilon\,\delta\tilde{\boldsymbol{\psi}}] = \exp[\varepsilon\,\delta\tilde{\boldsymbol{\theta}}]\ \exp[\tilde{\boldsymbol{\psi}}] \tag{2.96}$$

considering the fact that $\exp[\tilde{\boldsymbol{\psi}}]^{-1} = \exp[-\tilde{\boldsymbol{\psi}}]$ one obtains

$$\exp[\varepsilon\,\delta\tilde{\boldsymbol{\theta}}] = \exp[\tilde{\boldsymbol{\psi}} + \varepsilon\,\delta\tilde{\boldsymbol{\psi}}]\ \exp[-\tilde{\boldsymbol{\psi}}] \tag{2.97}$$

Finally, taking the derivative of equation (2.97) with respect to ε and setting $\varepsilon = 0$, it is still possible, using the Rodrigues' formula, to obtain the linearized relation between the incremental rotation vector $\delta\boldsymbol{\theta}$ and the increment of the total rotation vector $\delta\boldsymbol{\psi}$ as

$$\begin{aligned}
\delta\boldsymbol{\theta} &= D[\boldsymbol{\Lambda}(\varepsilon,\boldsymbol{\theta})] = D[\boldsymbol{\Lambda}_\varepsilon]\cdot\delta\tilde{\boldsymbol{\psi}} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\exp[\varepsilon\,\delta\tilde{\boldsymbol{\theta}}]\right]\Big|_{\varepsilon=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left[\exp[\tilde{\boldsymbol{\psi}} + \varepsilon\,\delta\tilde{\boldsymbol{\psi}}]\exp[-\tilde{\boldsymbol{\psi}}]\right]\Big|_{\varepsilon=0} \\
&= \boldsymbol{T}\delta\boldsymbol{\psi}
\end{aligned} \tag{2.98}$$

with

$$\boldsymbol{T} = \frac{\sin\psi}{\psi}\boldsymbol{I} + \frac{1-\cos\psi}{\psi^2}\tilde{\boldsymbol{\psi}} + \frac{\psi - \sin\psi}{\psi^3}\tilde{\boldsymbol{\psi}}^2 \tag{2.99}$$

$$\psi = ||\boldsymbol{\psi}||, \quad \boldsymbol{\Lambda} = \exp(\tilde{\boldsymbol{\psi}}), \quad \lim_{\boldsymbol{\psi}\to\boldsymbol{0}} \boldsymbol{T}(\boldsymbol{\psi}) = \boldsymbol{I} \tag{2.100}$$

where the spatial tangential transformation tensor $\boldsymbol{T}$ is the same linear operator as in the material form (2.93), and represents a mapping between vector spaces on the rotation manifold $\boldsymbol{T}\colon\ \mathcal{T}_{\boldsymbol{I}}^{spat} \to \mathcal{T}_{\boldsymbol{\Lambda}}^{spat}$.

2.6.5 Tangent vector spaces of rotation manifold

Material vector space. According to the previous results, it is possible to define the material vector space on the rotation manifold at any base point $\boldsymbol{\Lambda}$ as

$$\mathcal{T}_{\boldsymbol{\Lambda}}^{mat} = \{\boldsymbol{\Theta} := (\boldsymbol{\Psi}, \boldsymbol{\Theta}) \,|\, \boldsymbol{\Lambda} = \exp(\tilde{\Psi}) \in SO(3), \boldsymbol{\Theta} \in \mathbb{R}^3\} \tag{2.101}$$

where an element of the material vector space is $\boldsymbol{\Theta} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{mat}$, which is an affine space with the rotation vector $\boldsymbol{\Psi}$ as the base point and the incremental rotation vector $\delta\boldsymbol{\Theta}$ as a tangent vector. Note that the elements of this material vector space can be added by the parallelogram rule only if they occupy the same affine space, i.e. if their associated skew-symmetric tensors belong to the same tangential space of the rotation manifold. Definition reported in (2.101) for the material vector space should be considered as a useful and simple notation with equivalence relation with the material tensor space defined above.

Spatial vector space. By analogy with the material case, the spatial vector space on the rotation manifold at any base point $\boldsymbol{\Lambda}$ is defined as

$$\mathcal{T}_{\boldsymbol{\Lambda}}^{spat} = \{\boldsymbol{\theta} := (\boldsymbol{\psi}, \boldsymbol{\theta}) \,|\, \boldsymbol{\Lambda} = \exp(\tilde{\psi}) \in SO(3), \boldsymbol{\theta} \in \mathbb{R}^3\} \tag{2.102}$$

The spatial and the material vector spaces are related by the rotation operator via $\boldsymbol{\theta} = \boldsymbol{\Lambda}\boldsymbol{\Theta}$. From this follows that with the base point $\boldsymbol{I} \in SO(3)$

$$\boldsymbol{\psi}_{\boldsymbol{I}} = \boldsymbol{I}\boldsymbol{\Psi}_{\boldsymbol{I}} \longrightarrow \boldsymbol{\psi}_{\boldsymbol{I}} = \boldsymbol{\Psi}_{\boldsymbol{I}} \tag{2.103}$$

where $'='$ denotes the canonical isomorphism between the spatial and material vector spaces. The identity $\boldsymbol{I}$ maps between the vector spaces $\mathcal{T}_{\boldsymbol{I}}^{mat} \to \mathcal{T}_{\boldsymbol{I}}^{spat}$, and the relation between the spatial and the material vectors can be given as $(\boldsymbol{\psi}, \boldsymbol{\theta}) = (\boldsymbol{I}\boldsymbol{\Psi}, \boldsymbol{\Lambda}\boldsymbol{\Theta})$ where $\boldsymbol{\psi}$ and $\boldsymbol{\Psi}$ represent the base vectors in the spatial and material vector spaces, respectively. This relation can be rewritten compactly as $\boldsymbol{\theta} = \boldsymbol{\Lambda}\boldsymbol{\Theta}$, called the *push-forward* operation of $\boldsymbol{\Theta}_{\boldsymbol{\Lambda}}$ by $\boldsymbol{\Lambda}$, where the rotation operator should be considered as a mapping between the material and spatial vector spaces of rotation, i.e. $\boldsymbol{\Lambda} : \mathcal{T}_{\boldsymbol{\Lambda}}^{mat} \to \mathcal{T}_{\boldsymbol{\Lambda}}^{spat}$ (see [17] Figure 2.9 for a scheme of the connections between spatial and material configurations). Generally speaking, a push-forward operator [18] maps, by means of $\boldsymbol{\Lambda}$, a material vector space into a spatial vector space (one-to-one and onto). It makes sense since a rotation operator is a two-point tensor.

[17] The same diagram applies to admissible variations of rotation parameters, as well as their increments [58].

[18] For a rigorous account of push-forward and pull-back mappings, see [40].

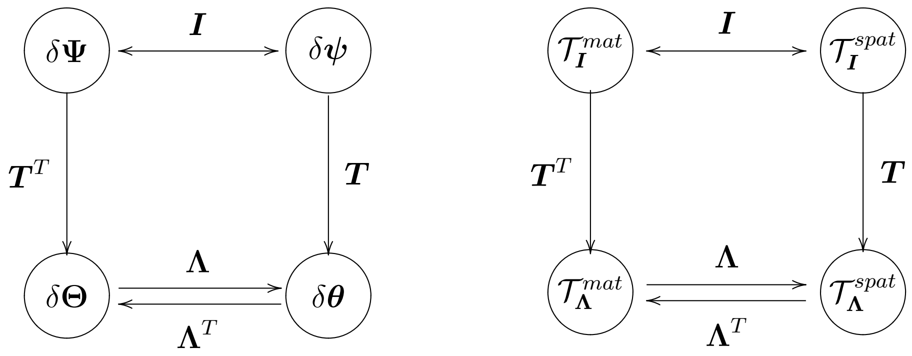


Figure 2.9: Commutative diagram of virtual material and spatial rotation vectors on the rotation manifold (on the left), and their corresponding vector spaces (on the right).

The corresponding push-forward operator for rotation tensors is $\tilde{\boldsymbol{\theta}} = \boldsymbol{\Lambda}\tilde{\boldsymbol{\Theta}}\boldsymbol{\Lambda}^T$, i.e. a mapping, performed by means of $\boldsymbol{\Lambda}$ between the material and spatial tangent spaces of rotation $\boldsymbol{\Lambda}(\cdot)\boldsymbol{\Lambda}^T : \mathcal{T}_{\boldsymbol{\Lambda}}^{mat} \to \mathcal{T}_{\boldsymbol{\Lambda}}^{spat}$.
Finally, it's noteworthy the correspondence between equation (2.93) and (2.99) due to the identity mapping between the vector spaces $\mathcal{T}_{\boldsymbol{I}}^{mat} \to \mathcal{T}_{\boldsymbol{I}}^{spat}$, such that $\boldsymbol{T}(\boldsymbol{\Psi}) = \boldsymbol{T}(\boldsymbol{\psi})$. For simplicity, in the forthcoming sections we refer to the second one expression, since there is no danger of confusion.

2.7 Time derivatives of rotation operator

Derivatives of rotation tensor and their axial vectors, with respect to the time variable, appear in applications of finite rotations to mechanics, where they play a crucial role in the development of kinematics. Hence, we introduce here such a topic, giving definitions in material as well as spatial description.

2.7.1 Angular velocity

Angular velocity in *material form*. Let consider a rotation $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(t)$ function of parameter $t \in \mathbb{R}$ which is taken as the independent variable and has the specific meaning of time. We indicate by $\dot{\boldsymbol{\Lambda}} = \mathrm{d}\boldsymbol{\Lambda}/\mathrm{d}t$ the time derivative of $\boldsymbol{\Lambda}$ with respect to time coordinate t. Taking derivative of the orthogonality condition $\boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \mathbf{I}$ (2.5), we get

$$\dot{\boldsymbol{\Lambda}}^T\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T\dot{\boldsymbol{\Lambda}} = \mathbf{0} \tag{2.104}$$

which given in the form (*skewness condition*)

$$\boldsymbol{\Lambda}^T\dot{\boldsymbol{\Lambda}} = -\dot{\boldsymbol{\Lambda}}^T\boldsymbol{\Lambda} \quad \to \quad \boldsymbol{\Lambda}^T\dot{\boldsymbol{\Lambda}} = -(\boldsymbol{\Lambda}^T\dot{\boldsymbol{\Lambda}})^T \tag{2.105}$$

shows that $\boldsymbol{\Lambda}^T\dot{\boldsymbol{\Lambda}}$ is a skew tensor. With this result in hand we define the *material angular velocity* tensor $\tilde{\boldsymbol{\Omega}}$, and consequently its axial vector $\boldsymbol{\Omega} = \mathrm{axial}(\tilde{\boldsymbol{\Omega}})$, as

$$\boxed{\tilde{\boldsymbol{\Omega}} = \mathrm{skew}\,[\boldsymbol{\Omega}] = [\boldsymbol{\Omega}\times] \triangleq \boldsymbol{\Lambda}^T\dot{\boldsymbol{\Lambda}} \quad \text{where} \quad \tilde{\boldsymbol{\Omega}} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{mat}}SO(3),\ \boldsymbol{\Omega} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{mat}} \subset \mathbb{R}^3} \tag{2.106}$$

In matrix notation we have

$$\tilde{\mathbf{\Omega}} = \begin{bmatrix} 0 & -\Omega_3 & \Omega_2 \\ \Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{bmatrix} \qquad \text{and} \qquad \mathbf{\Omega} = \begin{bmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{bmatrix} \tag{2.107}$$

where $\tilde{\mathbf{\Omega}}^T = -\tilde{\mathbf{\Omega}}$ is a spatial skew-symmetric tensor which defines the *spin* of the moving frame. The associated axial vector $\mathbf{\Omega}$, which satisfies relation $\tilde{\mathbf{\Omega}}\mathbf{\Omega} = \mathbf{0}$, gives the *vorticity* of the moving frame.

One can observe that $\mathbf{\Theta}$, $\dot{\mathbf{\Theta}}$, $\mathbf{\Omega} \in \mathcal{T}_{\mathbf{\Lambda}}^{mat}$, i.e. the material incremental rotation vector, its time derivative and the material angular vector, respectively, belong to the same tangent material vector space on the rotation manifold.

Angular velocity in *spatial form*. Similar expression and derivation can be accomplished for the spatial angular velocity tensor and vector, yielding the definition of the spatial skew tensor $\tilde{\boldsymbol{\omega}}$ and its axial vector $\boldsymbol{\omega} = \text{axial}(\tilde{\boldsymbol{\omega}})$

$$\boxed{\tilde{\boldsymbol{\omega}} = \text{skew}\,[\boldsymbol{\omega}] = [\boldsymbol{\omega}\times] \triangleq \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T \quad \text{where} \quad \tilde{\boldsymbol{\omega}} \in \mathcal{T}_{\mathbf{\Lambda}}^{\text{spat}} SO(3),\ \boldsymbol{\omega} \in \mathcal{T}_{\mathbf{\Lambda}}^{\text{spat}} \subset \mathbb{R}^3} \tag{2.108}$$

As before, the material incremental rotation vector $\boldsymbol{\theta}$, its time derivative vector $\dot{\boldsymbol{\theta}}$ and the spatial angular vector $\boldsymbol{\omega}$ belong to the same spatial vector space on the rotation manifold, i.e. $\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}, \boldsymbol{\omega} \in \mathcal{T}_{\mathbf{\Lambda}}^{spat}$.

Pre-multiplying and post-multiplying for $\mathbf{\Lambda}$ equations (2.106) and (2.108), we get another two forms of derivative $\dot{\mathbf{\Lambda}}$

$$\dot{\mathbf{\Lambda}} = \mathbf{\Lambda}\,\tilde{\mathbf{\Omega}} \tag{2.109}$$

$$\dot{\mathbf{\Lambda}} = \tilde{\boldsymbol{\omega}}\,\mathbf{\Lambda} \tag{2.110}$$

Note that $\dot{\mathbf{\Lambda}} \notin SO(3)$, i.e. the time derivative of a rotation is not a rotation. At contrary, recalling from previous section that the skew-tensor space $so(3)$ is the tangent space to the rotation group, equations (2.109) and (2.110) show that the derivative is a composition of the current rotation with an element of the rotation tangent space. Moreover in the case (2.109) the spin tensor $\tilde{\mathbf{\Omega}}$ precedes the rotation in the composition product, i.e. it lays in the space that has not been affected by any rotation. Accordingly, it belongs to the rotation tangent space at the identity $\boldsymbol{I}$, namely $\mathcal{T}_{\boldsymbol{I}}^{\text{mat}}$. Conversely, in the case (2.110), the spin tensor $\tilde{\boldsymbol{\omega}}$ follows the rotation in the composition product, i.e. it lays in a space that has been already rotated by $\mathbf{\Lambda}$.

Comparing equations (2.109) and (2.110) results

$$\dot{\mathbf{\Lambda}} = \tilde{\boldsymbol{\omega}}\,\mathbf{\Lambda} = \mathbf{\Lambda}\,\tilde{\mathbf{\Omega}} \;\Rightarrow$$
$$\tilde{\boldsymbol{\omega}} = \mathbf{\Lambda}\,\tilde{\mathbf{\Omega}}\,\mathbf{\Lambda}^T \qquad \text{and} \qquad \tilde{\mathbf{\Omega}} = \mathbf{\Lambda}^T\,\tilde{\boldsymbol{\omega}}\,\mathbf{\Lambda} \tag{2.111}$$

which shows that $\tilde{\boldsymbol{\omega}}$ is the rotated-forward expression (to the spatial configuration) of $\tilde{\mathbf{\Omega}}$, and $\tilde{\mathbf{\Omega}}$ is the rotated-back expression (to the material configuration) of $\tilde{\boldsymbol{\omega}}$.

An equivalent relation can be obtained for axial vectors $\boldsymbol{\omega}$ and $\mathbf{\Omega}$. Comparing again equations (2.109) and (2.110) it follows that

$$\mathbf{\Lambda}\,\tilde{\mathbf{\Omega}}\mathbf{h} = \tilde{\boldsymbol{\omega}}\,\mathbf{\Lambda}\mathbf{h} \qquad \forall\, \boldsymbol{h} \in \mathbb{R}^3 \tag{2.112}$$

Using relation (2.14) it can be recast in terms of $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ as

$$\boldsymbol{\Lambda}(\boldsymbol{\Omega} \times \boldsymbol{h}) = \boldsymbol{\omega} \times \boldsymbol{\Lambda h} \qquad \forall\, \boldsymbol{h} \in \mathbb{R}^3 \tag{2.113}$$

On the left-hand side one can use the distributivity property of cross product with respect to product with a rotation (see (A.9)), and obtain

$$\boldsymbol{\omega} \times \boldsymbol{\Lambda h} = \boldsymbol{\Lambda\Omega} \times \boldsymbol{\Lambda h} \qquad \forall\, \boldsymbol{h} \in \mathbb{R}^3 \tag{2.114}$$

which entails

$$\boldsymbol{\omega} = \boldsymbol{\Lambda\Omega} \tag{2.115}$$

and consequentially

$$\boldsymbol{\Omega} = \boldsymbol{\Lambda}^T \boldsymbol{\omega} \tag{2.116}$$

Such as for their skew tensors, $\boldsymbol{\omega}$ is the rotated-forward expression of $\boldsymbol{\Omega}$, and $\boldsymbol{\Omega}$ is the rotated-back expression of $\boldsymbol{\omega}$.

Angular velocity vectors by total rotation vector. The spin tensor $\tilde{\boldsymbol{\omega}} = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T$ in spatial coordinates, and its axial vector $\boldsymbol{\omega}$, as well as their rotated-back form $\tilde{\boldsymbol{\Omega}} = \boldsymbol{\Lambda}^T\tilde{\boldsymbol{\omega}}\boldsymbol{\Lambda}$ and $\boldsymbol{\Omega} = \boldsymbol{\Lambda}^T\boldsymbol{\omega}$ introduced in section 2.7.1, play a crucial role in the kinematics and equilibrium of beam model. Here we are interested in study their relations with the total rotation vector $\boldsymbol{\Psi}$ and $\boldsymbol{\psi}$, respectively in material and spatial form. The relations can be found developing the mapping or the equivalent Rodrigues formula. The computation shows that the angular velocity vectors $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$ are related *linearly* with the time derivative of the total rotation vectors $\dot{\boldsymbol{\Psi}}$ and $\dot{\boldsymbol{\psi}}$ respectively, through a *non-linear* function of $\boldsymbol{\Psi}$ and $\boldsymbol{\psi}$. For $\boldsymbol{\omega}$ and its back-rotated counterpart it results in fact that (see appendix D)

$$\boldsymbol{\Omega} = \boldsymbol{T}^T(\boldsymbol{\Psi}) \cdot \dot{\boldsymbol{\Psi}}, \qquad \boldsymbol{\Omega} \in \mathcal{T}^{mat}_{\boldsymbol{\Lambda}}, \qquad \boldsymbol{\Psi}, \dot{\boldsymbol{\Psi}} \in \mathcal{T}^{mat}_{\boldsymbol{I}} \tag{2.117}$$

$$\boldsymbol{\omega} = \boldsymbol{T}(\boldsymbol{\psi}) \cdot \dot{\boldsymbol{\psi}}, \qquad \boldsymbol{\omega} \in \mathcal{T}^{spat}_{\boldsymbol{\Lambda}}, \qquad \boldsymbol{\psi}, \dot{\boldsymbol{\psi}} \in \mathcal{T}^{spat}_{\boldsymbol{I}} \tag{2.118}$$

where, we made use of the identity $\boldsymbol{T}(\boldsymbol{\Psi}) = \boldsymbol{T}(\boldsymbol{\psi})$ with

$$\boldsymbol{T} = \boldsymbol{I} + \frac{1-\cos\psi}{\psi^2}\tilde{\boldsymbol{\psi}} + \frac{\psi - \sin\psi}{\psi^3}\tilde{\boldsymbol{\psi}}^2 \tag{2.119}$$

$$\boldsymbol{T}^T = \boldsymbol{I} - \frac{1-\cos\psi}{\psi^2}\tilde{\boldsymbol{\psi}} + \frac{\psi - \sin\psi}{\psi^3}\tilde{\boldsymbol{\psi}}^2 \tag{2.120}$$

where $\boldsymbol{T}^T$ is computed from $\boldsymbol{T}$ changing the sign to the coefficient of $\tilde{\boldsymbol{\psi}}$, since $\tilde{\boldsymbol{\psi}}$ is skew-symmetric tensor ($\tilde{\boldsymbol{\psi}}^T = -\tilde{\boldsymbol{\psi}}$), and making use of equations (A.2) and (A.5). In addition, the following relations hold

$$\boldsymbol{T}(\boldsymbol{\psi}) : \mathcal{T}^{spat}_{\boldsymbol{I}} \rightarrow \mathcal{T}^{spat}_{\boldsymbol{\Lambda}} \tag{2.121}$$

$$\boldsymbol{T}^T(\boldsymbol{\Psi}) : \mathcal{T}^{mat}_{\boldsymbol{I}} \rightarrow \mathcal{T}^{mat}_{\boldsymbol{\Lambda}} \tag{2.122}$$

and clearly for $\boldsymbol{\psi} \rightarrow \boldsymbol{0} \Rightarrow \boldsymbol{T}(\boldsymbol{\psi}) \rightarrow \boldsymbol{I}$, and consequently $\boldsymbol{\omega} = \dot{\boldsymbol{\psi}}$. This result is often given as definition for the angular velocity vector in elementary books.

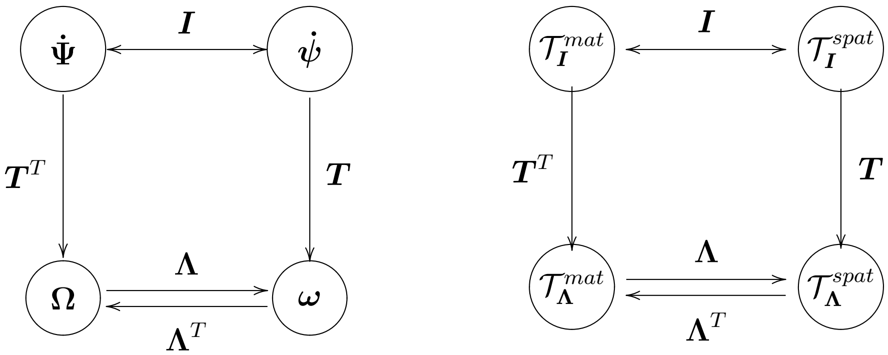


Figure 2.10: A commutative diagram of virtual material and spatial angular velocity vectors on the rotation manifold (on the left), and their corresponding vector spaces (on the right).

Using identity (2.22), the quadratic skew tensor $\tilde{\boldsymbol{\psi}}^2$ can be given in terms of vector $\boldsymbol{\psi}$, hence tensor $\boldsymbol{T}$ can be rearranged easily as

$$\boldsymbol{T} = \frac{\sin\psi}{\psi}\boldsymbol{I} + \frac{1-\cos\psi}{\psi^2}\tilde{\boldsymbol{\psi}} + \frac{\psi - \sin\psi}{\psi^3}\boldsymbol{\psi}\otimes\boldsymbol{\psi} \tag{2.123}$$

It's worthy of further attention that tensors $\boldsymbol{T}$ and $\boldsymbol{\Lambda}$ are linear combinations of the same elementary tensors. Moreover, it can be shown (see [31]) that they share the same eigenvalues, and hence they commute. To confirm this the following property holds. In fact, realizing that $\boldsymbol{\omega}$ is the spatial counterpart of $\boldsymbol{\Omega}$, from equations (2.118) and (2.115) one has

$$\begin{aligned} \boldsymbol{\omega} &= \boldsymbol{T}\dot{\boldsymbol{\psi}} \\ &= \boldsymbol{\Lambda}\boldsymbol{\Omega} \\ &= \boldsymbol{\Lambda}\boldsymbol{T}^T\dot{\boldsymbol{\Psi}} \end{aligned} \tag{2.124}$$

which, in turn, reveals the relationship

$$\boldsymbol{T} = \boldsymbol{\Lambda}\boldsymbol{T}^T \tag{2.125}$$

Finally, by inspecting equation (2.119) we notice that $\boldsymbol{T}(\boldsymbol{\psi})$ is *singular* for certain value of ψ. Indeed, calculating the determinant from expression (2.119) we obtain

$$\det(\boldsymbol{T}) = \frac{2(1-\cos\psi)}{\psi^2} \tag{2.126}$$

which is null for $\psi = 0 \pm 2k\pi$, where $k = 1, 2, 3, \dots$ is an integer. That is to say the parametrization presents a certain number of differentiability holes. This rank deficiency may lead the tangential stiffness tensor to be deficient too, and iterative solution procedures of Newton type may not converge or give spurious bifurcation because the determinant of the tangential stiffness matrix vanishes. In order to avoid this problem during computations, one can restrict the rotational vector to values in the range $0 \le \psi \le \pi$. Alternatively, [31] introduces a special incremental updating procedure when solving the nonlinear finite element equations, known as *updated Lagrangian.*

Proofs of equations (2.118) and (2.119) are given extensive in appendix D, however more details explanations are available in [31] and [53].

2.7.2 Angular acceleration

Angular acceleration in *material form.* For sake of completeness, further proceeding in derivation, the material angular acceleration tensor and the corresponding vector are defined as the time derivative of the angular velocity

$$\boxed{\tilde{\boldsymbol{A}} = \dot{\tilde{\boldsymbol{\Omega}}}, \quad \tilde{\boldsymbol{A}} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{mat}} SO(3), \quad \Longleftrightarrow \quad \boldsymbol{A} = \dot{\boldsymbol{\Omega}}, \quad \boldsymbol{A} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{mat}}} \tag{2.127}$$

The incremental material rotation vector $\boldsymbol{\Theta}$, the material angular velocity vector $\boldsymbol{\Omega}$ and the material angular acceleration vector $\boldsymbol{A}$ belong to the same material vector space in the rotation manifold, i.e. $\boldsymbol{\Theta}, \boldsymbol{\Omega}, \boldsymbol{A} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{mat}}$ with the base point $\boldsymbol{\Lambda} = \exp[\tilde{\boldsymbol{\Psi}}]$.

Angular acceleration in *spatial form.* As before, further proceeding in the derivation, the spatial angular acceleration tensor and the corresponding vector are defined as the time derivative of the angular velocity, i.e.

$$\boxed{\tilde{\boldsymbol{\alpha}} = \dot{\tilde{\boldsymbol{\omega}}}, \quad \tilde{\boldsymbol{\alpha}} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{spat}} SO(3), \quad \Longleftrightarrow \quad \boldsymbol{\alpha} = \dot{\boldsymbol{\omega}}, \quad \boldsymbol{\alpha} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{spat}}} \tag{2.128}$$

The incremental spatial rotation vector $\boldsymbol{\theta}$, the spatial angular velocity vector $\boldsymbol{\omega}$ and the spatial angular acceleration vector $\boldsymbol{\alpha}$ belong to the same spatial vector space in the rotation manifold, i.e. $\boldsymbol{\theta}, \boldsymbol{\omega}, \boldsymbol{\alpha} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{spat}}$.

Angular acceleration vectors by total rotation vector. In analogy with what showed for angular velocity, in turn angular acceleration vectors $\boldsymbol{\alpha}$ and $\boldsymbol{A}$ are related with the time derivative of the total rotation vectors $\dot{\boldsymbol{\Psi}}$ and $\dot{\boldsymbol{\psi}}$, as it can be shown simply by differentiating equations (2.117) and (2.118)

$$\boldsymbol{A} = \boldsymbol{T}^T \cdot \ddot{\boldsymbol{\Psi}} + \dot{\boldsymbol{T}}^T \cdot \dot{\boldsymbol{\Psi}}, \qquad \boldsymbol{A} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{mat}, \qquad \boldsymbol{\Psi}, \dot{\boldsymbol{\Psi}}, \ddot{\boldsymbol{\Psi}} \in \mathcal{T}_{\boldsymbol{I}}^{mat} \tag{2.129}$$

$$\boldsymbol{\alpha} = \boldsymbol{T} \cdot \ddot{\boldsymbol{\psi}} + \dot{\boldsymbol{T}} \cdot \dot{\boldsymbol{\psi}}, \qquad \boldsymbol{\alpha} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{spat}, \qquad \boldsymbol{\psi}, \dot{\boldsymbol{\psi}}, \ddot{\boldsymbol{\psi}} \in \mathcal{T}_{\boldsymbol{I}}^{spat} \tag{2.130}$$

Note that the tangential transformations $\boldsymbol{T}, \dot{\boldsymbol{T}} \in (\mathcal{T}_{\boldsymbol{I}}^{\mathrm{spat}}, \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{spat}})$ and $\boldsymbol{T}^T, \dot{\boldsymbol{T}}^T \in (\mathcal{T}_{\boldsymbol{I}}^{\mathrm{mat}}, \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{mat}})$ operate with different base point.

2.8 Spatial derivatives: curvature

Let consider a one-parameter rotation $\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(S)$, function of parameter S which is taken as the independent variable and has the specific meaning of arc-length measure. We indicate by $\boldsymbol{\Lambda}' = \mathrm{d}\boldsymbol{\Lambda}/\mathrm{d}S$ the spatial derivative of $\boldsymbol{\Lambda}$ with respect to the arc-length parameter S. Taking derivative of the orthogonally condition $\boldsymbol{\Lambda}^T\boldsymbol{\Lambda} = \boldsymbol{I}$ (2.5), we get

$$\boldsymbol{\Lambda}'^T\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T\boldsymbol{\Lambda}' = \boldsymbol{0} \tag{2.131}$$

which given in the form (*skewness condition*)

$$\mathbf{\Lambda}^T\mathbf{\Lambda}' = -\mathbf{\Lambda}'^T\mathbf{\Lambda} \quad \rightarrow \quad \mathbf{\Lambda}^T\mathbf{\Lambda}' = -(\mathbf{\Lambda}^T\mathbf{\Lambda}')^T \tag{2.132}$$

shows that $\mathbf{\Lambda}^T\mathbf{\Lambda}'$ is a skew tensor. With this result in hand we define the material and spatial curvature tensors $\tilde{\boldsymbol{K}}$ and $\tilde{\boldsymbol{\kappa}}$ of a parametrized line (beam) respectively as

$$\boxed{\tilde{\boldsymbol{K}} = \operatorname{skew}[\boldsymbol{K}] = [\boldsymbol{K}\times] \triangleq \mathbf{\Lambda}^T\mathbf{\Lambda}' \quad \text{where} \quad \tilde{\boldsymbol{K}} \in \mathcal{T}^{\text{mat}}_{\mathbf{\Lambda}}SO(3),\ \boldsymbol{K} \in \mathcal{T}^{\text{mat}}_{\mathbf{\Lambda}} \subset \mathbb{R}^3} \tag{2.133}$$

$$\boxed{\tilde{\boldsymbol{\kappa}} = \operatorname{skew}[\boldsymbol{\kappa}] = [\boldsymbol{\kappa}\times] \triangleq \mathbf{\Lambda}'\mathbf{\Lambda}^T \quad \text{where} \quad \tilde{\boldsymbol{\kappa}} \in \mathcal{T}^{\text{mat}}_{\mathbf{\Lambda}}SO(3),\ \boldsymbol{\kappa} \in \mathcal{T}^{\text{mat}}_{\mathbf{\Lambda}} \subset \mathbb{R}^3} \tag{2.134}$$

Therefore curvature could be regarded as the rate of change of the rotation matrix $\mathbf{\Lambda}$ with respect to the longitudinal coordinate. Taking advantage of the axial vector relation (2.14), the axial vectors $\boldsymbol{K}$ and $\boldsymbol{\kappa}$ are called the material and spatial curvature vector, respectively.
Revising (2.133) by post-multiplying for $\mathbf{\Lambda}$ we get

$$\mathbf{\Lambda}' = \tilde{\boldsymbol{\kappa}}\mathbf{\Lambda} \tag{2.135}$$

whereas by pre-multiplying for $\mathbf{\Lambda}$ we get

$$\mathbf{\Lambda}' = \mathbf{\Lambda}\tilde{\boldsymbol{K}} \tag{2.136}$$

Comparing equations (2.135) and (2.136), curvature tensors satisfy the push-forward relations

$$\begin{gathered} \mathbf{\Lambda}' = \tilde{\boldsymbol{\kappa}}\,\mathbf{\Lambda} = \mathbf{\Lambda}\,\tilde{\boldsymbol{K}} \;\Rightarrow \\ \tilde{\boldsymbol{\kappa}} = \mathbf{\Lambda}\,\tilde{\boldsymbol{K}}\,\mathbf{\Lambda}^T \qquad \text{and} \qquad \tilde{\boldsymbol{K}} = \mathbf{\Lambda}^T\,\tilde{\boldsymbol{\kappa}}\,\mathbf{\Lambda} \end{gathered} \tag{2.137}$$

Equivalent relations can be obtained for axial vectors $\boldsymbol{\kappa}$ and $\boldsymbol{K}$, entirely similar to those of angular velocities. Still from equations (2.135) and (2.136) it follows that

$$\tilde{\boldsymbol{\kappa}}\,\mathbf{\Lambda}\mathbf{h} = \mathbf{\Lambda}\,\tilde{\boldsymbol{K}}\mathbf{h} \qquad \forall\,\boldsymbol{h} \in \mathbb{R}^3 \tag{2.138}$$

which using the axial vector relation (2.14) can be given in term of $\boldsymbol{\kappa}$ and $\boldsymbol{K}$ as

$$\boldsymbol{\kappa} \times \mathbf{\Lambda}\boldsymbol{h} = \mathbf{\Lambda}(\boldsymbol{K} \times \boldsymbol{h}) \qquad \forall\,\boldsymbol{h} \in \mathbb{R}^3 \tag{2.139}$$

Using for the right-hand side the distributivity property of cross product under rotation (A.9), we obtain

$$\boldsymbol{\kappa} \times \mathbf{\Lambda}\boldsymbol{h} = \mathbf{\Lambda}\boldsymbol{K} \times \mathbf{\Lambda}\boldsymbol{h} \qquad \forall\,\boldsymbol{h} \in \mathbb{R}^3 \tag{2.140}$$

which entails

$$\boldsymbol{\kappa} = \mathbf{\Lambda}\boldsymbol{K} \tag{2.141}$$

and consequently

$$\boldsymbol{K} = \boldsymbol{\Lambda}^T \boldsymbol{\kappa} \tag{2.142}$$

Curvature vectors by rotation vector. Following the same procedure as the one discussed above (see for instance (2.118)), we can get an alternative form of the curvature strain measure as (see appendix D)

$$\boldsymbol{\kappa} = \boldsymbol{T}(\tilde{\boldsymbol{\theta}})\boldsymbol{\theta}' \tag{2.143}$$

$$\boldsymbol{K} = \boldsymbol{T}^T(\tilde{\boldsymbol{\Theta}})\boldsymbol{\Theta}' \tag{2.144}$$

Tensor $\boldsymbol{T}$ relates spin-like variables with the appropriate variation of the rotation vector. In particular, in expression (2.143) the tensor $\boldsymbol{T}$ maps the spatial variation of the rotation tensor into a generalized curvature vector, and therefore may be regarded as the compatibility equations, as they nonlinearly relate generalized strains with generalized displacement.

Table 2.2: Rotation field components in material and spatial description.

Quantity description	Coordinate	Spin tensor ($\in \mathcal{T}\,SO(3)$)[a]	Axial vector ($\in \mathbb{R}^3$)[b]
Total rotation	Material	$\tilde{\boldsymbol{\Psi}}$	$\boldsymbol{\Psi}$
	Spatial	$\tilde{\boldsymbol{\psi}}$	$\boldsymbol{\psi}$
Incremental rotation	Material	$\tilde{\boldsymbol{\Theta}} \stackrel{\text{def}}{=} \boldsymbol{\Lambda}^T \tilde{\boldsymbol{\theta}}\, \boldsymbol{\Lambda}$	$\boldsymbol{\Theta} = \boldsymbol{\Lambda}^T \boldsymbol{\theta} = \boldsymbol{T}^T \boldsymbol{\Psi}$[c,d,e]
	Spatial	$\tilde{\boldsymbol{\theta}} \stackrel{\text{def}}{=} \boldsymbol{\Lambda}\, \tilde{\boldsymbol{\Theta}}\, \boldsymbol{\Lambda}^T$	$\boldsymbol{\theta} = \boldsymbol{\Lambda}\, \boldsymbol{\Theta} = \boldsymbol{T}\, \boldsymbol{\psi}$
Angular velocity	Material	$\tilde{\boldsymbol{\Omega}} \stackrel{\text{def}}{=} \boldsymbol{\Lambda}^T \dot{\boldsymbol{\Lambda}}$[f]	$\boldsymbol{\Omega} = \boldsymbol{\Lambda}^T \boldsymbol{\omega} = \boldsymbol{T}^T \dot{\boldsymbol{\Psi}}$
	Spatial	$\tilde{\boldsymbol{\omega}} \stackrel{\text{def}}{=} \dot{\boldsymbol{\Lambda}}\, \boldsymbol{\Lambda}^T$	$\boldsymbol{\omega} = \boldsymbol{\Lambda}\, \boldsymbol{\Omega} = \boldsymbol{T}\, \dot{\boldsymbol{\psi}}$
Angular acceleration	Material	$\tilde{\boldsymbol{A}} \stackrel{\text{def}}{=} \boldsymbol{\Lambda}^T \ddot{\boldsymbol{\Lambda}} + \dot{\boldsymbol{\Lambda}}^T \dot{\boldsymbol{\Lambda}}$	$\boldsymbol{A} = \boldsymbol{T}^T \ddot{\boldsymbol{\Psi}} + \dot{\boldsymbol{T}}^T \dot{\boldsymbol{\Psi}}$
	Spatial	$\tilde{\boldsymbol{\alpha}} \stackrel{\text{def}}{=} \ddot{\boldsymbol{\Lambda}}\, \boldsymbol{\Lambda}^T + \dot{\boldsymbol{\Lambda}}\, \dot{\boldsymbol{\Lambda}}^T$	$\boldsymbol{\alpha} = \boldsymbol{T} \ddot{\boldsymbol{\Psi}} + \dot{\boldsymbol{T}} \dot{\boldsymbol{\Psi}}$
Curvature	Material	$\tilde{\boldsymbol{K}} \stackrel{\text{def}}{=} \boldsymbol{\Lambda}^T \boldsymbol{\Lambda}'$[g]	$\boldsymbol{K} = \boldsymbol{\Lambda}^T \boldsymbol{\kappa} = \boldsymbol{T}^T \boldsymbol{\Psi}'$
	Spatial	$\tilde{\boldsymbol{\kappa}} \stackrel{\text{def}}{=} \boldsymbol{\Lambda}'\, \boldsymbol{\Lambda}^T$	$\boldsymbol{\kappa} = \boldsymbol{\Lambda}\, \boldsymbol{K} = \boldsymbol{T}\, \boldsymbol{\psi}'$

[a] The relation between a skew-symmetric tensor and its axial vector is formally written as $\boldsymbol{\Theta} = \text{axial}[\tilde{\boldsymbol{\Theta}}]$.

[b] The relation between an axial vector and its skew-symmetric tensor is formally written as $\tilde{\boldsymbol{\Theta}} = \text{skew}[\boldsymbol{\Theta}]$.

[c] Rotation tensor is a mapping $\boldsymbol{\Lambda} : \mathcal{T}^{mat}_{\boldsymbol{\Lambda}} \rightarrow \mathcal{T}^{spat}_{\boldsymbol{\Lambda}}$, such as $\boldsymbol{\Lambda} = \exp[\tilde{\boldsymbol{\psi}}] = \boldsymbol{I} + \frac{\sin\psi}{\psi} \tilde{\boldsymbol{\psi}} + \frac{1-\cos\psi}{\psi^2} \tilde{\boldsymbol{\psi}}^2$.

[d] Transformation tensor is a mapping $\boldsymbol{T} : \mathcal{T}^{spat}_{\boldsymbol{I}} \rightarrow \mathcal{T}^{spat}_{\boldsymbol{\Lambda}}$ such as $\boldsymbol{T} = \frac{\sin\psi}{\psi} \boldsymbol{I} + \frac{1-cos\psi}{\psi^2} \tilde{\boldsymbol{\psi}} + \frac{\psi-\sin\psi}{\psi^3} \tilde{\boldsymbol{\psi}}^2$.

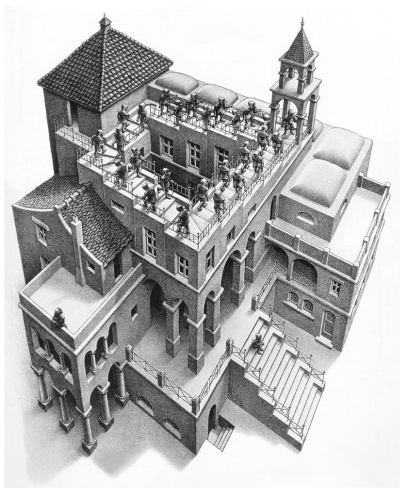
[e] Commutative property hold for the transformation tensor $\boldsymbol{T}^T = \boldsymbol{\Lambda}^T \boldsymbol{T}$, as expected since by inspection $\boldsymbol{\omega} = \boldsymbol{T}\, \dot{\boldsymbol{\psi}} = \boldsymbol{\Lambda}\, \boldsymbol{\Omega} = \boldsymbol{\Lambda}\, \boldsymbol{T}^T \dot{\boldsymbol{\psi}}$.

[f] Time derivative is indicated as $\dot{(\cdot)} = \mathrm{d}(\cdot)/\mathrm{d}t$.

[g] Spatial derivative is indicated as $(\cdot)' = \mathrm{d}(\cdot)/\mathrm{d}s$, where s represent the axial-length parameter.

Chapter 3

Geometrically exact beam theory: kinematics and strain



Come forth into the light of things,
Let nature be your teacher.

W. Wordsworth

This chapter is the first of three devoted to present the geometrically exact formulation for beams capable of reproducing large displacements and rotations. The present formulation is based on that originally proposed by Simo [56] and extended by Simo & Vu-Quoc [57], [58], [59] and [64], which generalize to the full three-dimensional dynamic case the formulation originally developed by Reissner [50] and [51] for the plane static problem. This approach allows to consider finite shearing, extension, flexure and torsion. In the present case, an initially straight and unstressed beam is considered as the reference configuration, as in the analogous approach by Simo [56]. The overall theory is addressed within the framework of classical continuum mechanics, considering the basic definitions of body, configuration space, deformation, etc. as presented in [40] and [28].

The Chapter is organized as follow. Firstly, the initial and current reference placements for an arbitrary body is introduced, and subsequently its specialization for beam-like bodies is addressed. The straight and unstrained reference configuration of the beam is then described in detail with the help of a reference frame, assumed for simplicity coincident to the inertial coordinate system. The kinematic of the classic geometrically exact beam theory is gradually introduced, starting from the statement of the basic kinematic assumptions, and continuing with the description of the moving reference system and its relation with the reference frame. Finally, the configurational description of current configuration of the beam in the three-dimensional space and the statement of the configuration space as a nonlinear differentiable manifold, constituted by a parameter family of 3D Eucledian vectors and orthogonal matrices. Proceeding, the consistent derivation of the deformation gradient tensor is explicitly obtained, and following the mathematical form of strain measures involved at any point of the current cross-section of the beam is determined. Their physical meaning is also addressed in the three-dimensional space. Lastly, the linearization of the most important kinematical quantities is performed basing on the concept of Gateaux directional derivative.

3.1 Basic kinematics

For an appropriated description of the three-dimensional motion of beam in finite deformation it is necessary to deal with the finite rotation of a unit triad; therefore, the results of Chapter 1 are used repeatedly here to describe the Rissner-Simo geometrically exact formulation for beams.

3.1.1 Initial and current reference placements

Let $\chi\colon \mathcal{B} \to \mathbb{R}^3$ be a smooth time-dependent embedding of material body $\mathcal{B}$ into Euclidean space $\mathbb{R}^3$ defined by a *global* spatially fixed [1] reference system $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. For each fixed time $t \in \mathbb{R}^+$, the mapping $\chi(\cdot, t)$ is defined as the *current placement* of the body $\mathcal{B}$ along with the current place vector $\boldsymbol{x}$ of a body-point, namely

$$\mathcal{B}_t \subset \mathbb{R}^3 := \chi(\boldsymbol{X}, t), \quad \boldsymbol{x} = \chi(\boldsymbol{X}, t), \quad \forall \boldsymbol{X} \in \mathcal{B} \tag{3.1}$$

The initial reference placement $\mathcal{B}_0$ is defined as the special case of the current placement $\mathcal{B}_t$ by setting $t = 0$, giving

$$\mathcal{B}_0 \subset \mathbb{R}^3 := \chi(\boldsymbol{X}, t = 0), \quad \boldsymbol{X} = \chi(\boldsymbol{X}, t = 0), \quad \forall \boldsymbol{X} \in \mathcal{B} \tag{3.2}$$

where $\boldsymbol{X}$ is an initial reference place vector. Since the initial reference placement $\mathcal{B}_0$ is unaffected by observation transformation (see e.g. [47]), it is possible to call vectors and tensors defined on the initial reference placement $\mathcal{B}_0$ as *material quantities*. For example, the reference place vector $\boldsymbol{X}$ is called material place vector, and $\mathcal{B}_0$ the material placement of the body. Sometimes the material description is named referential, global, natural, original or *Lagrangian* description. Contrary to the material placement $\mathcal{B}_0$, the current placement $\mathcal{B}_t$, along with vectors and tensors defined on it, are concerned on the observation transformation. Vectors and tensors defined on the current placement $\mathcal{B}_t$ are called *spatial quantities*, e.g. a current place vector $\boldsymbol{x}$ is also

[1] By *spatial fixed* means it is fixed in an arbitrarily chosen orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ that has no acceleration nor rotation in the three-dimensional inertial physical space.

named as a spatial place vector, and $\mathcal{B}_t$ as a spatial placement. A spatial description is sometimes referred also as present or *Eulerian* description.

In this work the terms *material* and *spatial* will be applied for placements, vectors, tensors, fields, spaces, and descriptions. A geometric interpretation of the material body $\mathcal{B}$, the material placement $\mathcal{B}_0$ and the spatial placement $\mathcal{B}_t$, as well as for material and spatial vectors $\boldsymbol{V}$ and $\boldsymbol{v}$ respectively, is given in Figure 3.1 [2].

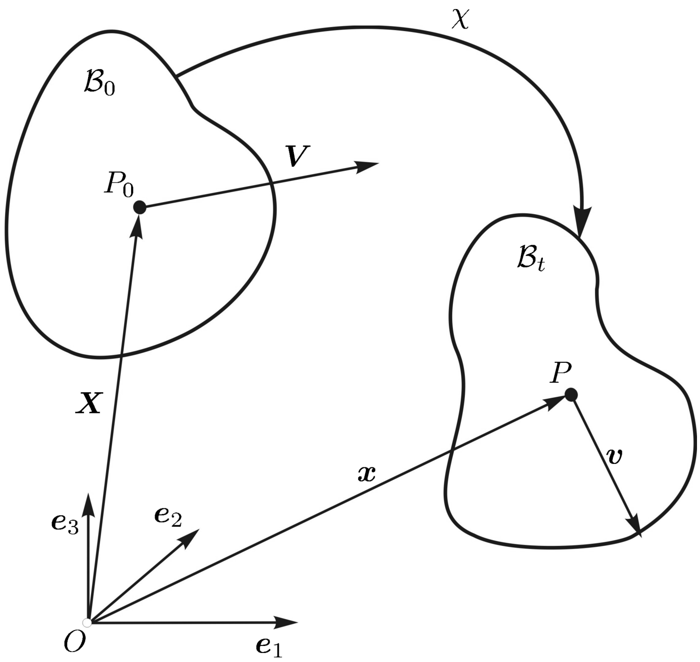


Figure 3.1: Configurational description of the motion.

3.1.2 Beam configurational description

While in the previous section was introduced the configurational description for a generic body $\mathcal{B}$, here we want to specialize this view to the case of beams.

Beam reference configuration. For sake of simplicity, without losing generality, let assume that, in the reference configuration, the beam has a *straight axis* and *uniform cross-sections* [3], laying in an *unstrained* and *unstressed* state. Adopting a classical approach, a beam is viewed as a three-dimensional body, whose material placement can be described by a straight line $\mathscr{C}_0 \subset \mathbb{R}^3$, which we refer to as the *line of centroids*, that has attached at each point a planar domain Ω_0, called *cross-section*. The line $\mathscr{C}_0$ is assumed normal to the plane of each cross-section, with intersection point the centroid of Ω_0. Accordingly, the reference configuration of the beam could be specified in terms of a straight line $\mathscr{C}_0$ parametrized by a map $\boldsymbol{\varphi}_0(S)\colon [0, L] \to \mathbb{R}^3$

[2]Note that placement, likewise place vectors, should be regarded as mappings, not the image of these maps.

[3]It is assumed a prismatic beam, where the cross-section doesn't change along the axis. From a practical point of view, this hypothesis ensures the geometrical quantities as cross-section area or moment of inertia, remain constant along the element.

and constituted by all the physical points $\boldsymbol{X}_0$ which are occupied by material particles at the initial time, conventionally designed by t_0, i.e.

$$\mathscr{C}_0 = \left\{ \boldsymbol{X}_0 \in \mathbb{R}^3 \,|\, \boldsymbol{X}_0 = \boldsymbol{\varphi}_0(S) \quad \forall S \in [0, L] \right\} \tag{3.3}$$

where L is the total length of the beam, and S the relative position of the centroidal particle with respect to the curve (arc-length coordinate).

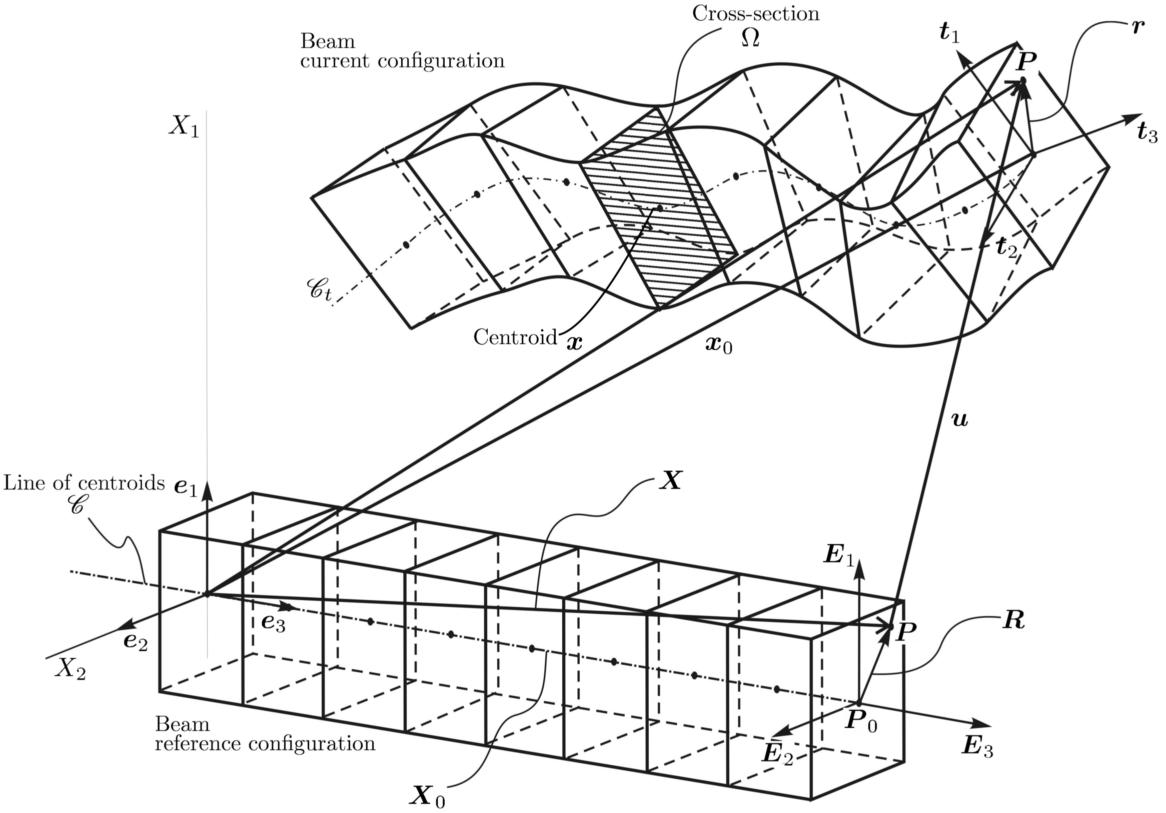


Figure 3.2: Three-dimensional representation of the coordinate system, beam reference configuration Ω_0 and beam current configuration (Ω_t): global reference system $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ and set of coordinates $\{X_1, X_2, X_3\}$; reference frame $\{\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$, reference axis position vector $\mathbf{X_0}$, reference cross-section position vector $\mathbf{R}$ and reference position vector $\mathbf{X}$; moving frame $\{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$, current axis position vector $\boldsymbol{x}_0$, current cross-section position vector $\mathbf{r}$ and current position vector $\boldsymbol{x}$.

In order to describe this configuration, we conventionally introduce a right-handed orthonormal frame $\{O, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$, called *reference frame*, with origin O located on the axis and $\boldsymbol{E}_i = \{\boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$ oriented such that $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ lay parallel to a generic cross-section and give the directions of the principal axis of inertia, whereas $\boldsymbol{E}_3$ is tangent to the beam axis (line of centroids $\mathscr{C}_0$) in its initial straight configuration. Assuming for simplicity that the reference frame coincides with the global one $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ for all stages, as shown in Figure 3.2, the beam reference configuration is described by the reference position vector field $\boldsymbol{X} \in \mathbb{R}^3$

$$\mathbf{X}(S, t) = \mathbf{X}_0(X_3) + \mathbf{R}(X_1, X_2) \tag{3.4}$$

where

$$\mathbf{X}_0(X_3) = X_3\,\mathbf{E_3} \qquad \mathbf{R} = X_\alpha\,\mathbf{E}_\alpha = X_1\mathbf{E}_1(S) + X_2\mathbf{E}_2(S) \tag{3.5}$$

and t being an evolution time parameter. Being X_3 the reference axis coordinate, $\mathbf{X}_0$ describes the position of a point along the beam axis, while being X_1 and X_2 the reference cross-section coordinates, $\mathbf{R}$ represents the relative position of a point within a cross-section with respect to the axis, also named *reference cross-section position vector*. Vector $\mathbf{E}_\alpha$ defines accordingly the cross-section plane.

Kinematic assumptions. The definition of a beam deformation map is the basic step to build a beam theory from a three-dimensional continuum theory using a principle of virtual work. We start here to introduce a set of restrictions on the kinematics of the displacement field, which doesn't affect the adequate reproduction of finite strains, large displacements and rotations, as well as shear distortion in bending.
The basic kinematic assumptions are

- The cross-section remain plane in the current (spatial) configuration, i.e. warping effects are not allowed;
- The cross-sections remain undeformed in their plane during the deformation, i.e. cross-sections don't undergo any change of shape or size [4];
- The cross-section may only rotate as a rigid body and it doesn't remain necessarily normal to the deformed line of centroids $\mathscr{C}$.

The first hypothesis follows from the fact that the deformation map is assumed independent of any function of the sectional coordinates X_1, X_2, i.e. plane cross-sections remain plane, whereas the second hypothesis precludes any changes in the cross-sectional area and shape. Lastly, the third hypothesis states that cross-sections are invariants under any deformation and they do not remain normal to the deformed longitudinal axis in any possible deformed state of the beam. The deviation from normality is produced by a transverse shear that is assumed to the constant over the cross-section. The reader will note the close analogy with the Timoshenko beam hypothesis for small displacement theory. The geometric significance of kinematical assumptions just introduced is illustrated for the plane case in Figure 3.3.

Beam current configuration. During the motion the beam deforms from the undisturbed straight configuration at time t_0 to the current beam configuration at time t. To describe the beam current configuration, we introduce a right-handed floating orthonormal frame, $\{o, \boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$, called *moving* or *current frame*, locally attached to the beam, with origin o located on the current axis and $\boldsymbol{t}_i := \{\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$ oriented such that $\boldsymbol{t}_1$ and $\boldsymbol{t}_2$ lay parallel to a generic cross-section in the current configuration and point to the direction of the principal axis of inertia, while $\boldsymbol{t}_3$ is normal to each cross-section in the current configuration. We assume that this spatial frame coincides with the material frame at the initial stage, i.e. $\boldsymbol{t}_i = \boldsymbol{E}_i$.
The introduced moving frame has the following properties

[4] The *rigid cross-section* assumption is valid in practice for slender beams, thin beams, or rods.

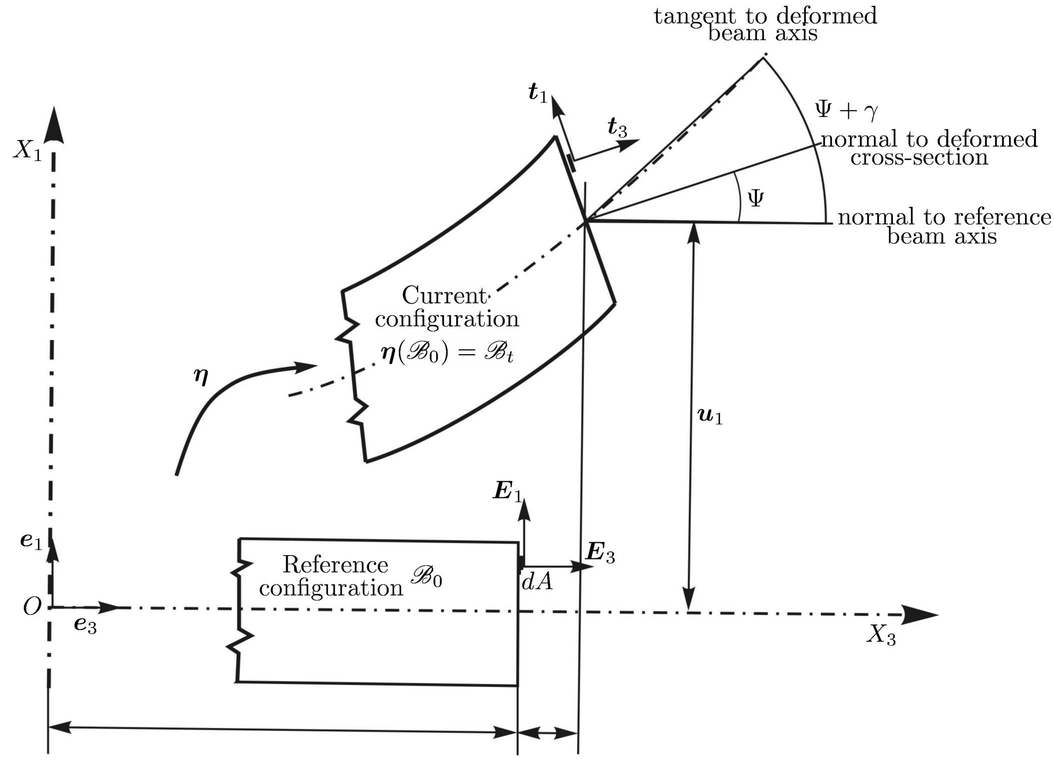


Figure 3.3: Kinematic assumptions: reference and current configurations in the plane problem.

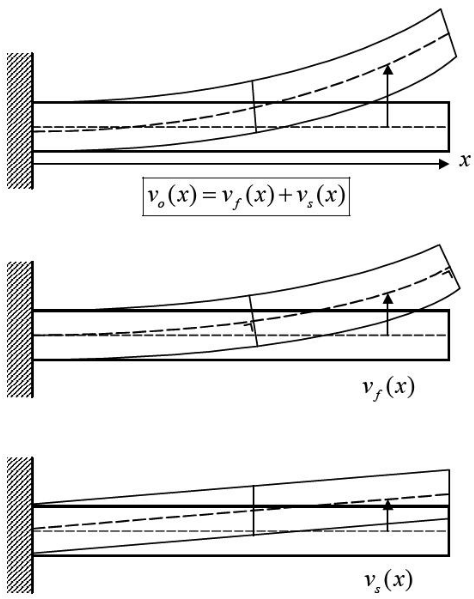


Figure 3.4: Displacements decomposition of Timoshenko beam: v_f represents the classical flexural displacement referred to Euler-Bernoulli beam theory, whereas v_s stands for the shear displacement referred to Timoshenko beam theory. Additive formula is assumed to be valid.

i. Normality condition $||\boldsymbol{t}_i(S)|| = 1 \quad$ with $\quad i = 1, 2, 3;$

ii. Orthogonality condition $\boldsymbol{t}_i(S) \cdot \boldsymbol{t}_j(S) = 0 \quad$ with $\quad i \neq j;$

iii. $\boldsymbol{t}_3(S) \equiv \boldsymbol{n}(S) = \boldsymbol{t}_1(S) \times \boldsymbol{t}_2(S) \qquad \forall\, S \in [0, L] \subset \mathbb{R};$

where the parameter S represents the arc-length of the line of centroids in the reference unloaded configuration, and $\boldsymbol{n}(S)$ indicates the unit vector normal to the generic cross-section.

An arbitrary cross-section of the beam, initially laying in a plane normal to $\boldsymbol{E}_3$ is assumed to experience a finite rotation defined by specifying the current orientation of the orthonormal basis $\{\boldsymbol{t}_i\}$ attached to the cross sections and initially coincident with $\{\boldsymbol{E}_i\}$. This is equivalent to prescribing a one-parameter family of orthogonal transformations $\boldsymbol{\Lambda}\colon S \in [0, L] \to SO(3)$ which uniquely define the orientation of the moving frame. Pointing out that the moving frame is function only of the reference axis coordinate X_3, i.e. $\boldsymbol{t}_i = \boldsymbol{t}_i(X_3)$, and observing that the moving frame and the reference frame are both orthonormal, we may introduce a one-parameter rotation tensor $\boldsymbol{\Lambda} \in SO(3)$, relating the reference and the moving frame at time $t \in \mathbb{R}^+$ as

$$\boxed{\boldsymbol{t}_i(S, t) = \boldsymbol{\Lambda}(S, t)\, \boldsymbol{E}_i} \tag{3.6}$$

i.e. the rotation operator $\boldsymbol{\Lambda}$ maps the Cartesian reference frame $\{O, \boldsymbol{E}_1, \boldsymbol{E}_2, \boldsymbol{E}_3\}$ into the orthonormal moving frame $\{o, \boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$. Physically, we may define the moving frame as the rotated reference frame.

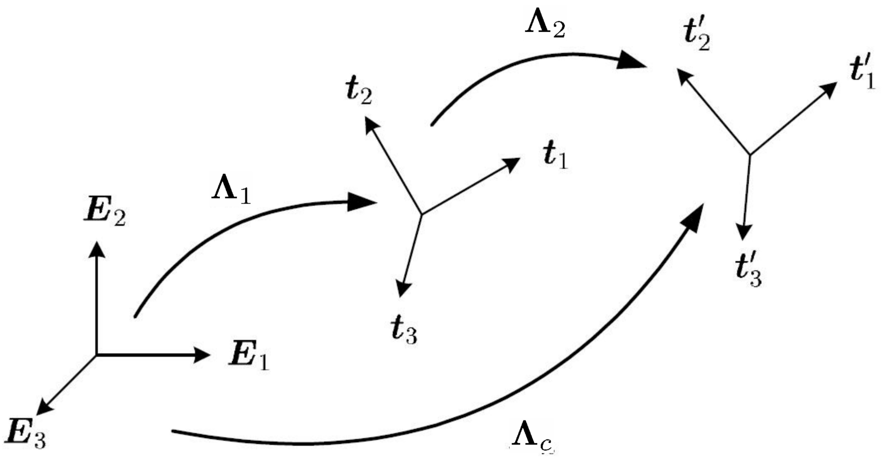


Figure 3.5: Incremental rotation frame.

Rearranging equation (3.6) we can obtain the following expression for $\boldsymbol{\Lambda}$

$$\boldsymbol{\Lambda}(X_3) = \boldsymbol{t}_i(X_3) \otimes \boldsymbol{E}_i \tag{3.7}$$

which shows that $\boldsymbol{\Lambda}$ is a two-point orthogonal tensor field. Here we used $\otimes$ to denote the standard tensor product of vectors. From a computational point of view, it is interesting to note that taking $\boldsymbol{E}_i$ as the standard basis in $\mathbb{R}^3$, the column vectors constituting $\boldsymbol{\Lambda}$ are respectively $\boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3$ in the standard basis.

With the help of the introduced quantities, following [56], we assume to describe the placement of a material point in the deformed configuration by the position vector field $\boldsymbol{x} \in \mathbb{R}^3$

$$\boldsymbol{x}(S,t) = \boldsymbol{x}_0(X_3) + \boldsymbol{r}(X_\alpha, X_3) \tag{3.8}$$

where

$$\boldsymbol{x}_0(X_3) = X_3\,\boldsymbol{t}_3 \qquad \boldsymbol{r} = X_\alpha\,\boldsymbol{t}_\alpha(X_3) \tag{3.9}$$

Vector $\boldsymbol{x}_0 = \boldsymbol{x}_0(X_3)$ represents the position of the cross-section centroid in the current configuration, that is the position of the moving frame's origin o (see Figure 3.2). From Figure 3.2 it can be concluded also that the position vectors of the material point $\boldsymbol{X}_0$ moves to the spatial location $\boldsymbol{x}_0$ through a *translational displacement* vector of the centroidal axis, such that $\boldsymbol{x}_0 = \boldsymbol{X}_0 + \boldsymbol{u}$. The components of the displacement vector $\boldsymbol{u}$ are regarded as the cross-section translational degrees of freedom.
On the other side, vector $\boldsymbol{r}$ represents the position of a point within a cross-section in the current configuration, reason why we call it *current cross-section position vector.* According to equation (3.6) we have

$$\boldsymbol{r} = \boldsymbol{\Lambda}(X_3)\,\boldsymbol{R}(X_\alpha) \tag{3.10}$$

that is, $\boldsymbol{r}$ is obtained by the rotation of the reference cross-section position vector and, since any $\boldsymbol{R}$ of a same cross-section rotates of the same quantity $\boldsymbol{\Lambda}(X_3)$, it follows that the cross-section rigidly moves from the reference to the current configuration and that $\boldsymbol{\Lambda}$ represents the so called *cross-section rigid rotation.*
With the help of the introduced quantities, we may now rewrite the deformation map (3.8) as

$$\begin{aligned}\boldsymbol{x}(X_3) &= \boldsymbol{x}_0(X_3) + \boldsymbol{r}(X_3)\\ &= [\boldsymbol{X}_0(X_3) + \boldsymbol{u}(X_3)] + \boldsymbol{\Lambda}(X_3)\,\boldsymbol{R}(X_\alpha)\end{aligned} \tag{3.11}$$

which clearly shows how the current configuration of the beam is uniquely defined by the spatial position of the centroidal line $\boldsymbol{x}_0(X_3)$ and the orientation of the moving frame $\boldsymbol{\Lambda}(X_3)$. As a consequence of that, the three-dimensional kinematics is reduced to a one-dimensional kinematics with parameter X_3.
Since a three-dimensional configuration is uniquely determined by prescribing the function $\boldsymbol{\eta} = (\boldsymbol{x}_0, \boldsymbol{\Lambda})$ on the domain $[0, L]$ (taking values on $\mathbb{R}^3 \times SO(3)$), it becomes natural to refer to the set

$$\mathscr{C} = \left\{\boldsymbol{\eta} = (\boldsymbol{u}(s,t), \boldsymbol{\Lambda}(s,t)) : [0,L] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3 \times SO(3)\right\} \tag{3.12}$$

$$\begin{cases} \boldsymbol{u} : [0,L] \times \mathbb{R}^+ \longrightarrow \mathbb{R}^3 \\ \boldsymbol{\Lambda} : [0,L] \times \mathbb{R}^+ \longrightarrow SO(3) \end{cases} \tag{3.13}$$

as the current configuration state of the beam at time $t \in \mathbb{R}^+$.

According to equation (3.11), in the current configuration the beam can be physically seen as a line, i.e. the axis individuated by $\boldsymbol{x}_0$, and a set of attached cross-sections obtained by a rigid rotation of the cross-section itself in the reference configuration.

According to that, the current placement of the beam resulting from 3D motion is completely characterized through the position of the centroidal line and the local orientation of the moving floating frame, as the result of a composition of a *translational motion* and a *rotational motion*. The former governs the change of position vector via displacement $\boldsymbol{u}$, whereas the latter governs the change of orientation via orthogonal rotation tensor $\boldsymbol{\Lambda}$. Hence, the current placement is completely determined by the pair $(\boldsymbol{u}, \boldsymbol{\Lambda}) \in \mathbb{R}^3 \times SO(3)$ at any point of the beam, where $\mathbb{R}^3$ refers to the translational displacement field $\boldsymbol{u}$ and $SO(3)$ to the rotational displacement field $\boldsymbol{\Psi}$. Respectively, the element of $\mathbb{R}^3$ is the cross-section centroid displacement vector, whereas the element of $SO(3)$ is the rotation operator $\boldsymbol{\Lambda}$ parametrized by the rotation vector $\boldsymbol{\Psi}$.
We also note that equation (3.11) states esplicitly the kinematic assumption for which the displacement and rotation fields are independent variables.

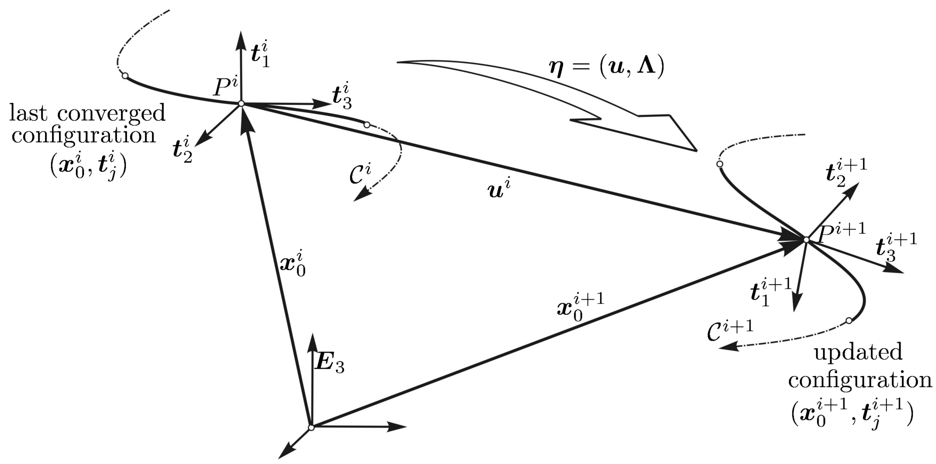


Figure 3.6: Deformation process.

3.2 Deformation gradient and strain measures

Before proceeding with the computation of deformation gradient, we recall from (2.106) that the time derivative of a general one-parameter rotation matrix is $\dot{\boldsymbol{\Lambda}}(t) = \tilde{\boldsymbol{\omega}}(t)\boldsymbol{\Lambda}(t)$, where $\tilde{\boldsymbol{\omega}}(t)$ is the skew-symmetric spin tensor (material angular tensor). Considering that also here we are dealing with a one-parameter rotation, the cross-section rotation $\boldsymbol{\Lambda}(X_3)$, we can naturally write

$$\frac{\mathrm{d}\boldsymbol{\Lambda}}{\mathrm{d}X_3} = \boldsymbol{\Lambda}' = \tilde{\boldsymbol{\kappa}}\boldsymbol{\Lambda} \qquad \text{with} \qquad \tilde{\boldsymbol{\kappa}} = \tilde{\boldsymbol{\kappa}}(X_3) \qquad \tilde{\boldsymbol{\kappa}} \in so(3) \tag{3.14}$$

From the three-dimensional theory of non-linear continuum mechanics, the deformation gradient $\boldsymbol{F}$ is defined as the gradient of the deformation map $\boldsymbol{x}$ (3.11) as

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}} \tag{3.15}$$

where the symbol $\partial(\cdot)$ stands for a partial derivative. It determines the strain measures at any point of the beam cross-section.

Expressing a component of the reference position vector field $\boldsymbol{X}$ as $X_i = \boldsymbol{X} \cdot \boldsymbol{E}_i$, we get

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial X_i} \otimes \boldsymbol{E}_i \tag{3.16}$$

where the notation $\otimes$ stands for the tensor product of vectors. Now, noting that in equation (3.11) $\boldsymbol{x}_0$ and $\boldsymbol{t}_\alpha$ depends on axial-length parameter X_3, and remind equation (3.6), we obtain

$$\begin{aligned}\boldsymbol{F} &= \frac{\partial \boldsymbol{x}}{\partial X_\alpha} \otimes \boldsymbol{E}_\alpha + \frac{\partial \boldsymbol{x}}{\partial X_3} \otimes \boldsymbol{E}_3 \\ &= \boldsymbol{t}_\alpha \otimes \boldsymbol{E}_\alpha + \left(\frac{\partial \boldsymbol{x_0}}{\partial X_3} + X_\alpha \frac{\partial \boldsymbol{t}_\alpha}{\partial X_3}\right) \otimes \boldsymbol{E}_3 \\ &= \boldsymbol{t}_\alpha \otimes \boldsymbol{E}_\alpha + \left(\frac{\partial \boldsymbol{x_0}}{\partial X_3} + X_\alpha \frac{\partial \boldsymbol{\Lambda}}{\partial X_3} \boldsymbol{E}_\alpha\right) \otimes \boldsymbol{E}_3\end{aligned} \tag{3.17}$$

Using the spin tensor $\tilde{\boldsymbol{\kappa}}$ to express the cross-section rotation derivative $\frac{\mathrm{d}\boldsymbol{\Lambda}}{\mathrm{d}X_3}$ (see equation (3.14)), and recalling equation (3.6), the term $X_\alpha \frac{\partial \boldsymbol{\Lambda}}{\partial X_3} \boldsymbol{E}_\alpha$ can be rearranged as

$$X_\alpha \frac{\partial \boldsymbol{\Lambda}}{\partial X_3} \boldsymbol{E}_\alpha = X_\alpha \tilde{\boldsymbol{\kappa}} \boldsymbol{\Lambda} \boldsymbol{E}_\alpha = X_\alpha \tilde{\boldsymbol{\kappa}} \boldsymbol{t}_\alpha \tag{3.18}$$

By substitution of this term, the deformation gradient becomes

$$\boldsymbol{F} = \boldsymbol{t}_\alpha \otimes \boldsymbol{E}_\alpha + \left(\frac{\partial \boldsymbol{x_0}}{\partial X_3} + X_\alpha \, \tilde{\boldsymbol{\kappa}} \, \boldsymbol{t}_\alpha\right) \otimes \boldsymbol{E}_3 \tag{3.19}$$

Adding and subtracting the tensor $\boldsymbol{t}_3 \otimes \boldsymbol{E}_3$ to the right-hand side, and recognizing that $\boldsymbol{t}_i \otimes \boldsymbol{E}_i = \boldsymbol{\Lambda}$ when $\boldsymbol{E}_i$ is the standard basis of $\mathbb{R}$, and making use of property (A),the Lie algebra (2.14) and equation (3.10), we may compactly rewrite equation (3.19) as

$$\begin{aligned}\boldsymbol{F} &= \boldsymbol{\Lambda} + \left[\left(\frac{\partial \boldsymbol{x_0}}{\partial X_3} - \boldsymbol{t}_3\right) + X_\alpha \, \tilde{\boldsymbol{\kappa}} \, \boldsymbol{t}_\alpha\right] \otimes \boldsymbol{E}_3 \\ &= \boldsymbol{\Lambda} \left\{\boldsymbol{I} + \left[\boldsymbol{\Lambda}^T \left(\frac{\partial \boldsymbol{x_0}}{\partial X_3} - \boldsymbol{t}_3\right) + \boldsymbol{\Lambda}^T X_\alpha \, \tilde{\boldsymbol{\kappa}} \, \boldsymbol{t}_\alpha\right] \otimes \boldsymbol{E}_3\right\} \\ &= \boldsymbol{\Lambda} \left[\boldsymbol{I} + (\boldsymbol{\Lambda}^T \boldsymbol{\gamma} + \boldsymbol{\Lambda}^T \tilde{\boldsymbol{\kappa}} \, \boldsymbol{r}) \otimes \boldsymbol{E}_3\right] \\ &= \boldsymbol{\Lambda} \left[\boldsymbol{I} + (\boldsymbol{\Gamma} + \tilde{\boldsymbol{K}} \boldsymbol{R}) \otimes \boldsymbol{E}_3\right] \\ &= \boldsymbol{\Lambda} \left[\boldsymbol{I} + (\boldsymbol{\Gamma} + \boldsymbol{K} \times \boldsymbol{R}) \otimes \boldsymbol{E}_3\right] \\ &= \boldsymbol{\Lambda} \boldsymbol{F}^r\end{aligned} \tag{3.20}$$

where [5] $\boldsymbol{F}^r$ represents the deformation gradient in material (back-rotated) coordinates, and

[5]Note the upper case convention for material quantities and lower case for spatial quantities. The material and spatial strain components are connected by the relation $\Gamma_i \boldsymbol{t}_i = \gamma_i \boldsymbol{E}_i$, how depicted in Figure 3.7.

$$\mathbf{\Gamma} = \mathbf{\Lambda}^T \boldsymbol{\gamma} \qquad \text{with} \qquad \boldsymbol{\gamma} = \frac{\partial \boldsymbol{x_0}}{\partial X_3} - \boldsymbol{t}_3 \tag{3.21}$$

$$\boldsymbol{K} = \mathbf{\Lambda}^T \boldsymbol{\kappa} \qquad \text{with} \qquad \boldsymbol{\kappa} = \boldsymbol{T}\boldsymbol{\Psi'} \tag{3.22}$$

It's worth to note that equation (3.20) clearly express the decomposition of the deformation gradient into one term corresponding to the deformation normal to the cross-section plus a second term giving the rotation of the moving frame.
We remark also that even when the deformation gradient is intrinsically *two point* tensor, mathematically it is possible to obtain its spatial counterpart by means of push-forward operation by $\mathbf{\Lambda}$ its material leg to the spatial configuration.

From these relations, using distributivity of cross product with respect to a rotation tensor (see equation (A.9)), we observe that

$$\begin{aligned} \mathbf{\Gamma} &= \mathbf{\Lambda}^T \boldsymbol{\gamma} \\ &= \mathbf{\Lambda}^T \left(\frac{\partial \boldsymbol{x_0}}{\partial X_3} - \boldsymbol{t}_3 \right) \\ &= \mathbf{\Lambda}^T \left(\frac{\partial \boldsymbol{x_0}}{\partial X_3} - \mathbf{\Lambda} \boldsymbol{E}_3 \right) \\ &= \mathbf{\Lambda}^T \frac{\partial \boldsymbol{x_0}}{\partial X_3} - \boldsymbol{E}_3 \end{aligned} \tag{3.23}$$

$$\begin{aligned} \boldsymbol{K} &= \mathbf{\Lambda}^T \boldsymbol{\kappa} \\ &= \mathbf{\Lambda}^T \boldsymbol{T} \boldsymbol{\Psi'} \\ &= \boldsymbol{T}^T \boldsymbol{\Psi'} \end{aligned} \tag{3.24}$$

where (3.24) is formally equivalent to equation (2.144).

Note that parameters $\mathbf{\Gamma}$ and $\boldsymbol{K}$ can be regarded as translational and rotational generalized strains, respectively. In fact we recognized in (3.24) the curvature strain vector (bending and torsional strain measure), whereas equation (3.23) aggregate together the transverse shear and axial strain measure.

Table 3.1: Strain measures in spatial and material form.

Strain measure	Spatial form	Material form
Translational strain	$\boldsymbol{\gamma} = \boldsymbol{x}_0' - \boldsymbol{t}_3$	$\mathbf{\Gamma} = \mathbf{\Lambda}^T(\boldsymbol{x}_0' - \boldsymbol{E}_3)$
Rotational strains	$\boldsymbol{\kappa} = \boldsymbol{T}\boldsymbol{\Psi'}$	$\boldsymbol{K} = \boldsymbol{T}^T\boldsymbol{\Psi'}$

In particular, consider first vector $\boldsymbol{\gamma} = \partial \boldsymbol{x_0} / \partial X_3 - \boldsymbol{t_3}$, we recognize in $\boldsymbol{\gamma}$ as the difference between the vector tangent to the current line of centroids $\partial \boldsymbol{x_0} / \partial X_3$ and the unit vector orthogonal to the cross-section in the current configuration (see Figure 3.8) It is clear that the components of $\boldsymbol{\gamma}$ with respect to the current moving frame $\{o, \boldsymbol{t}_1, \boldsymbol{t}_2, \boldsymbol{t}_3\}$ can be interpreted as follow

- The components in directions $\boldsymbol{t_1}$ and $\boldsymbol{t_2}$, $\boldsymbol{\gamma_1}$ and $\boldsymbol{\gamma_2}$ respectively, represent the centroidal shear flow between sections;

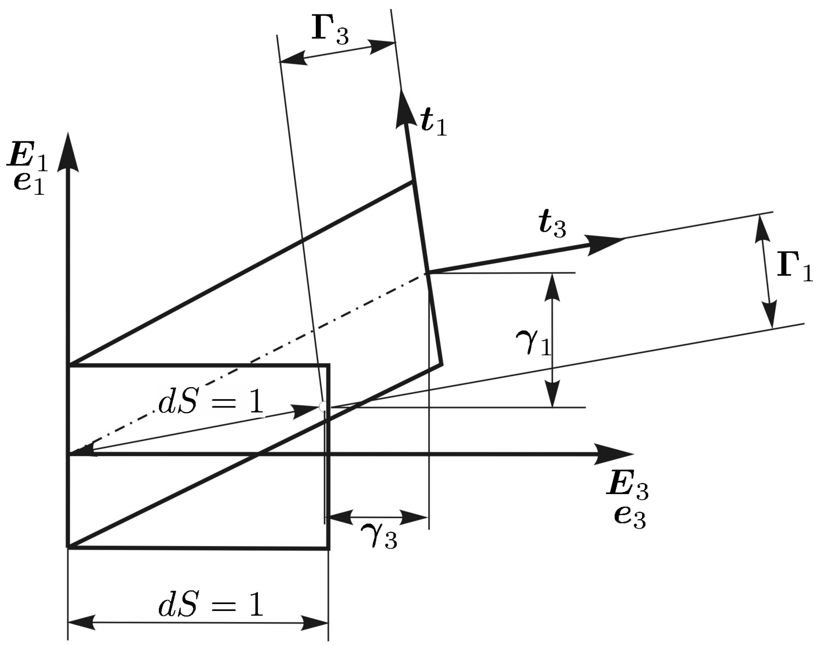


Figure 3.7: A geometric interpretation of the shear strain components of a beam in the finite strain plane case. $\boldsymbol{\gamma}$ measures the difference between the slope of the deformed axis of the beam and the normal to the cross-section defined by $\boldsymbol{t}_1$.

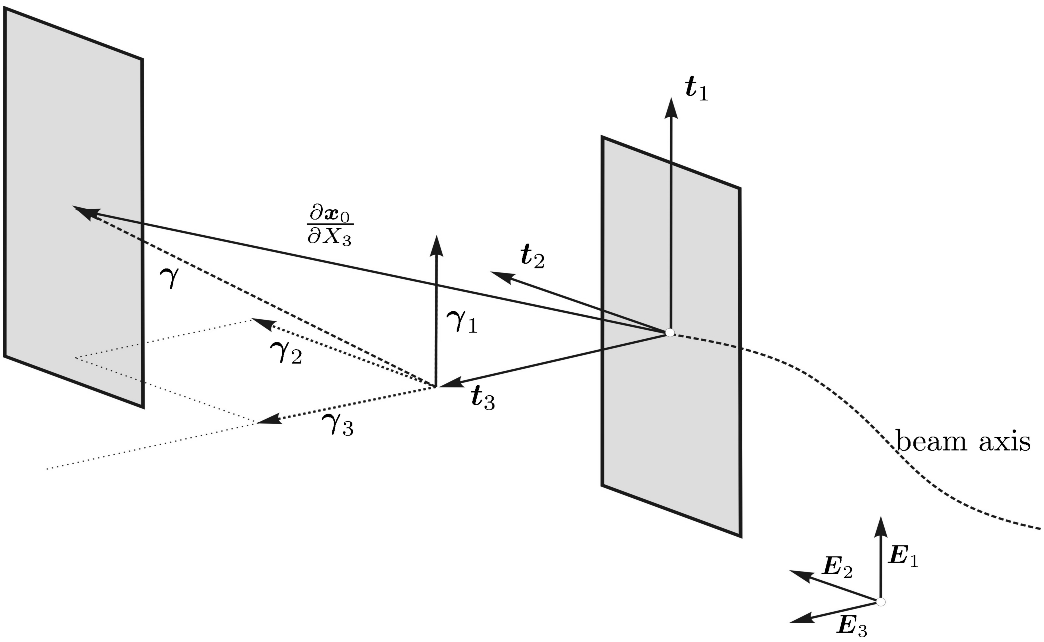


Figure 3.8: Beam strain measure $\boldsymbol{\gamma}$ in three-dimensional space: physical meaning and components in the current configuration.

- The component in direction $\boldsymbol{t_3}$, i.e. $\gamma_{\mathbf{3}}$, represents the centroidal elongation, or shrinkage, of an infinitesimal fiber in the direction normal to the cross-section.

It means that these components are the physical true shear and axial strain measures.

Consider now $\boldsymbol{\kappa}_\alpha = \boldsymbol{\kappa} \times \boldsymbol{t}_\alpha$. Assuming $\boldsymbol{\kappa}$ as a spin vector parallel to the direction $\boldsymbol{t_3}$, we expect that it controls the variation of rotation in this direction. In fact, let consider the case when $\boldsymbol{\kappa}$ lines up with $\boldsymbol{t_3}$ as depicted in Figure 3.9. The cross products $\boldsymbol{\kappa} \times \boldsymbol{t_1}$ and $\boldsymbol{\kappa} \times \boldsymbol{t_2}$ are vectors laying in plane $\boldsymbol{t_1} - \boldsymbol{t_2}$ and turning around $\boldsymbol{t_3}$. It means that they represent the physical variation of rotation around $\boldsymbol{t_3}$, i.e. the cross-section torsion. Similar observation can be done in the case when $\boldsymbol{\kappa}$ lines up with $\boldsymbol{t_1}$ or $\boldsymbol{t_2}$. In these cases the respectively cross products represent the variation of rotation around $\boldsymbol{t_1}$ and $\boldsymbol{t_2}$, i.e. the physical cross-section bending around $\boldsymbol{t_1}$ and $\boldsymbol{t_2}$. For the exposed reasons we can say that the components of $\boldsymbol{\kappa}$ represent the true *bending* and *torsional strain measures*. More in general, vector $\boldsymbol{\kappa}$ is a combination of all these variations of rotations, i.e. it represents the global curvature of the beam.

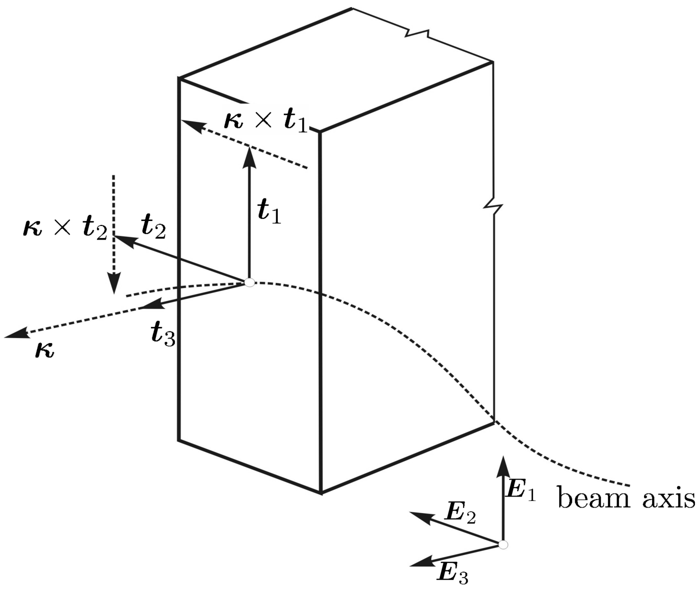


Figure 3.9: Beam strain measure $\boldsymbol{\kappa}$: physical meaning and components in the current configuration.

The generalized spatial strains $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ are affected by superposed rigid body motions, hence for constitutive description are preferred the rotated-back material strains $\boldsymbol{\Gamma}$ and $\boldsymbol{K}$. Figure 3.10 shows a schematic and compact representation of the strain measurements expressing their components in the material reference frame by simplicity.

Now, since spin vector $\boldsymbol{K}$ governs the bending and torsional (rotational) strain measures, we focus here on the strain pair $(\boldsymbol{\Gamma}, \boldsymbol{K})$. Moreover, since for this parametrization the beam configuration is completely defined by knowing, for each cross-section, $\boldsymbol{u}$ and $\boldsymbol{\Psi}$, both belonging to vector spaces, for convenience they are grouped together. In analogy with definition (3.12), let hence introduce the cross-section *generalized*

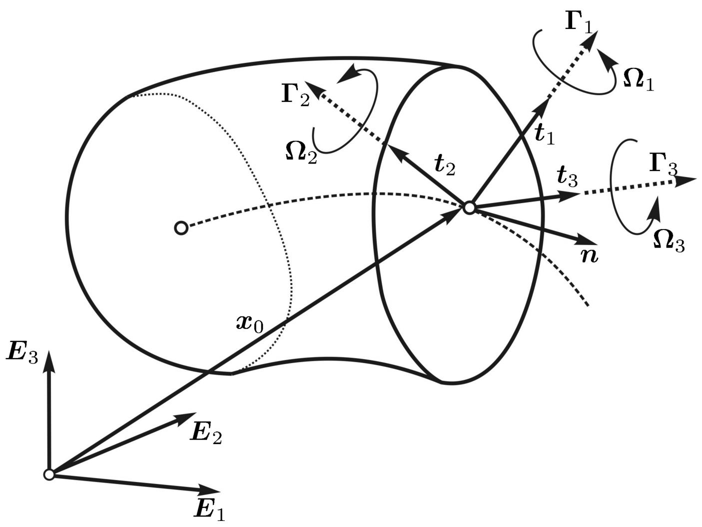


Figure 3.10: Geometric representation of the reduced strain quantities in material coordinates.

displacement vector $\boldsymbol{\eta} = \boldsymbol{\eta}(S, t)$ given by

$$\boldsymbol{\eta} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{\Psi} \end{bmatrix} \tag{3.25}$$

which collects the *kinematic variables* $\boldsymbol{u}$ and $\boldsymbol{\Psi}$, where the former denotes the displacements experienced by the material point $\boldsymbol{X}_0 : [0, L]$ and the latter describes the rotation of floating frame attached at that point.
Recalling equation (3.11), we note that the spatial derivative of the axis position vector in spatial coordinates is

$$\begin{aligned} \boldsymbol{x}_{\boldsymbol{0}}' &= (\boldsymbol{X}_{\boldsymbol{0}} + \boldsymbol{u})' \\ &= (X_3 \boldsymbol{E}_3)' + \boldsymbol{u}' \\ &= \boldsymbol{E}_3 + \boldsymbol{u}' \end{aligned} \tag{3.26}$$

we can place $\boldsymbol{\Gamma}$ and $\boldsymbol{K}$ within a generalized strain vector $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{\eta})$ given by

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \boldsymbol{\Gamma} \\ \boldsymbol{K} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}^T \boldsymbol{x}_{\boldsymbol{0}}' - \boldsymbol{E}_3 \\ \boldsymbol{T}^T \boldsymbol{\Psi}' \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}^T (\boldsymbol{E}_3 + \boldsymbol{u}'(X_3, t)) - \boldsymbol{E}_3 \\ \boldsymbol{T}^T \boldsymbol{\Psi}' \end{bmatrix} \tag{3.27}$$

3.2.1 Time derivative of deformation gradient

In this section we calculate the material time derivative of deformation gradient $\boldsymbol{F}$, that will be used in the next sections for the presentation of the balance laws for beam-like bodies.

Let hence equation (2.108), which provides $\dot{\boldsymbol{\Lambda}} = \tilde{\boldsymbol{\omega}} \boldsymbol{\Lambda}$, and noting that the time derivative of cross-section position vector in material coordinates vanishes $\dot{\boldsymbol{R}} = \boldsymbol{0}$, we can take the material time derivative of deformation gradient expressed by equation

(3.20), so-called *velocity gradient*

$$\begin{aligned}\dot{\boldsymbol{F}} &= \dot{\boldsymbol{\Lambda}}[\boldsymbol{I} + (\boldsymbol{\Gamma} + \boldsymbol{K} \times \boldsymbol{R}) \otimes \boldsymbol{E}_3] + \boldsymbol{\Lambda}[(\dot{\boldsymbol{\Gamma}} + \dot{\boldsymbol{K}} \times \boldsymbol{R}) \otimes \boldsymbol{E}_3] \\ &= \tilde{\boldsymbol{\omega}}\boldsymbol{\Lambda}[\boldsymbol{I} + (\boldsymbol{\Gamma} + \boldsymbol{K} \times \boldsymbol{R}) \otimes \boldsymbol{E}_3] + \boldsymbol{\Lambda}[(\dot{\boldsymbol{\Gamma}} + \dot{\boldsymbol{K}} \times \boldsymbol{R}) \otimes \boldsymbol{E}_3] \\ &= \tilde{\boldsymbol{\omega}}\boldsymbol{F} + \boldsymbol{\Lambda}[(\dot{\boldsymbol{\Gamma}} + \dot{\boldsymbol{K}} \times \boldsymbol{R}) \otimes \boldsymbol{E}_3] \end{aligned} \tag{3.28}$$

where the differentiation with respect to time is indicated, as usual, by a superposed dot. Time differentiation of the strain measures provides the generalized strain rate vectors. Calculating separately each term we obtain

$$\begin{aligned}\dot{\boldsymbol{\Gamma}} &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{\Lambda}^T \frac{\partial \boldsymbol{x_0}}{\partial X_3}) - \frac{\mathrm{d}\boldsymbol{E}_3}{\mathrm{d}t} \\ &= \dot{\boldsymbol{\Lambda}}^T \boldsymbol{x}_0' + \boldsymbol{\Lambda}^T \dot{\boldsymbol{x}}_0' \\ &= -\boldsymbol{\Lambda}^T \tilde{\boldsymbol{\omega}}\, \boldsymbol{x}_0' + \boldsymbol{\Lambda}^T \dot{\boldsymbol{x}}_0' \\ &= \boldsymbol{\Lambda}^T (\dot{\boldsymbol{x}}_0' - \tilde{\boldsymbol{\omega}} \boldsymbol{x}_0') \\ &= \boldsymbol{\Lambda}^T (\dot{\boldsymbol{x}}_0' - \boldsymbol{\omega} \times \boldsymbol{x}_0') \\ &= \boldsymbol{\Lambda}^T (\dot{\boldsymbol{x}}_0' + \boldsymbol{x}_0' \times \boldsymbol{\omega}) \end{aligned} \tag{3.29}$$

where the notation $(\cdot)'$ stands for the differentiation with respect to axial-length parameter X_3. The term $\dot{\boldsymbol{K}}$ is obtained by means of equations (2.108) and (2.134), in fact one can observe

$$\begin{aligned} \tilde{\boldsymbol{\omega}} = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T \;\longrightarrow\; & \dot{\boldsymbol{\Lambda}} = \tilde{\boldsymbol{\omega}}\boldsymbol{\Lambda} \\ & \dot{\boldsymbol{\Lambda}}' = \tilde{\boldsymbol{\omega}}'\boldsymbol{\Lambda} + \tilde{\boldsymbol{\omega}}\boldsymbol{\Lambda}' \end{aligned} \tag{3.30}$$

$$\begin{aligned} \tilde{\boldsymbol{\kappa}} = \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T \;\longrightarrow\; & \boldsymbol{\Lambda}' = \tilde{\boldsymbol{\kappa}}\boldsymbol{\Lambda} \\ & \dot{\boldsymbol{\Lambda}}' = \dot{\tilde{\boldsymbol{\kappa}}}\boldsymbol{\Lambda} + \tilde{\boldsymbol{\kappa}}\dot{\boldsymbol{\Lambda}} \end{aligned} \tag{3.31}$$

and equation (3.30) and (3.31)

$$\begin{aligned} \tilde{\boldsymbol{\omega}}'\boldsymbol{\Lambda} + \tilde{\boldsymbol{\omega}}\boldsymbol{\Lambda}' &= \dot{\tilde{\boldsymbol{\kappa}}}\boldsymbol{\Lambda} + \tilde{\boldsymbol{\kappa}}\dot{\boldsymbol{\Lambda}} \\ \tilde{\boldsymbol{\omega}}'\boldsymbol{\Lambda} &= \dot{\tilde{\boldsymbol{\kappa}}}\boldsymbol{\Lambda} + \tilde{\boldsymbol{\kappa}}\dot{\boldsymbol{\Lambda}} - \tilde{\boldsymbol{\omega}}\boldsymbol{\Lambda}' \\ \tilde{\boldsymbol{\omega}}' &= \dot{\tilde{\boldsymbol{\kappa}}} + \tilde{\boldsymbol{\kappa}}\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T - \tilde{\boldsymbol{\omega}}\boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T \\ \tilde{\boldsymbol{\omega}}' &= \dot{\tilde{\boldsymbol{\kappa}}} + \tilde{\boldsymbol{\kappa}}\tilde{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\kappa}} \end{aligned} \tag{3.32}$$

now, focusing on the last two terms, the parallel use of Lie algebra (2.14) and Lie brackets (A.16) to the generic skew-symmetric tensors $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{C}}$ and their relative axial vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ provides the following correspondences

$$\begin{aligned} &\text{Lie algebra:} \quad \tilde{\boldsymbol{C}}\boldsymbol{h} = \boldsymbol{c} \times \boldsymbol{h} \\ &\text{Lie brackets:} \; (\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}})\boldsymbol{h} = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{h} \\ &\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}} \;\longleftrightarrow\; \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \tilde{\boldsymbol{A}}\,\boldsymbol{b} \end{aligned} \tag{3.33}$$

such that

$$\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{\kappa}}\tilde{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\kappa}} \;\longleftrightarrow\; \boldsymbol{c} = \boldsymbol{\kappa} \times \boldsymbol{\omega} \tag{3.34}$$

hence the raletive axial vector becomes

$$\boldsymbol{\omega}' = \text{axial}[\tilde{\boldsymbol{\omega}}'] = \dot{\boldsymbol{\kappa}} - \boldsymbol{\omega} \times \boldsymbol{\kappa} \quad \Longrightarrow \quad \dot{\boldsymbol{\kappa}} = \boldsymbol{\omega}' + \boldsymbol{\omega} \times \boldsymbol{\kappa} \tag{3.35}$$

which together with equation (2.142) provides

$$\begin{aligned} \boldsymbol{K} = \boldsymbol{\Lambda}^T \boldsymbol{\kappa} \quad \longrightarrow \quad \dot{\boldsymbol{K}} &= \dot{\boldsymbol{\Lambda}}^T \boldsymbol{\kappa} + \boldsymbol{\Lambda}^T \dot{\boldsymbol{\kappa}} \\ &= \boldsymbol{\Lambda}^T \dot{\boldsymbol{\kappa}} + \boldsymbol{\Lambda}^T \tilde{\boldsymbol{\omega}}^T \boldsymbol{\kappa} \\ &= \boldsymbol{\Lambda}^T \dot{\boldsymbol{\kappa}} - \boldsymbol{\Lambda}^T \tilde{\boldsymbol{\omega}} \boldsymbol{\kappa} \\ &= \boldsymbol{\Lambda}^T (\dot{\boldsymbol{\kappa}} - \tilde{\boldsymbol{\omega}} \boldsymbol{\kappa}) \\ &= \boldsymbol{\Lambda}^T (\boldsymbol{\omega}' + \boldsymbol{\omega} \times \boldsymbol{\kappa} - \tilde{\boldsymbol{\omega}} \boldsymbol{\kappa}) \\ &= \boldsymbol{\Lambda}^T (\boldsymbol{\omega}' + \boldsymbol{\omega} \times \boldsymbol{\kappa} - \boldsymbol{\omega} \boldsymbol{\kappa}) \\ &= \boldsymbol{\Lambda}^T \boldsymbol{\omega}' \end{aligned} \tag{3.36}$$

where we the skewness property $\tilde{\boldsymbol{\omega}}^T = -\tilde{\boldsymbol{\omega}}$.
Alternative expression for (3.29) and (3.36) are obtained using equations (2.118) and (2.143) (see appendix D for further details), and recasting the terms involving time variations as function of the generalized displacements one get

$$\begin{aligned} \dot{\boldsymbol{\Gamma}} &= \boldsymbol{\Lambda}^T (\dot{\boldsymbol{x}}_0' + \boldsymbol{x}_0' \times \boldsymbol{T}(\boldsymbol{\Psi}) \dot{\boldsymbol{\Psi}}) \\ &= \boldsymbol{\Lambda}^T (\dot{\boldsymbol{u}}' + \boldsymbol{x}_0' \times \boldsymbol{T} \dot{\boldsymbol{\Psi}}) \end{aligned} \tag{3.37}$$

$$\begin{aligned} \dot{\boldsymbol{K}} &= \boldsymbol{\Lambda}^T (\boldsymbol{T}(\boldsymbol{\Psi}) \dot{\boldsymbol{\Psi}})' \\ &= \boldsymbol{\Lambda}^T (\boldsymbol{T}' \dot{\boldsymbol{\Psi}} + \boldsymbol{T} \dot{\boldsymbol{\Psi}}') \end{aligned} \tag{3.38}$$

where by direct derivation of equation (2.119) with respect to X_3 one get the tensor

$$\begin{aligned} \boldsymbol{T}' =& \frac{1 - \cos \Psi}{\Psi^2} \tilde{\boldsymbol{\Psi}}' + \frac{\Psi - \sin \Psi}{\Psi^3} (\tilde{\boldsymbol{\Psi}} \tilde{\boldsymbol{\Psi}}' + \tilde{\boldsymbol{\Psi}}' \tilde{\boldsymbol{\Psi}}) \\ &+ \frac{\Psi \sin \Psi - 2 + 2 \cos \Psi}{\Psi^4} (\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}') \tilde{\boldsymbol{\Psi}} \\ &+ \frac{3 \sin \Psi - 2\Psi - \Psi \cos \Psi}{\Psi^5} (\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}') \tilde{\boldsymbol{\Psi}}^2 \end{aligned} \tag{3.39}$$

Further details are available in [49].

3.2.2 Time derivative of strain rate

With the aid of (3.37) and (3.38), as well as apposition (3.25) for generalized displacement vector, the generalized material strain rate back-rotated $\dot{\boldsymbol{\varepsilon}}^r$ may be written by means of the vector

$$\dot{\boldsymbol{\varepsilon}}^r = \begin{bmatrix} \dot{\boldsymbol{\Gamma}} \\ \dot{\boldsymbol{K}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Lambda}^T (\dot{\boldsymbol{x}}_0' + \boldsymbol{x}_0' \times \boldsymbol{T} \dot{\boldsymbol{\Psi}}) \\ \boldsymbol{\Lambda}^T (\boldsymbol{T}' \dot{\boldsymbol{\Psi}} + \boldsymbol{T} \dot{\boldsymbol{\psi}}') \end{bmatrix} = \boldsymbol{S} \boldsymbol{\Delta} \dot{\boldsymbol{\eta}} \tag{3.40}$$

where $\boldsymbol{\Delta}$ is the differential operator

$$\boldsymbol{\Delta} = \begin{bmatrix} \boldsymbol{I}\frac{\partial}{\partial X_3} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \\ \mathbf{0} & \boldsymbol{I}\frac{\partial}{\partial X_3} \end{bmatrix} \tag{3.41}$$

and $\boldsymbol{S}$ is the matrix given by

$$\boldsymbol{S} = \boldsymbol{\Lambda}^T \boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\Lambda}^T & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \tilde{\boldsymbol{x}}_{\mathbf{0}}' \boldsymbol{T} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{T}' & \boldsymbol{T} \end{bmatrix} \tag{3.42}$$

where $\mathbf{0}$ is the the null tensor and $\tilde{\boldsymbol{x}_{\mathbf{0}}}'$ is the skew-symmetric tensor whose axial vector is $\boldsymbol{x_0}'$, i.e. more compactly $\tilde{\boldsymbol{x}}_{\mathbf{0}}' = \text{skew}[\boldsymbol{x}_{\mathbf{0}}']$.

3.3 Linearizations of kinematical quantities

The strain quantities defined in the previous section are nonlinear expressions in terms of the generalized motion vector $\boldsymbol{\eta}$, and will lead to a nonlinear governing equations. In this section we deal with the linearization procedures of the fundamental kinematic measures, in order to provide quantities afterwards needed to develop the beam principle of virtual work.

3.3.1 Admissible variations

The current configuration of beam at time t is specified by the position vector of its line of centroids and the orientation of the moving frame by

$$\mathcal{C}_t = \left\{ (\boldsymbol{x}_0(s,t), \boldsymbol{\Lambda}(s,t)) : [0,L] \times [0,T] \Longrightarrow \mathbb{R}^3 \times SO(3) \right\} \tag{3.43}$$

which is a nonlinear differentiable manifold. Let now introduce the notation $\boldsymbol{\delta}(\cdot)$ for a virtual variation [6]. with this on hand, it is possible to construct a *perturbed* (or varied) configuration relative to $\mathcal{C}$, denoted by $\mathcal{C}_\varepsilon$ as follow

$$\mathcal{C}_\varepsilon = \left\{ (\boldsymbol{x}_{0,\varepsilon}(s,t), \boldsymbol{\Lambda}_\varepsilon(s,t)) : [0,L] \times [0,T] \Longrightarrow \mathbb{R}^3 \times SO(3) \right\} \tag{3.44}$$

obtained by setting

$$\boldsymbol{x}_{0,\varepsilon} = \boldsymbol{x}_0 + \varepsilon\, \boldsymbol{\delta x}_0 \tag{3.45}$$

$$\boldsymbol{\Lambda}_\varepsilon = \exp[\varepsilon\, \boldsymbol{\delta}\tilde{\theta}]\, \boldsymbol{\Lambda} \tag{3.46}$$

where $\varepsilon \in \mathbb{R}$, $\boldsymbol{\delta x}_0 \in \mathbb{R}^3$ could be viewed as a superimposed infinitesimal displacement onto the line of centroids defined by $\boldsymbol{x}_0$, and $\boldsymbol{\delta}\tilde{\theta} \in \mathcal{T}^{spat}_{\boldsymbol{\Lambda}} SO(3) \approx so(3)$ with the corresponding axial vector $\boldsymbol{\delta}\theta \in \mathcal{T}^{spat}_{\boldsymbol{\Lambda}}$. Similarly, $\exp[\varepsilon\, \boldsymbol{\delta}\tilde{\theta}]$ for $\varepsilon > 0$ represents a superimposed infinitesimal rotation onto the moving frame defined by $\boldsymbol{\Lambda}$. It should be recalled that *finite rotations* are defined by *orthogonal* transformations, whereas infinitesimal rotations are obtained through *skew-symmetric* transformations. By exponentiation

[6] It should be noted that the virtual displacement could be any size, infinitesimal or finite.

of a skew-symmetric matrix (infinitesimal rotation) one obtains an orthogonal matrix (finite rotation). Thus, equation (3.46) is constructed so that $\boldsymbol{\Lambda}_\varepsilon$ remains orthogonal and thus defines a possible orientation of the moving frame. Hence, by construction, $\mathcal{C}_\varepsilon$ constitutes a possible current configuration of the beam.

Alternatively, it is possible to work with the field $\boldsymbol{\delta\eta} = (\boldsymbol{\delta x}_0, \boldsymbol{\delta}\tilde{\theta}) \in \mathcal{T}^{spat}$ which defines the field of *kinematically admissible variations*. Accordingly, the set of kinematically admissible variations, denoted by $\mathcal{TC}_t$, is given by

$$\mathcal{TC}_t = \{\boldsymbol{\delta\eta} = (\boldsymbol{\delta x}_0, \boldsymbol{\delta}\tilde{\theta}) \in \mathbb{R}^3 \times \mathbb{R}^3 |\ \boldsymbol{\delta x}_0|_{\partial\Omega_u} = \boldsymbol{\delta}\theta|_{\partial\Omega_u} = \mathbf{0}\} \tag{3.47}$$

where $\partial\Omega_u$ is the part of the boundary where displacements and/or rotations are prescribed.

3.3.2 Linearization of displacement field

One can note that we can operate on variations as commonly used with differentials, despite the fact that both quantities have different meaning: variation is something more general then differential, but it can reduces to it in particular cases. It is important to mention that, if variations and derivatives are assumed to commute, this implies that all the manifold constraints are *holomorphic* (more details can be consulted in [36]).

The centroidal position vector $\boldsymbol{x}_0$ and its displacement vector $\boldsymbol{u}$ live in the same vector space, and the additive rule applies to them (see (3.11)). Noting that, by definition, the reference position vector remains unchanged during the deformation process, the classical variation of the material centroidal position vector reads

$$\begin{aligned} \boldsymbol{x}_0 = \boldsymbol{X}_0 + \boldsymbol{u} \quad \longrightarrow \quad \boldsymbol{\delta x}_0 &= \boldsymbol{\delta}(\boldsymbol{X}_0 + \boldsymbol{u}) \\ &= \boldsymbol{\delta X}_0 + \boldsymbol{\delta u} \\ &= \boldsymbol{\delta u} \end{aligned} \tag{3.48}$$

With this result in hand, recalling equation (3.54), we can calculate the variation of general position vector of a material point as $\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{r}$

$$\begin{aligned} \boldsymbol{\delta x} &= \boldsymbol{\delta}(\boldsymbol{x}_0 + \boldsymbol{r}) \\ &= \boldsymbol{\delta u} + \boldsymbol{\delta}(\boldsymbol{\Lambda R}) \\ &= \boldsymbol{\delta u} + \boldsymbol{\delta\Lambda}\, \boldsymbol{R} \\ &= \boldsymbol{\delta u} + \boldsymbol{\delta}\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda R} \\ &= \boldsymbol{\delta u} + \boldsymbol{\delta}\tilde{\boldsymbol{\theta}}\, \boldsymbol{r} \\ &= \boldsymbol{\delta u} + \boldsymbol{\delta\theta} \times \boldsymbol{r} \end{aligned} \tag{3.49}$$

Alternatively, by making use of equation (3.53), we can derive a slight different expression

$$\begin{aligned} \boldsymbol{\delta x} &= \boldsymbol{\delta u} + \boldsymbol{\Lambda}\boldsymbol{\delta}\tilde{\boldsymbol{\Theta}}\, \boldsymbol{R} \\ &= \boldsymbol{\delta u} + \boldsymbol{\Lambda}\boldsymbol{\delta}\tilde{\boldsymbol{\Theta}} \times \boldsymbol{R} \end{aligned} \tag{3.50}$$

3.3.3 Linearizations of rotation tensor

Let consider the rotation tensor $\mathbf{\Lambda} \in SO(3)$ and introducing a parameter $\varepsilon \in \mathbb{R}^3$, we indicate the perturbed infinitesimal rotation by $\mathbf{\Lambda}_\varepsilon \in SO(3)$. The tangent operator of a generic function $F = F(\mathbf{\Lambda})$ at the base point $\mathbf{\Lambda}$, $\boldsymbol{\delta F}(\mathbf{\Lambda})$, can be constructed in two different ways. They depend on the structure of $\mathbf{\Lambda}_\varepsilon$, which is strictly related with the choice to perform the linearization directly on the manifold, or indirectly into the linear space of rotation vectors. Therefore these two ways or forms of linearization are called

- **Direct linearization form**, where $\mathbf{\Lambda}_\varepsilon$ is constructed via an infinitesimal variation of the manifold element $\mathbf{\Lambda}$, on the rotation manifold $SO(3)$, i.e. where the linearization is done directly on the *manifold*;
- **Indirect linearization form**, where $\mathbf{\Lambda}_\varepsilon$ is constructed via an infinitesimal variation of the rotational vector space element $\mathbf{\Psi}$, into the rotational vector linear space, i.e. the linearization is done on the *linear space* of vectors which parametrize the rotation tensor.

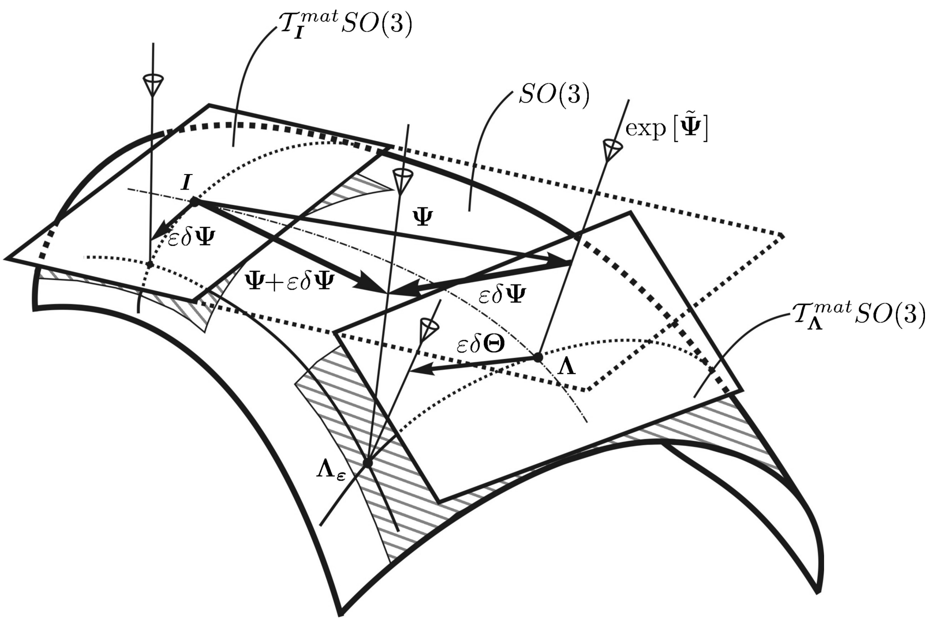


Figure 3.11: Geometrical interpretation of SO(3) manifold, its tangent spaces in material description and finite rotation decomposition.

3.3.3.1 Direct linearization

Following standard usage for the direct linearization, the perturbed rotation tensor $\mathbf{\Lambda}_\varepsilon$ can be defined in two ways, according to which description is adopted

$$\text{Material description:} \quad \mathbf{\Lambda}_\varepsilon \triangleq \mathbf{\Lambda}\exp[\varepsilon\boldsymbol{\delta}\tilde{\mathbf{\Theta}}] \tag{3.51}$$

$$\text{with} \quad \varepsilon\boldsymbol{\delta}\tilde{\mathbf{\Theta}} \in \mathcal{T}_{\mathbf{\Lambda}}^{mat}SO(3)$$

$$\text{Spatial description:} \quad \mathbf{\Lambda}_\varepsilon \triangleq \exp[\varepsilon\boldsymbol{\delta}\tilde{\boldsymbol{\theta}}]\mathbf{\Lambda} \tag{3.52}$$

$$\text{with} \quad \varepsilon\boldsymbol{\delta}\tilde{\boldsymbol{\theta}} \in \mathcal{T}_{\mathbf{\Lambda}}^{spat}SO(3)$$

Note that these perturbed rotations respect the definition of the direct linearization form. It is also an admissible variation because $\lim_{\varepsilon\to 0}\boldsymbol{\Lambda}_\varepsilon = \boldsymbol{\Lambda}$ holds.

Material form. Equation (3.51) provides the quantity $\boldsymbol{\Lambda}_\varepsilon$ used in the material form of direct linearization. The order of product matrices into the expression of the perturbed rotation indicates that the new infinitesimal rotation $\exp[\varepsilon\delta\tilde{\boldsymbol{\Theta}}]$ precedes in the sequence of rotations the actual one $\boldsymbol{\Lambda}$. The incremental rotation vector $\delta\boldsymbol{\Theta}$ could be seen as a rotation applied to the material frame $\{\boldsymbol{E}_i\}$, and for this reason the linearization form arisen is called material. This kind of rotation sequence, where the rotation vectors do not suffer of previous rotations, is called a compound rotations around *fixed axes* [7] (see [1]).

The direct material variation of the rotation tensor $\delta\boldsymbol{\Lambda}$ is computed as stated in equation (2.2) by means of Gateaux derivative by making use of exponential mapping (see equation (2.57)) and its series expansion, obtaining

$$
\begin{aligned}
\delta\boldsymbol{\Lambda} &= \lim_{\varepsilon\to 0}\frac{\boldsymbol{\Lambda}_\varepsilon - \boldsymbol{\Lambda}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{\boldsymbol{\Lambda}\exp(\varepsilon\delta\tilde{\boldsymbol{\Theta}}) - \boldsymbol{\Lambda}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{\boldsymbol{\Lambda}[\exp(\varepsilon\delta\tilde{\boldsymbol{\Theta}}) - \boldsymbol{I}]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{\boldsymbol{\Lambda}(\tilde{\boldsymbol{I}} + \varepsilon\delta\tilde{\boldsymbol{\Theta}} + \varepsilon^2\delta\tilde{\boldsymbol{\Theta}}^2/2 + \varepsilon^3\delta\tilde{\boldsymbol{\Theta}}^3/3! + \ldots - \boldsymbol{I})}{\varepsilon} \\
&= \boldsymbol{\Lambda}\,\delta\tilde{\boldsymbol{\Theta}}
\end{aligned} \tag{3.53}
$$

Finally, observe what we have anticipated: the direct linearization depends on two measures, $\boldsymbol{\Lambda}$ and $\delta\tilde{\boldsymbol{\Theta}}$ which do not belong to the same space.

Spatial form. Equation (3.52) provides the infinitesimal perturbed configuration $\boldsymbol{\Lambda}_\varepsilon$ used in the spatial form of direct linearization. The order of product matrices into the expression of the perturbed rotation indicates that the new infinitesimal rotation $\exp[\varepsilon\delta\tilde{\boldsymbol{\theta}}]$ is superimposed on the current rotation $\boldsymbol{\Lambda}$. The incremental rotation vector $\delta\boldsymbol{\theta}$ could be seen as a rotation applied to the spatial frame $\{\boldsymbol{t}_i\}$, and for this reason the linearization form arisen is called spatial. This kind of rotation sequence, where each rotation is affected by preceding rotation, is called a compound of rotations around *follower axes*, i.e. axes rigidly attached to the body itself and hence rotated by the previous rotations in the sequence (see [1]).
Argyris (see [1]) proved that a sequence of two rotations around follower axes is equal to the inverted sequence around fixed axes and viceversa (see equation (3.51) and (3.52)). This confirms that the spatial and material direct perturbed rotations $\boldsymbol{\Lambda}_\varepsilon$ are indeed equivalent.

The direct spatial variation of the rotation tensor $\delta\boldsymbol{\Lambda}$ is computed once again as stated in equation (2.2) by means of Gateaux derivative, by making use of the

[7] According to the enlighten paper [2], a fixed axes do not deviate from their original direction with respect to a fixed coordinate system in the course of a rotation.

exponential mapping (see equation (2.57)) and its series expansion, obtaining

$$
\begin{aligned}
\delta\boldsymbol{\Lambda} &= \lim_{\varepsilon\to 0}\frac{\boldsymbol{\Lambda}_\varepsilon - \boldsymbol{\Lambda}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{\exp(\varepsilon\delta\tilde{\boldsymbol{\theta}})\boldsymbol{\Lambda} - \boldsymbol{\Lambda}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{[\exp(\varepsilon\delta\tilde{\boldsymbol{\theta}}) - \boldsymbol{I}]\boldsymbol{\Lambda}}{\varepsilon} \\
&= \lim_{\varepsilon\to 0}\frac{(\tilde{\boldsymbol{I}} + \varepsilon\delta\tilde{\boldsymbol{\theta}} + \varepsilon^2\delta\tilde{\boldsymbol{\theta}}^2/2 + \varepsilon^3\delta\tilde{\boldsymbol{\theta}}^3/3! + \ldots - \boldsymbol{I})\boldsymbol{\Lambda}}{\varepsilon} \\
&= \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda}
\end{aligned}
\tag{3.54}
$$

where $\delta\boldsymbol{\Lambda}$ can be interpreted as a perturbation of $\boldsymbol{\Lambda}$ by an infinitesimal rotation $\delta\tilde{\boldsymbol{\theta}}$. Here we note again that the direct linearization depends on two measures, $\boldsymbol{\Lambda}$ and $\delta\tilde{\boldsymbol{\theta}}$ which do not belong to the same space. Indeed, it is an element of the tangent space of $SO(3)$ at point $\boldsymbol{\Lambda}$, or in other words, it is a tensor of infinitesimal rotation superposed on the existing rotation $\boldsymbol{\Lambda}$.
Finally, the variation operator $\delta\boldsymbol{\Lambda}^T$ can be obtained from $\delta\boldsymbol{\Lambda}$ since $\delta\boldsymbol{\Lambda}^T = (\delta\boldsymbol{\Lambda})^T$, hence just changing sign to the skew-symmetric tensor such as

$$
\begin{aligned}
(\delta\boldsymbol{\Lambda})^T &= (\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda})^T \\
&= \boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}^T \\
&= -\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}
\end{aligned}
\tag{3.55}
$$

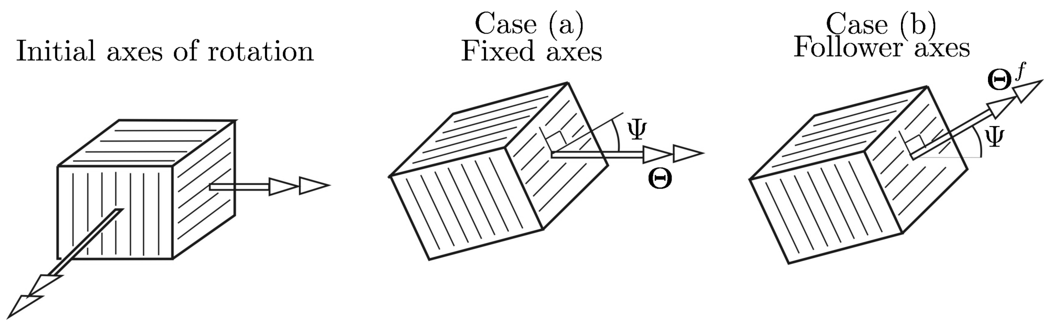


Figure 3.12: Rotation vector for fixed and follower axes: a pair of rotations are supposed to be applied to the body with the order $\boldsymbol{\Psi}$, $\boldsymbol{\Theta}$ forming a compound rotation. The second rotation vector applies to a fixed axis independent on the previous rotation (case (a)), whereas in case (b) the rotation vector $\boldsymbol{\Theta}^f$ applies to the follower axis.

3.3.3.2 Indirect (Lagrangian) linearization

Within the framework of indirect linearization performed with reference to the linear space, the quantity $\boldsymbol{\Lambda}_\varepsilon$ is defined as

$$
\boldsymbol{\Lambda}_\varepsilon = \exp[\tilde{\boldsymbol{\Psi}} + \varepsilon\delta\tilde{\boldsymbol{\Psi}}] \tag{3.56}
$$
$$
\text{with} \qquad \tilde{\boldsymbol{\Psi}}, \delta\tilde{\boldsymbol{\Psi}} \in \mathcal{T}_{\boldsymbol{I}}SO(3)
$$

and represents an admissible infinitesimal perturbed configuration of $\mathbf{\Lambda}$ because respects the condition $\lim_{\varepsilon\to 0}\mathbf{\Lambda}_\varepsilon = \mathbf{\Lambda}$. In this case, the perturbed rotation is not anymore a rotation sequence, since the linearization is not carried out into the group of rotation $SO(3)$. On the contrary, it is carried out into the *linear space* which parametrizes the rotation, i.e. the space of rotation vectors (or equivalently, the space of skew-symmetric tensor associated to rotation vectors), via the chart which links $so(3)$ with $SO(3)$, that is the exponential mapping [8]. As the space of variation is linear, variations are carried out by means of usual *additive operations*, so the condition (3.56) turns into

$$\tilde{\mathbf{\Psi}}_\varepsilon = \tilde{\mathbf{\Psi}} + \varepsilon\delta\tilde{\mathbf{\Psi}} \qquad \text{where} \qquad \tilde{\mathbf{\Psi}}_\varepsilon \in SO(3) \tag{3.57}$$

where the condition $\lim_{\varepsilon\to 0}\tilde{\mathbf{\Psi}}_\varepsilon = \tilde{\mathbf{\Psi}}$ is ensured.

The indirect linearization of the rotation tensor $\delta\mathbf{\Lambda}$ is again calculated by the directional derivative

$$\begin{aligned}\delta\mathbf{\Lambda} &= \left.\frac{\mathrm{d}\mathbf{\Lambda}_\varepsilon}{\mathrm{d}\varepsilon}\right|_{\varepsilon=0} \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\exp[\tilde{\mathbf{\Psi}} + \varepsilon\delta\tilde{\mathbf{\Psi}}]\right|_{\varepsilon=0}\end{aligned} \tag{3.58}$$

where in this case we cannot use anymore the property of exponential map derivative, since it refers to the case $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\exp[\varepsilon\delta\tilde{\mathbf{\Psi}}]$. Instead we must substitute the perturbed rotation $\tilde{\mathbf{\Psi}}_\varepsilon$ into the explicit expression of the exponential map (2.47), and carry out derivation.

3.3.4 Linearization of angular velocity (spin)

Let recall the definition of spatial spin tensor $\tilde{\boldsymbol{\omega}} = \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T$ given in equation (2.108), and on its axial vector $\boldsymbol{\omega}$. Its direct spatial linearization can be computed taking the directional derivative of the infinitesimal perturbed spin tensor $\tilde{\boldsymbol{\omega}}_\varepsilon = \dot{\mathbf{\Lambda}}_\varepsilon\mathbf{\Lambda}_\varepsilon^T$, where $\mathbf{\Lambda}_\varepsilon$ is the spatial perturbed rotation tensor defined by (3.52), exactly how was done for linearization of rotation tensor.

Alternatively, with an only formal difference, we can base our calculation taking advantage of properties of variation operator (i.e. linearity and chain rule), and using the previous results of variation of rotation tensor. Hence, using the skew-symmetry condition (3.55), the orthogonality conditions (2.3) for rotation matrix, and the com-

[8]We recall that since $so(3)$ is isomorphic in $\mathbb{R}^3$, we can refer either to the skew-symmetric tensor or its axial vector.

mutativity property of variation operator [9], the variation of spin tensor is

$$\begin{aligned}
\delta\tilde{\boldsymbol{\omega}} &= \delta(\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T) \\
&= \delta\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T + \dot{\boldsymbol{\Lambda}}\,\delta\boldsymbol{\Lambda}^T \\
&= (\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda})^{\bullet}\boldsymbol{\Lambda}^T + \boldsymbol{\Lambda}'(-\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}) \\
&= (\delta\dot{\tilde{\boldsymbol{\theta}}}\boldsymbol{\Lambda} + \delta\tilde{\boldsymbol{\theta}}\dot{\boldsymbol{\Lambda}})\boldsymbol{\Lambda}^T - \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}} \\
&= \delta\dot{\tilde{\boldsymbol{\theta}}}\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T + \delta\tilde{\boldsymbol{\theta}}\dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T - \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}} \\
&= \delta\dot{\tilde{\boldsymbol{\theta}}} + \delta\tilde{\boldsymbol{\theta}}\,\tilde{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}\,\delta\tilde{\boldsymbol{\theta}}
\end{aligned} \tag{3.59}$$

Equation (3.59) can be rewritten in terms of axial vector. In fact, focusing on the last two terms, the parallel use of Lie algebra (2.14) and Lie brackets (2.20) to the generic skew-symmetric tensors $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{C}}$ and their relative axial vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, provides the following correspondences

$$\begin{aligned}
&\text{Lie algebra:} \quad \tilde{\boldsymbol{C}}\,\boldsymbol{h} = \boldsymbol{c} \times \boldsymbol{h} \\
&\text{Lie brackets: } (\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}})\boldsymbol{h} = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{h} \\
&\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}} \;\longleftrightarrow\; \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \tilde{\boldsymbol{A}}\,\boldsymbol{b}
\end{aligned} \tag{3.60}$$

such that

$$\tilde{\boldsymbol{C}} = \delta\tilde{\boldsymbol{\theta}}\,\tilde{\boldsymbol{\omega}} - \tilde{\boldsymbol{\omega}}\,\delta\tilde{\boldsymbol{\theta}} \;\longleftrightarrow\; \boldsymbol{c} = \delta\boldsymbol{\theta} \times \boldsymbol{\omega} = \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\omega} \tag{3.61}$$

and consequently the axial vector associated to tensor (3.59) results

$$\delta\boldsymbol{\omega} = \delta\dot{\boldsymbol{\theta}} + \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\omega} \tag{3.62}$$

Now, with this result in hand and with the help of (2.116), we can derive the material counterpart as

$$\begin{aligned}
\delta\boldsymbol{\Omega} &= \delta(\boldsymbol{\Lambda}^T\boldsymbol{\omega}) \\
&= \delta\boldsymbol{\Lambda}^T\boldsymbol{\omega} + \boldsymbol{\Lambda}^T\delta\boldsymbol{\omega} \\
&= (\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda})^T\boldsymbol{\omega} + \boldsymbol{\Lambda}^T(\delta\dot{\boldsymbol{\theta}} + \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\omega}) \\
&= -\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\omega} + \boldsymbol{\Lambda}^T\delta\dot{\boldsymbol{\theta}} + \boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\omega} \\
&= \boldsymbol{\Lambda}^T\delta\dot{\boldsymbol{\theta}}
\end{aligned} \tag{3.63}$$

Alternatively from equation (2.117), the variation of material angular velocity vector becomes

$$\begin{aligned}
\delta\boldsymbol{\Omega} &= \delta(\boldsymbol{T}^T\dot{\boldsymbol{\Psi}}) \\
&= \delta\boldsymbol{T}^T\dot{\boldsymbol{\Psi}} + \boldsymbol{T}^T\delta\dot{\boldsymbol{\Psi}}
\end{aligned} \tag{3.64}$$

[9] Although before was noted that variation and differentiation lead to different mathematical results, however we point out how variation symbol δ and differential symbol d swap. In other words one can proof that $\mathrm{d}/\mathrm{d}x\,\delta y(x) = \delta\,\mathrm{d}y(x)/\mathrm{d}x$. Even though this constitutes a common assumption in continuum mechanics, however, it implies that all the considered restrictions are *holonomic*. More details can be consulted in [36].

An interesting analogy between spin vector and its variation. By reversing equation (3.54) we obtain

$$\boldsymbol{\delta\Lambda} = \boldsymbol{\delta\tilde{\theta}\Lambda} \quad \longrightarrow \quad \boldsymbol{\delta\tilde{\theta}} = \boldsymbol{\delta\Lambda\Lambda}^T \tag{3.65}$$

which is exactly the same expression one can obtain formally linearizing the orthogonality condition $\boldsymbol{\Lambda\Lambda}^T = \boldsymbol{I}$. We recall that also the definition of the angular velocity tensor

$$\tilde{\boldsymbol{\omega}} = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T \tag{3.66}$$

has been obtained formally time differentiating the same orthogonality condition with respect to the arbitrary parameter t from which $\boldsymbol{\Lambda}$ depends. Setting an analogy between the operation of linearization and of time derivation, we can see the perfect analogy between the spatial spin tensor $\tilde{\boldsymbol{\omega}}$ and the spatial variation $\boldsymbol{\delta\tilde{\theta}}$. Thereby, the angular velocity vector given by (2.118)

$$\boldsymbol{\omega} = \boldsymbol{T}\,\dot{\boldsymbol{\psi}} \tag{3.67}$$

because of the analogy just introduced, we can heuristically state that the same relation holds between the variation of rotation vector $\boldsymbol{\delta\theta}$ and the variation of total rotation vector $\boldsymbol{\delta\psi}$, i.e.

$$\boldsymbol{\delta\theta} = \boldsymbol{T}\,\boldsymbol{\delta\psi} \tag{3.68}$$

3.3.5 Linearization of shear strain (translational strain)

The linearization of translational strain parameter can be obtained basing on the variation of rotation tensor just introduced, and applying the chain rule for partial derivatives.
Let recall the strain measures given in table 3.1. Considering $\boldsymbol{\gamma} = \boldsymbol{x}_0' - \boldsymbol{t}_3$ and noticing equation (3.54) and equation (3.6), one has the usual variation operation

$$\begin{aligned}
\boldsymbol{\delta\gamma} &= \delta(\boldsymbol{x}_0' - \boldsymbol{t}_3) \\
&= \delta(\boldsymbol{x}_0' - \boldsymbol{\Lambda E}_3) \\
&= \boldsymbol{\delta x}_0' - \delta(\boldsymbol{\Lambda E}_3) \\
&= \boldsymbol{\delta x}_0' - \boldsymbol{\delta\tilde{\theta}\Lambda E}_3 \\
&= \boldsymbol{\delta x}_0' - \boldsymbol{\delta\tilde{\theta} t}_3
\end{aligned} \tag{3.69}$$

and finally making use of the first equation (3.23) which relates the shear strain in spatial and material form, the skew-symmetry condition (3.55), and displacement variation (3.48), one get

$$
\begin{aligned}
\delta\boldsymbol{\Gamma} &= \delta(\boldsymbol{\Lambda}^T\boldsymbol{\gamma}) \\
&= \delta\boldsymbol{\Lambda}^T\boldsymbol{\gamma} + \boldsymbol{\Lambda}^T\delta\boldsymbol{\gamma} \\
&= -\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\gamma} + \boldsymbol{\Lambda}^T[\delta\boldsymbol{x}_0' - \delta\tilde{\boldsymbol{\theta}}\,\boldsymbol{t}_3] \\
&= \boldsymbol{\Lambda}^T[-\delta\tilde{\boldsymbol{\theta}}(\boldsymbol{x}_0' - \boldsymbol{t}_3) + \delta\boldsymbol{x}_0' - \delta\tilde{\boldsymbol{\theta}}\,\boldsymbol{t}_3] \\
&= \boldsymbol{\Lambda}^T(\delta\boldsymbol{x}_0' - \delta\tilde{\boldsymbol{\theta}}\boldsymbol{x}_0') \\
&= \boldsymbol{\Lambda}^T[(\delta\boldsymbol{x}_0)' - \delta\tilde{\boldsymbol{\theta}}\boldsymbol{x}_0'] \\
&= \boldsymbol{\Lambda}^T[(\delta\boldsymbol{u})' - \delta\tilde{\boldsymbol{\theta}}\boldsymbol{x}_0'] \\
&= \boldsymbol{\Lambda}^T[(\delta\boldsymbol{u})' + \boldsymbol{x}_0' \times \delta\boldsymbol{\theta}]
\end{aligned}
\tag{3.70}
$$

3.3.6 Linearization of curvature

The linearization of rotational strain parameter can be obtained basing once again on the variation of rotation tensor, and applying the chain rule for partial derivatives. Let recall the definition of curvature tensor $\tilde{\boldsymbol{\kappa}} = \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T$ given in equation (2.134). Using the skew-symmetry condition (3.55), the orthogonality conditions (2.3) for rotation matrix, and the commutativity property of variation operator, the variation of curvature tensor is

$$
\begin{aligned}
\delta\tilde{\boldsymbol{K}} &= \delta(\boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T) \\
&= \delta\boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T + \boldsymbol{\Lambda}'\,\delta\boldsymbol{\Lambda}^T \\
&= (\delta\boldsymbol{\Lambda})'\boldsymbol{\Lambda}^T + \boldsymbol{\Lambda}'(-\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}) \\
&= (\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda})'\boldsymbol{\Lambda}^T - \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}} \\
&= (\delta\tilde{\boldsymbol{\theta}}'\boldsymbol{\Lambda} + \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda}')\boldsymbol{\Lambda}^T - \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}} \\
&= \delta\tilde{\boldsymbol{\theta}}'\boldsymbol{\Lambda}\boldsymbol{\Lambda}^T + \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T - \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}} \\
&= \delta\tilde{\boldsymbol{\theta}}' + \delta\tilde{\boldsymbol{\theta}}\,\tilde{\boldsymbol{\kappa}} - \tilde{\boldsymbol{\kappa}}\,\delta\tilde{\boldsymbol{\theta}}
\end{aligned}
\tag{3.71}
$$

where focusing on the last two terms, the parallel use of Lie algebra (2.14) and Lie brackets (2.20) to the generic skew-symmetric tensors $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{C}}$ and their relative axial vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$, provides the following correspondences

$$
\begin{aligned}
&\text{Lie algebra:} \quad \tilde{\boldsymbol{C}}\,\boldsymbol{h} = \boldsymbol{c} \times \boldsymbol{h} \\
&\text{Lie brackets:} \ (\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}})\boldsymbol{h} = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{h} \\
&\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}} \ \longleftrightarrow \ \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \tilde{\boldsymbol{A}}\,\boldsymbol{b}
\end{aligned}
\tag{3.72}
$$

such that

$$
\tilde{\boldsymbol{C}} = \delta\tilde{\boldsymbol{\theta}}\,\tilde{\boldsymbol{\kappa}} - \tilde{\boldsymbol{\kappa}}\,\delta\tilde{\boldsymbol{\theta}} \ \longleftrightarrow \ \boldsymbol{c} = \delta\boldsymbol{\theta} \times \boldsymbol{\kappa} = \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\kappa}
\tag{3.73}
$$

and consequently the axial vector associated to tensor (3.71) results

$$
\delta\boldsymbol{\kappa} = \delta\boldsymbol{\theta}' + \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\kappa}
\tag{3.74}
$$

Now, with this result in hand and with the help of the first of (3.24)

$$\begin{aligned}\delta\boldsymbol{K} &= \delta(\boldsymbol{\Lambda}^T\boldsymbol{\kappa})\\ &= \delta\boldsymbol{\Lambda}^T\boldsymbol{\kappa} + \boldsymbol{\Lambda}^T\delta\boldsymbol{\kappa}\\ &= (\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\Lambda})^T\boldsymbol{\kappa} + \boldsymbol{\Lambda}^T(\delta\boldsymbol{\theta}' + \delta\tilde{\boldsymbol{\theta}}\boldsymbol{\kappa})\\ &= -\boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\kappa} + \boldsymbol{\Lambda}^T\delta\boldsymbol{\theta}' + \boldsymbol{\Lambda}^T\delta\tilde{\boldsymbol{\theta}}\boldsymbol{\kappa}\\ &= \boldsymbol{\Lambda}^T\delta\boldsymbol{\theta}'\end{aligned} \tag{3.75}$$

Alternatively from equation (2.144), the variation of material curvature vector becomes

$$\begin{aligned}\delta\boldsymbol{K} &= \delta(\boldsymbol{T}^T\boldsymbol{\Psi}')\\ &= \delta\boldsymbol{T}^T\boldsymbol{\Psi}' + \boldsymbol{T}^T\delta\boldsymbol{\Psi}'\end{aligned} \tag{3.76}$$

3.3.7 Linearization of strain

Summarizing the above results in matrix form, we can compactly rewrite equations (3.69), (3.70), (3.74) and (3.75) respectively as (see [44])

$$\begin{bmatrix}\delta\boldsymbol{\Gamma}\\ \delta\boldsymbol{K}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\Lambda}^T\boldsymbol{I}\frac{\partial}{\partial X_3} & \boldsymbol{\Lambda}^T\tilde{\boldsymbol{x}}_0'\\ \boldsymbol{0} & \boldsymbol{\Lambda}^T\boldsymbol{I}\frac{\partial}{\partial X_3}\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{u}\\ \delta\boldsymbol{\theta}\end{bmatrix} \tag{3.77}$$

and

$$\begin{bmatrix}\delta\boldsymbol{\gamma}\\ \delta\boldsymbol{\kappa}\end{bmatrix} = \begin{bmatrix}\boldsymbol{I}\frac{\partial}{\partial X_3} & \tilde{\boldsymbol{t}}_3\\ \boldsymbol{0} & (\boldsymbol{I}\frac{\partial}{\partial X_3} - \tilde{\boldsymbol{\kappa}})\end{bmatrix}\begin{bmatrix}\delta\boldsymbol{u}\\ \delta\boldsymbol{\theta}\end{bmatrix} \tag{3.78}$$

Equation (3.77) could be recasted in a more compact form, and pointing out the dependence on total rotation vector $\boldsymbol{\Psi}$ by means of equation (2.98), one get

$$\delta\boldsymbol{\varepsilon}^r = \boldsymbol{S}\boldsymbol{\Delta}\delta\boldsymbol{\eta} \tag{3.79}$$

which is in perfect accordance with equation (3.40) having introduced the virtual generalized displacement vector $\delta\boldsymbol{\eta}$.

3.3.8 Linearization of deformation gradient

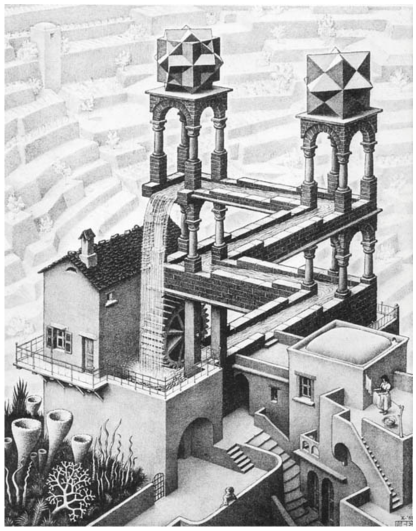
To write the forthcoming principle of virtual work using as internal work the first Piola-Kirchhoff stress tensor $\boldsymbol{P}$, we need to compute the work conjugate virtual deformation $\delta\boldsymbol{F}^r$. It can be obtained with the help of the aforementioned variations. With reference to (3.20) and making use of distributivity property of variation operator, the deformation gradient back-rotated assumes the form

$$\begin{aligned}\delta\boldsymbol{F}^r &= \delta[(\boldsymbol{\Gamma} + \boldsymbol{K}\times\boldsymbol{R})\otimes\boldsymbol{E}_3]\\ &= (\delta\boldsymbol{\Gamma} + \delta\boldsymbol{K}\times\boldsymbol{R})\otimes\boldsymbol{E}_3\end{aligned} \tag{3.80}$$

where the variation of fixed basis is assumed null for hypothesis.

Chapter 4

Geometrically exact beam theory: balance equations



We cannot command nature except by obeying her.

F. Bacon

This chapter contains a summary of the continuum mechanics background which is needed for the finite element formulation of solid mechanics and structural problems. The chapter opens with a brief recall to Cauchy's Theorem and then balance laws are introduced in spatial and material description. Finally the strong form of equilibrium equations is derived along with the equation of motion written in terms of resultants.

4.1 Balance equations

Within the classical continuum mechanics, three universal conservation principles should be satisfied in any motion. The principle of *conservation of mass* is identically fulfilled by an assignment of the mass density to the body itself and not to the volume occupied by this body in any configuration. Moving on, the fundamental relations which govern global balances of linear and angular momenta lead to the known local Cauchy equations of motion.

4.1.1 Cauchy's Theorem

Consider a general deformable continuum body $\mathscr{B}$ occupying currently, at time t, an arbitrary region Ω of physical space with boundary surface $\partial\Omega$ and, at time $t = 0$, the region Ω_0 with boundary surface $\partial\Omega_0$, as shown in Figure 4.1. We refer to Ω as *current* (*spatial*) configuration and to Ω_0 as *reference* (*material*) configuration.

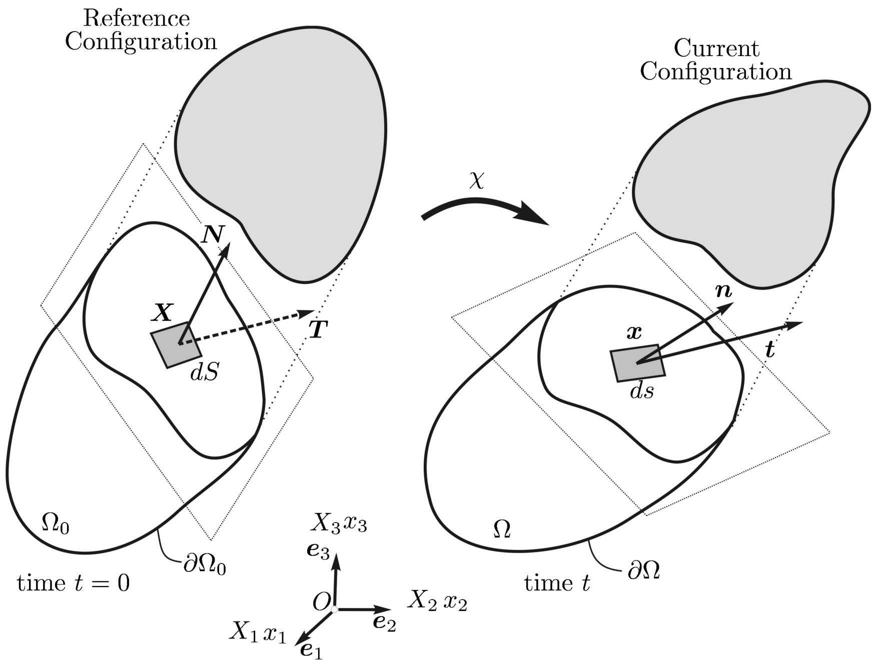


Figure 4.1: An interpretation of the traction vectors acting on infinitesimal surface elements with outward normals. Note that $\mathbf{t_N} \in \boldsymbol{\Omega}$, although it is drawn in the material placement.

The position of body material points with respect to a fixed reference system are denoted by the vector field $\boldsymbol{x}$ in the current configuration Ω and by the vector field $\boldsymbol{X}$ in the reference configuration Ω_0. In the framework of Newtonian mechanics, we postulate that arbitrary forces act on parts or the whole of the boundary surface $\partial\Omega$ (called external forces), and on an (imaginary) surface within the interior of the body (called internal forces) in some distribute manner.

Let the body now be cut by a plane surface which passes any given point $P \equiv \boldsymbol{x} \subset \Omega$ at time t. As illustrated in Figure 4.1, the plane surface separates the deformable body into two portions. Since we consider interaction of the two portions, forces and couples are transmitted across the internal plane surface. We denote an infinitesimal resultant (actual) force acting on a surface element ΔS as $\mathrm{d}\mathbf{f}$, and infinitesimal resultant couple $\mathrm{d}\mathbf{m}$, respectively

$$\boldsymbol{t_{P,n}} = \mathrm{d}\mathbf{f} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{f}}{\Delta S} \tag{4.1}$$

$$\boldsymbol{m}_{\boldsymbol{P},\boldsymbol{n}} = \mathtt{dm} = \lim_{\Delta S \to 0} \frac{\Delta \mathbf{m}}{\Delta S} \triangleq 0 \tag{4.2}$$

For sake of simplicity, we omit distributed resultant couples, not occurring in the classical formulation of continuum mechanics. Consequently equation (4.2) is identically null. Consider the actual infinitesimal force, $\mathrm{d}\mathbf{f}$, acting in the current configuration Ω on an infinitesimal surface plane element, $\mathrm{d}s$, internal to the body with normal unit vector $\mathbf{n}$, and located at point $\boldsymbol{x}$, see Figure 4.1. When considered in the reference configuration Ω_0, the spatial objects $\mathrm{d}s$, $\mathbf{n}$ and $\boldsymbol{x}$ are denoted respectively by $\mathrm{d}S$, $\mathbf{N}$ and $\boldsymbol{X}$. According to Figure 4.1, we claim that for every surface element

$$\mathrm{d}\mathbf{f} = \mathbf{t_n}\,\mathrm{d}s = \mathbf{t_N}\,\mathrm{d}S \tag{4.3}$$

where the Cauchy's stress theorem, with a slight abuse in notation

$$\mathbf{t_n}(\boldsymbol{x}, \mathbf{n}, t) = \boldsymbol{\sigma}(\boldsymbol{x}, t) \cdot \mathbf{n} \tag{4.4}$$

$$\mathbf{t_N}(\boldsymbol{X}, \mathbf{N}, t) = \mathbf{P}(\boldsymbol{X}, t) \cdot \mathbf{N} \tag{4.5}$$

with $\boldsymbol{\sigma}$ denoting a symmetric spatial tensor field called the *Cauchy stress tensor*, and $\mathbf{P}$ characterizing a tensor field called the *first Piola-Kirchhoff stress tensor*. It is worth noting that $\boldsymbol{\sigma}$ is defined in the current configuration and therefore it is also called *true stress* since it is the physical stress of the true-current configuration. Whereas $\mathbf{P}$ is a two-point tensor since maps a vector defined in the reference configuration into a vector defined in the current configuration. In addition, one can show that the former linearly maps the current unit area vector $\mathbf{n}\,\mathrm{d}s$ into the current infinitesimal force $\mathrm{d}\mathbf{f}$, while the latter linearly maps the reference unit area vector $\mathbf{N}\,\mathrm{d}S$ into the current infinitesimal force $\mathrm{d}\mathbf{f}$. Alternatively, one can say that Cauchy's theorem relates the stress vector $\mathbf{t_n}$ to the surface normal $\mathbf{n}$ via the linear mapping $\boldsymbol{\sigma}$ in spatial description, whereas relates the traction vector $\mathbf{t_N}$ to the surface normal $\mathbf{N}$ via the linear mapping $\mathbf{P}$ in material description. Finally, substituting the Cauchy's theorem into equation (4.3) we get

$$\mathrm{d}\mathbf{f} = \boldsymbol{\sigma}(\boldsymbol{x}, t) \cdot \mathbf{n}\,\mathrm{d}s = \mathbf{P}(\boldsymbol{X}, t) \cdot \mathbf{N}\,\mathrm{d}S \tag{4.6}$$

4.1.2 Weak (*integral*) form of equilibrium equations

In this section we present translational and rotational integral equilibrium equations for the three-dimensional continuum body $\mathscr{B}$ in the dynamic regime. These equations are directly derived from momentum balance principles (linear and angular), which are valid for the whole or arbitrary parts of a continuum body $\mathscr{B}$. These equations are clearly valued on the real current configuration Ω and involve $\mathrm{d}\mathbf{f}$, but since the differential infinitesimal current force $\mathrm{d}\mathbf{f}$ can be expressed in term of both the *true* Cauchy's tensor $\boldsymbol{\sigma}$ and the *two-point* first Piola-Kirchhoff tensor $\mathbf{P}$, they can be formulated in two ways, one for each stress measure adopted.

4.1.2.1 Balance of linear and angular momentum

The momentum principle for a collection of particles states that the time rate of change of the total momentum of a given set of particles equals the vector sum of all the *external forces* acting on the particles of the set, provided Newton's Third Law of action and reaction governs the internal forces. The continuum form of this principle is a basic *postulate* of continuum mechanics.

Spatial description. Consider a continuum body $\mathscr{B}$ with a set of particles occupying an arbitrary region Ω bounded by surface $\partial\Omega$ at time t. We consider a region $\mathcal{P}$, subset of the reference region Ω with boundary surface $\partial\mathcal{P}$. In addition, assume a closed system with a given motion $\boldsymbol{x} = \varphi(\boldsymbol{X}, t)$, spatial mass density $\rho = \rho(\boldsymbol{x}, t)$, and spatial velocity field $\mathbf{v} = \mathbf{v}(\boldsymbol{x}, t)$. We define the total linear momentum $\mathbf{Q}$ (translational momentum) by the vector-valued function

$$\boldsymbol{Q}(t) = \int_{\mathcal{P}} \rho(\boldsymbol{x}, t)\, \boldsymbol{v}(\boldsymbol{x}, t)\, \mathrm{d}v = \int_{\mathcal{P}_0} \rho_0(\boldsymbol{X})\, \boldsymbol{V}(\boldsymbol{X}, t)\, \mathrm{d}V \tag{4.7}$$

and the total angular momentum $\mathbf{L}$ (also referred as rotational momentum) relative to a fixed point (characterized by the position vector $\boldsymbol{x}_0$) as

$$\boldsymbol{L}(t) = \int_{\mathcal{P}} \boldsymbol{r} \times \rho(\boldsymbol{x}, t)\, \boldsymbol{v}(\boldsymbol{x}, t)\, \mathrm{d}v = \int_{\mathcal{P}_0} \boldsymbol{r} \times \rho_0(\boldsymbol{X})\, \boldsymbol{V}(\boldsymbol{X}, t)\, \mathrm{d}V \tag{4.8}$$

where $\mathbf{r} = \boldsymbol{x} - \boldsymbol{x}_\mathbf{0}$ is the position vector, computed with respect to a generic momentum pole o localized by $\boldsymbol{x}_0$. Momentum equations (4.7) and (4.8) are formulated with respect to the current (spatial) and reference (material) configurations with associated quantities $\rho, \boldsymbol{v}, \mathrm{d}\boldsymbol{v}$ and $\rho_0, \boldsymbol{V}, \mathrm{d}\boldsymbol{x}$, respectively.

The rate of change of the linear and angular momentum (4.7) and (4.8) of the particles which fill an arbitrary region $\mathcal{P}$ results in fundamental axiom called momentum balance principles of a continuum body. In particular the postulate of balance of linear momentum state

$$\frac{\mathrm{d}\mathbf{Q}}{dt} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}} \rho\, \mathbf{v}\, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}_0} \rho_0\, \mathbf{V}\, \mathrm{d}V = \mathbf{f}(t) \tag{4.9}$$

and the balance of angular momentum (or balance of rotational momentum) holds

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}} \mathbf{r} \times \rho\, \mathbf{v}\, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathcal{P}_0} \mathbf{r} \times \rho_0\, \mathbf{V}\, \mathrm{d}V = \mathbf{m}(t) \tag{4.10}$$

which are given in both the spatial and material description.

In relation to (4.9) and (4.10), $\mathbf{f}(t)$ and $\mathbf{m}(t)$ characterize respectively the resultant force and the resultant moment, i.e. the moment of $\mathbf{f}$ about $\boldsymbol{x}_\mathbf{0}$. The momentum balance principles are generalizations of Newton's first and second principle of motion to the context of continuum mechanics, as introduced by Cauchy and Euler. In fact, the classical Euler laws of motion assert that the total external force $\mathbf{f}$ and the total external moment $\mathbf{m}$ respectively equal the time rate of change in the total linear and angular momentum in the system. If the external sources vanish, linear and angular momentum of the body are said to be conserved.

By virtue of differentiation under the integral sign rule, we may rewrite the balance principles (4.9) and (4.10) as

$$\dot{\mathbf{Q}}(t) = \int_{\mathcal{P}} \rho\, \dot{\mathbf{v}}\, \mathrm{d}v = \int_{\mathcal{P}_0} \rho_0\, \dot{\mathbf{V}}\, \mathrm{d}V = \mathbf{f}(t) \tag{4.11}$$

$$\dot{\mathbf{L}}(t) = \int_{\mathcal{P}} \mathbf{r} \times \rho\, \dot{\mathbf{v}}\, \mathrm{d}v = \int_{\mathcal{P}_0} \mathbf{r} \times \rho_0\, \dot{\mathbf{V}}\, \mathrm{d}V = \mathbf{m}(t) \tag{4.12}$$

In the following we define the structure of forces acting on a continuum body. Consider a boundary surface $\partial\mathcal{P}$ of a generical part $\mathcal{P}$ of the current region Ω, which is subjected to the oriented Cauchy traction vector[1] $\mathbf{t} = \mathbf{t}(\boldsymbol{x}, \mathbf{n}, t)$, where the unit vector $\mathbf{n}$ is the outward normal to an infinitesimal surface element $\mathrm{d}s$ of $\partial\mathcal{P}$. In addition, let $\mathbf{b} = \mathbf{b}(\boldsymbol{x}, t)$ denote a spatial vector field called the *body force*[2], defined per unit current volume of region $\mathcal{P}$ acting on a particle, as illustrated in Figure 4.2. A body force is, for instance, self-weight or gravity loading per unit volume, i.e. $\mathbf{b} = \rho\mathbf{g}$ with the spatial mass density ρ and the gravitational acceleration $\mathbf{g}$.

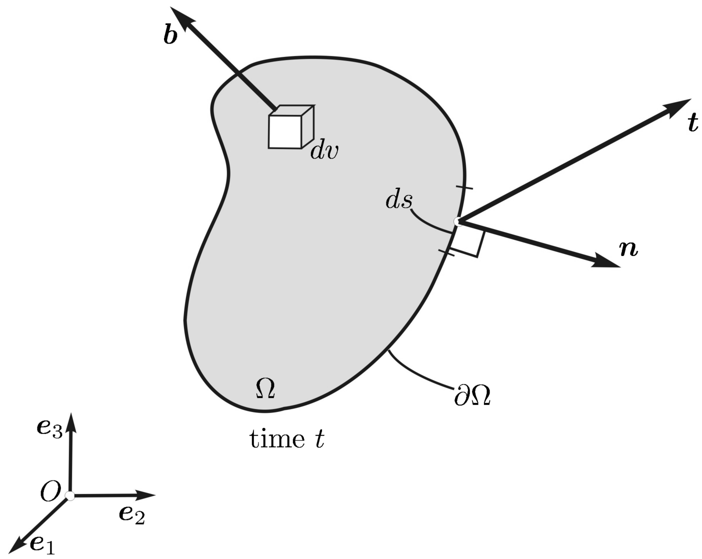


Figure 4.2: Structure of forces acting on the current configuration.

Hence, the resultant force $\mathbf{f}$ and the resultant moment $\mathbf{m}$ about a point $\boldsymbol{x}_0$ on the body in the current configuration have the additive forms

$$\mathbf{f}(t) = \int_{\mathcal{P}} \mathbf{b}\, \mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{t}_{\mathbf{n}}\, \mathrm{d}s \tag{4.13}$$

$$\mathbf{m}(t) = \int_{\mathcal{P}} \mathbf{r} \times \mathbf{b}\, \mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{r} \times \mathbf{t}_{\mathbf{n}}\, \mathrm{d}s \tag{4.14}$$

where

- $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{x})$ is the vector field of body force per unit of *current* volume;
- $\mathbf{r} = \boldsymbol{x} - \boldsymbol{x}_0$ is the position vector computed with respect to a generic momentum pole o localized by $\boldsymbol{x}_0$;
- $\mathbf{t}_{\mathbf{n}}$ is the traction vector introduced in the previous section (4.4);
- $dv \subset \mathcal{P}$ is the current infinitesimal volume;

[1] Force measured per unit current surface area of $\partial\mathcal{P}$.
[2] Note that the symbol $\mathbf{b}$ should not be confused with the left Cauchy-Green strain tensor.

- $ds \subset \partial\mathcal{P}$ is the current infinitesimal area.

Finally, by virtue of (4.9) and (4.10) the global forms of balance of linear momentum and balance of angular momentum may be given in the spatial description as

$$\begin{aligned}\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \rho\,\mathbf{v}\,\mathrm{d}v \\ &= \int_{\mathcal{P}} \mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{t_n}\,\mathrm{d}s \qquad \forall\,\mathcal{P}\subset\Omega\end{aligned} \tag{4.15}$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}} \mathbf{r}\times\rho\,\mathbf{v}\,\mathrm{d}v \\ &= \int_{\mathcal{P}} \mathbf{r}\times\mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{r}\times\mathbf{t_n}\,\mathrm{d}s \qquad \forall\,\mathcal{P}\subset\Omega\end{aligned} \tag{4.16}$$

where for the balance of angular momentum we have assumed the restriction that distributed resultant couples are neglected. The dynamic equilibrium axiom postulates that a deformable body is in equilibrium if and only if the external force resultant and the external momentum resultant equalize the rate of change of the linear and angular momentum, respectively, on each portion $\mathcal{P}$ of the body. The specialization of linear and angular momentum balance laws to the static regime are known respectively as *translational equilibrium* and *rotational equilibrium* equations.

Material description. For solid bodies, it is sometimes more convenient to work with the material description. In order to express the momentum balance principles in terms of material coordinates, we introduce the pseudo-body force called the reference body force $\mathbf{B} = \mathbf{B}(\boldsymbol{X}, t)$, which denotes the *spatial* body force parametrized in the material configuration. Hence, it acts on the region $\mathcal{P}_0$, subset of the reference region Ω_0 with boundary surface $\partial\mathcal{P}_0$. From the relation between the current and reference volume[3], $\mathrm{d}v = J\cdot\mathrm{d}V$, the transformation of the body force in reference form becomes

$$\int_{\mathcal{P}} \mathbf{b}(\boldsymbol{x}, t)\,\mathrm{d}v = \int_{\mathcal{P}_0} \mathbf{B}(\boldsymbol{X}, t)\,\mathrm{d}V \tag{4.17}$$

where $\mathbf{B} = J\mathbf{b}$ are called reference body forces and $J = \det\mathbf{F}$. From the equivalence of infinitesimal force $\mathrm{d}\mathbf{f}$ (4.3), the surface integral of the traction forces can be given as

$$\int_{\partial\mathcal{P}} \mathbf{t_n}(\boldsymbol{x})\,\mathrm{d}s = \int_{\partial\mathcal{P}_0} \mathbf{t_N}(\boldsymbol{X})\,\mathrm{d}S \tag{4.18}$$

Using the first Piola-Kirchhoff traction vector $\mathbf{t_N} = \mathbf{t}(\boldsymbol{X}, \mathbf{N}, t)$ introduced in (4.5), in analogy with (4.15) and (4.16), the global forms of balance of linear and angular momentum in the material description respectively are

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}_0} \rho_0\,\mathbf{V}\,\mathrm{d}V = \int_{\mathcal{P}_0} \mathbf{B}\,\mathrm{d}V + \int_{\partial\mathcal{P}_0} \mathbf{t_N}\,\mathrm{d}S \tag{4.19}$$

[3] See [28] page 74

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}_0} \mathbf{r}\times\rho_0\,\mathbf{V}\,\mathrm{d}V = \int_{\mathcal{P}_0} \mathbf{r}\times\mathbf{B}\,\mathrm{d}V + \int_{\partial\mathcal{P}_0} \mathbf{r}\times\mathbf{t}_{\mathbf{N}}\,\mathrm{d}S \tag{4.20}$$

and finally rearranging by recalling (4.9) and (4.10) we get

$$\mathbf{f}(t) = \int_{\mathcal{P}_0} \mathbf{B}\,\mathrm{d}V + \int_{\partial\mathcal{P}_0} \mathbf{t}_{\mathbf{N}}\,\mathrm{d}S \qquad \forall\,\mathcal{P}_0\subset\Omega_0 \tag{4.21}$$

$$\mathbf{m}(t) = \int_{\mathcal{P}_0} \mathbf{r}\times\mathbf{B}\,\mathrm{d}V + \int_{\partial\mathcal{P}_0} \mathbf{r}\times\mathbf{t}_{\mathbf{N}}\,\mathrm{d}S \qquad \forall\,\mathcal{P}_0\subset\Omega_0 \tag{4.22}$$

It must be emphasized that force and moment resultants, even though described with respect to the reference coordinate system, are still valued for the current configuration. Also note that the position vector $\mathbf{r}$ has not been affected by any transformation.

4.1.3 Strong (*differential*) form of equilibrium equations

In this section we derive the differential form of equations of motion describing the dynamic equilibrium of elastic body. These equations are directly derived specializing the momentum balance principles to the local (infinitesimal) point.

4.1.3.1 Equation of motion

Spatial description. Here we want to derive the equation of motion in spatial description referring to the *true* Cauchy's stress $\boldsymbol{\sigma}$. By computing the integral form of Cauchy's stress theorem (4.4) and by applying the divergence theorem, the surface integral in the translational balance is converted into a volume integral, we get

$$\int_{\partial\mathcal{P}} \mathbf{t}(\mathbf{x},t,\mathbf{n})\,\mathrm{d}s = \int_{\partial\mathcal{P}} \boldsymbol{\sigma}(\mathbf{x},t)\cdot\mathbf{n}\,\mathrm{d}s = \int_{\mathcal{P}} \mathrm{div}\boldsymbol{\sigma}(\mathbf{x},t)\,\mathrm{d}v \tag{4.23}$$

where $\boldsymbol{\sigma}$ is the symmetric Cauchy stress tensor and the operator $\mathrm{div}(\cdot)$ is the spatial divergence operator. Knowing that the spatial velocity field $\mathbf{v}$ may be expressed as the time rate of change of the displacement field $\mathbf{u}$, by substituting this result into the balance linear momentum (4.15), and using (4.11) we get the Cauchy's first equation of motion as follow

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{P}}\rho\,\mathbf{v}\,\mathrm{d}v = \int_{\mathcal{P}}\rho\,\dot{\mathbf{v}}\mathrm{d}v &= \int_{\mathcal{P}}\rho\ddot{\mathbf{u}}\,\mathrm{d}v\\ &= \int_{\mathcal{P}}\mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}}\mathbf{t}\,\mathrm{d}s\\ &= \int_{\mathcal{P}}\mathbf{b}\,\mathrm{d}v + \int_{\mathcal{P}}\mathrm{div}\boldsymbol{\sigma}\,\mathrm{d}v\\ &= \int_{\mathcal{P}}(\mathrm{div}\boldsymbol{\sigma}+\mathbf{b})\,\mathrm{d}v\end{aligned} \tag{4.24}$$

$$\int_{\mathcal{P}}(\mathrm{div}\boldsymbol{\sigma}+\mathbf{b}-\rho\,\ddot{\mathbf{u}})\,\mathrm{d}v = \mathbf{0} \qquad \forall\,\mathcal{P}\subset\Omega \tag{4.25}$$

where (4.25) is represented in the integral (*weak*) form.

Because this relation is supposed to hold for any part $\mathcal{P}$ of the region Ω, and since the integral function is continuous, we may deduce Cauchy's first equation in the local form

$$\boxed{\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\,\ddot{\mathbf{u}}} \tag{4.26}$$

for each point $\boldsymbol{x}$ of volume v and for all time instants t. Notice that the differential equation of motion is here presented with respect to the current (spatial) configuration, and governs the elastodynamic behavior of deformable body. Rearranging (4.26) in index notation we get

$$\begin{aligned}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_x y}{\partial y} + \frac{\partial\tau_x z}{\partial z} + b_x &= \rho\ddot{u}_x\\ \frac{\partial\tau_y x}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_y z}{\partial z} + b_y &= \rho\ddot{u}_y\\ \frac{\partial\tau_z x}{\partial x} + \frac{\partial\tau_z y}{\partial y} + \frac{\partial\sigma_z}{\partial z} + b_z &= \rho\ddot{u}_z\end{aligned} \tag{4.27}$$

Generally, relation (4.26) is nonlinear with respect to the displacement field $\boldsymbol{u}$. The nonlinearities are implicitly present due to *geometric* sources, i.e. the kinematics of motion of the body, and *material* sources, i.e. the material itself, since the Cauchy stress $\boldsymbol{\sigma}$ may, in general, depend on $\boldsymbol{u}$. In the present work, only geometrical nonlinearities are addressed, and thereby balance equations for the beam are established with reference to the *deformed configuration*.
Term $\rho\,\ddot{u}$ characterizes the inertial force per unit of current volume. If the acceleration, i.e. the second derivative of displacement $\boldsymbol{u}$, is assumed to be zero for all $\boldsymbol{x} \subset \Omega$, equation (4.26) becomes

$$\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0} \tag{4.28}$$

which is referred to as Cauchy's equation of equilibrium in purely static investigation (elastostatic).

From the rotational equilibrium (4.16), manipulations using the Cauchy's theorem and the divergence theorem shows that

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^T} \tag{4.29}$$

Material description. Using the integral form of Cauchy's stress theorem (4.5) and applying the divergence theorem, the surface integral is converted into a volume integral, we get

$$\int_{\partial\mathcal{P}_0} \mathbf{t_N}\,\mathrm{d}S = \int_{\partial\mathcal{P}_0} \boldsymbol{P}\cdot\mathbf{N}\,\mathrm{d}S = \int_{\mathcal{P}_0} \mathrm{DIV}\boldsymbol{P}\,\mathrm{d}V \tag{4.30}$$

where the operator DIV$(\cdot)$ is the material divergence operator. In analogy with (4.25), by substituting this result into the balance linear momentum in material description (4.19) we obtain

$$\int_{\mathcal{P}_0} (\mathrm{DIV}\boldsymbol{P} + \mathbf{B} - \rho_0\,\ddot{\mathbf{U}})\,\mathrm{d}V = \mathbf{0} \qquad \forall\,\mathcal{P}_0 \subset \Omega_0 \tag{4.31}$$

which, holding for any part $\mathcal{P}_0$ of the region Ω_0, yields the reference form of Cauchy's equation of equilibrium

$$\boxed{\mathrm{DIV}\boldsymbol{P} + \mathbf{B} = \rho_0\,\ddot{\mathbf{U}}} \tag{4.32}$$

From the rotational equilibrium (4.16), manipulations using the Cauchy's theorem and the divergence theorem, show that

$$\boxed{\boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T} \tag{4.33}$$

4.1.3.2 Equation of motion in terms of resultant

Spatial description. Consider a cross-section $s_t = \varphi(s,t)$ in the current configuration, and let[4] $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{P}(s,\boldsymbol{r})$ denote the first Piola-Kirchhoff stress tensor at the point detected by the cross-section position vector $\boldsymbol{r}$. By recalling (4.5) we may express the two-point tensor $\boldsymbol{P}(s,\boldsymbol{r})$ in terms of its column-vectors by

$$\begin{aligned}\boldsymbol{P}(s,\boldsymbol{r}) &\doteq \boldsymbol{T_1}(s,\boldsymbol{r})\otimes\boldsymbol{E_1} + \boldsymbol{T_2}(s,\boldsymbol{r})\otimes\boldsymbol{E_2} + \boldsymbol{T_3}(s,\boldsymbol{r})\otimes\boldsymbol{E_3}\\ &= \boldsymbol{T}_i\otimes\boldsymbol{E}_i\end{aligned} \tag{4.34}$$

where clearly, $\boldsymbol{T_3}(s,\boldsymbol{r}) = \boldsymbol{P}(s,\boldsymbol{r})\,\boldsymbol{E_3}$ represents the stress vector (per unit of reference area) acting on a point of the beam cross-section in the reference configuration. Since the cross-sections remains plane and undistorted, and being $\boldsymbol{E_3}$ the normal vector to the undeformed cross-section, we notice that $\boldsymbol{T_3}$ corresponds to the cross-section stress vector (arbitrarily orientated). The *stress resultant of distributed internal force* per unit of reference length $\boldsymbol{f}(s,t)$ over the cross-section a in the current coordinates is then given by integration over the cross-sectional reference area $a \subset \mathbb{R}^2$

$$\boldsymbol{f}(s,t) \doteq \int_a \boldsymbol{P}(s,\boldsymbol{r})\,\boldsymbol{E_3}\,\mathrm{d}\xi = \int_a \boldsymbol{T_3}(s,\boldsymbol{r})\,\mathrm{d}\xi \tag{4.35}$$

Similarly, the *stress resultant internal torque* per unit of reference arc-length $\boldsymbol{m}(s,t)$ over the cross-section s in the current configuration is given by

$$\boldsymbol{m}(s,t) \doteq \int_a (\boldsymbol{x} - \boldsymbol{x_0})\times\boldsymbol{T_3}(s,\boldsymbol{r})\,\mathrm{d}\xi \tag{4.36}$$

One may note the small difference in font of (4.35), (4.36) and (4.11), (4.12). The resultant quantities (4.35) and (4.36) represent the exact static one-dimensional equivalents of the three-dimensional problem.

Focusing to the nonlinear beam model, to develop equations of motion expressed in terms of the resultant force $\boldsymbol{f}(s,t)$ and the resultant momentum $\boldsymbol{m}(s,t)$, we proceed from the material form of the balance of linear momentum principle of the 3-dimensional theory, which recalling (4.32) may be expressed as

$$\mathrm{DIV}\,\boldsymbol{P} + \mathbf{B} = \rho_0\,\ddot{\mathbf{U}} \tag{4.37}$$

[4] See equation (3.8).

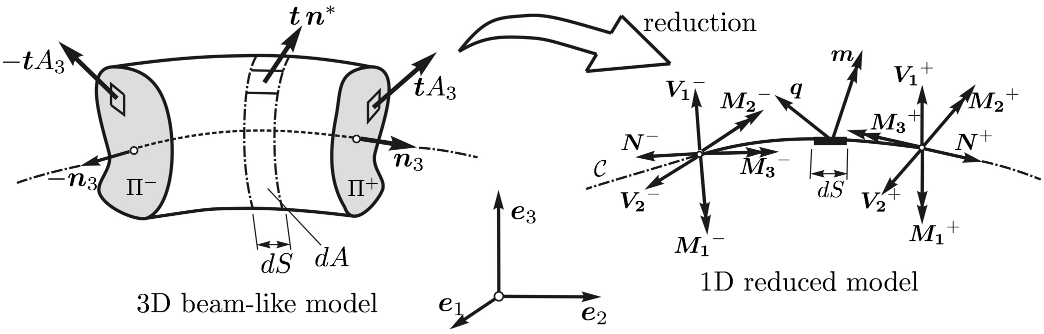


Figure 4.3: Reduction of the three-dimensional statics of the beam-like body to the one-dimensional 2-nodes resultant model.

where the divergence operator, in the present case, involves derivatives with respect to the coordinates of the undeformed configuration, because it follows from

$$\begin{aligned} \mathrm{DIV}\,\boldsymbol{P} &= \lim_{\Omega_0 \to 0} \frac{1}{\Omega_0} \int_{\partial\Omega_0} \mathbf{t_N}\,\mathrm{d}S \\ &= \lim_{\Omega_0 \to 0} \frac{1}{\Omega_0} \int_{\partial\Omega_0} \boldsymbol{P} \cdot \boldsymbol{N}\,\mathrm{d}S \end{aligned} \tag{4.38}$$

where $\partial\Omega_0$ is the boundary of the region Ω_0 and has unit normal vector field $\mathbf{N}$. Recalling (4.5), it is straightforward to demonstrate that the expression for the divergence of $\mathbf{P}$ in Cartesian coordinates is given by the formula

$$\begin{aligned} \mathrm{DIV}\boldsymbol{P} &= \sum_{i=1}^{3} \frac{\partial \boldsymbol{P}(x)}{\partial x_i} E_i \\ &= \sum_{i=1}^{3} \frac{\partial \boldsymbol{T}_i}{\partial x_i} \end{aligned} \tag{4.39}$$

and substituting this into (4.37) we get

$$\begin{aligned} \mathrm{DIV}\boldsymbol{P} + \mathbf{B} &= \rho_0\,\ddot{\mathbf{U}} \\ \sum_{i=1}^{3} \frac{\partial \boldsymbol{T}_i}{\partial x_i} + \mathbf{B} &= \rho_0\,\ddot{\mathbf{U}} \\ \left[\frac{\partial \boldsymbol{T_1}}{\partial x_1} + \frac{\partial \boldsymbol{T_2}}{\partial x_2} + \frac{\partial \boldsymbol{T_3}}{\partial x_3}\right] + \mathbf{B} &= \rho_0\,\ddot{\mathbf{U}} \end{aligned} \tag{4.40}$$

and rearranging

$$\frac{\partial \boldsymbol{T_3}}{\partial x_3} = \rho_0\,\ddot{\mathbf{U}} - \left[\frac{\partial \boldsymbol{T_1}}{\partial x_1} + \frac{\partial \boldsymbol{T_2}}{\partial x_2} + \mathbf{B}\right] \tag{4.41}$$

Deriving with respect to arc-length curvilinear ordinate s equation (4.35), making use of (4.41) we have

$$\begin{aligned} \frac{\partial}{\partial s}\boldsymbol{f}(s,t) &= \frac{\partial}{\partial s}\int_{\Omega_0} \boldsymbol{T_3}\,\mathrm{d}x \\ &= \int_{\Omega_0} \frac{\partial}{\partial s}\boldsymbol{T_3}\,\mathrm{d}x \\ &= \int_{\Omega_0} \rho_0\,\ddot{\mathbf{U}}\,\mathrm{d}x - \int_{\Omega_0}\left[\sum_{\alpha=1}^{2}\frac{\partial \boldsymbol{T_\alpha}}{\partial x_\alpha} + \mathbf{B}\right]\mathrm{d}x \end{aligned} \tag{4.42}$$

Applying the divergence theorem to the first term under the integral sign

$$\begin{aligned} \int_{\Omega_0}\sum_{\alpha=1}^{2}\frac{\partial \boldsymbol{T_\alpha}}{\partial x_\alpha}\mathrm{d}x &= \sum_{\alpha=1}^{2}\int_{\Omega_0}\frac{\partial \boldsymbol{T_\alpha}}{\partial x_\alpha}\mathrm{d}x \\ &= \sum_{\Gamma=1}^{2}\int_{\partial\Omega_0}\boldsymbol{T_\Gamma}\cdot\boldsymbol{N_\Gamma}\,\mathrm{d}\Gamma \end{aligned} \tag{4.43}$$

where $\boldsymbol{N_\Gamma}$ is the vector field normal to the lateral contour $\partial\Omega_0$ of the body. Let A_ρ be the mass density per unit of length of the undeformed straight beam, i.e.

$$A_\rho = \int_{\Omega_0}\rho_0\,\mathrm{d}x \tag{4.44}$$

and assume that the origin of the x coordinate coincides with the center of mass of a cross-section, i.e. the first mass moment density per unit of length of the undeformed straight beam vanishes

$$S_\rho = \int_{\Omega_0}\rho_0\xi\,\mathrm{d}x = 0 \tag{4.45}$$

hence, the inertial part can be written as

$$\int_{\Omega_0}\rho_0\ddot{\mathbf{U}}\,\mathrm{d}x = A_\rho\,\ddot{\mathbf{U}} \tag{4.46}$$

Next, defining the *applied external forces* acting on a part of the boundary of the beam, by unit length in the reference configuration as

$$\bar{\boldsymbol{f}}(s,t) = \sum_{\Gamma=1}^{2}\int_{\partial\Omega_0}\boldsymbol{T_\Gamma}\cdot\boldsymbol{N_\Gamma}\,\mathrm{d}\Gamma + \int_{\Omega_0}\boldsymbol{B}\,\mathrm{d}x \tag{4.47}$$

we obtain the balance equation in integral form

$$\boxed{\frac{\partial}{\partial s}\boldsymbol{f}(s,t) + \bar{\boldsymbol{f}}(s,t) = A_\rho\,\ddot{\mathbf{U}} = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}t}} \tag{4.48}$$

Now, focusing on equation (4.36) we get

$$\begin{aligned} \boldsymbol{m}(s,t) &= \int_{\Omega_0}(\boldsymbol{x}-\boldsymbol{x_0})\times\boldsymbol{T_3}(S,\boldsymbol{r})\,\mathrm{d}x \\ &= \int_{\Omega_0}\boldsymbol{x}\times\boldsymbol{T_3}\,\mathrm{d}x - \int_{\Omega_0}\boldsymbol{x_0}\times\boldsymbol{T_3}\,\mathrm{d}x \end{aligned} \tag{4.49}$$

and deriving with respect to arc-length curvilinear ordinate s

$$
\begin{aligned}
\frac{\partial \boldsymbol{m}}{\partial s} &= \int_{\Omega_0} \left[\frac{\partial \boldsymbol{x}}{\partial s} \times \boldsymbol{T_3} + \boldsymbol{x} \times \frac{\partial \boldsymbol{T_3}}{\partial s} \right] \mathrm{d}x - \int_{\Omega_0} \left[\frac{\partial \boldsymbol{x_0}}{\partial s} \times \boldsymbol{T_3} + \boldsymbol{x_0} \times \frac{\partial \boldsymbol{T_3}}{\partial s} \right] \mathrm{d}x \\
&= \int_{\Omega_0} \frac{\partial \boldsymbol{x}}{\partial s} \times \boldsymbol{T_3} \,\mathrm{d}x + \int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \frac{\partial \boldsymbol{T_3}}{\partial s} \mathrm{d}x - \int_{\Omega_0} \frac{\partial \boldsymbol{x_0}}{\partial s} \times \boldsymbol{T_3} \,\mathrm{d}x \\
&= \int_{\Omega_0} \frac{\partial \boldsymbol{x}}{\partial s} \times \boldsymbol{T_3} \,\mathrm{d}x + \int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \frac{\partial \boldsymbol{T_3}}{\partial s} \mathrm{d}x - \frac{\partial \boldsymbol{x_0}}{\partial s} \times \boldsymbol{f}
\end{aligned}
\tag{4.50}
$$

where use has been made of (4.35). Now, the second term can be rewritten by means of equation (4.41)

$$
\begin{aligned}
\int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \frac{\partial \boldsymbol{T_3}}{\partial s} \mathrm{d}x &= \int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \left[\rho_0 \, \ddot{\mathbf{U}} - \sum_{\alpha=1}^{2} \frac{\partial \boldsymbol{T_\alpha}}{\partial x_\alpha} + \mathbf{B} \right] \mathrm{d}x \\
= \int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \rho_0 \, \ddot{\mathbf{U}} \,\mathrm{d}x &- \int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \mathbf{B} \,\mathrm{d}x - \int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \sum_{\alpha=1}^{2} \frac{\partial \boldsymbol{T_\alpha}}{\partial x_\alpha} \mathrm{d}x \\
= \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} - \int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \mathbf{B} \,\mathrm{d}x &- \sum_{\Gamma=1}^{2} \int_{\partial\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times (\boldsymbol{T_\Gamma} \cdot \boldsymbol{N_\Gamma}) \,\mathrm{d}\Gamma
\end{aligned}
\tag{4.51}
$$

where we used relation (4.12) and applied the divergence theorem as above. Introducing the following notation for the *applied external moment* field, acting on a part of the boundary of the beam (by unit length of the current configuration)

$$
\bar{\boldsymbol{m}}(s,t) = \sum_{\Gamma=1}^{2} \int_{\partial\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times (\boldsymbol{T_\Gamma} \cdot \boldsymbol{N_\Gamma}) \,\mathrm{d}\Gamma + \int_{\Omega_0} (\boldsymbol{x} - \boldsymbol{x_0}) \times \mathbf{B} \,\mathrm{d}x
\tag{4.52}
$$

finally we obtain

$$
\begin{aligned}
\frac{\partial}{\partial s} \boldsymbol{m}(s,t) &= \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} + \int_{\Omega_0} \frac{\partial \boldsymbol{x}}{\partial x_i} \times \boldsymbol{T_i} \,\mathrm{d}x - \frac{\partial \boldsymbol{x_0}}{\partial s} \times \boldsymbol{f} - \boldsymbol{m}(s,t) \\
&= \frac{\mathrm{d}\boldsymbol{L}}{\mathrm{d}t} - \frac{\partial \boldsymbol{x_0}}{\partial s} \times \boldsymbol{f} - \boldsymbol{m}(s,t)
\end{aligned}
\tag{4.53}
$$

where should be noted that $\partial \boldsymbol{x} / \partial x_i \times \boldsymbol{T_i} = \mathbf{0}$, and rearranging

$$
\boxed{\frac{\partial}{\partial s} \boldsymbol{m}(s,t) + \frac{\partial \boldsymbol{x_0}}{\partial s} \times \boldsymbol{f} + \bar{\boldsymbol{m}}(s,t) = \frac{\partial \boldsymbol{L}}{dt} \doteq \boldsymbol{I_O} \, \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I_O} \, \boldsymbol{\omega}}
\tag{4.54}
$$

Summarizing, the linear and angular momentum balance equations then take the differential form

$$
\boxed{
\begin{aligned}
&\frac{\partial}{\partial s} \boldsymbol{f} + \bar{\boldsymbol{f}} = \frac{d\mathbf{Q}}{dt} = A_\rho \ddot{\mathbf{U}} \\
&\frac{\partial}{\partial s} \boldsymbol{m} + \frac{\partial \boldsymbol{x_0}}{\partial S} \times \boldsymbol{f} + \bar{\boldsymbol{m}} = \frac{\partial \boldsymbol{L}}{dt} \doteq \boldsymbol{I_O} \, \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I_O} \, \boldsymbol{\omega}
\end{aligned}
}
\tag{4.55}
$$

where $\bar{\boldsymbol{f}}$ and $\bar{\boldsymbol{m}}$ are the *applied external force* per unit of current arc-length. This set of equations simply express the equilibrium on the domain and on the static boundary between the applied external generalized force (quantity with bar superscript) and the internal generalized forces (quantity without bar superscript). These are the usual set of equation imposed in a strong sense in the traditional FEM (besides the pointwise imposition of essential boundary values).
Basically, the construction of the dynamic weak formulation of the set of differential equations to be solved by the finite element method, is obtained by considering 4 steps: (i) take the dot product of differential equation (4.55) with a test function [5], (ii) integrate the result over the length of the reference beam $[0, L]$, (iii) use the divergence theorem (integration by parts) to reduce derivatives to their minimum order, and finally replace the boundary conditions. Results are reported in the forthcoming sections.

Material description. In applications, the material form of the aforementioned equations is often more convenient. The vector fields $\boldsymbol{f}(s,t)$ and $\boldsymbol{m}(s,t)$ are parametrized by the current axial-length s, take values on the *current* (spatial) configuration, with basis say or $\{\boldsymbol{t_i}\}$. These resultants are affected by superposed rigid body motions and are, for this sake, not convenient for constitutive description. As an alternative, we define the material vector fields $\boldsymbol{N}$ and $\boldsymbol{M}$ by the rotate-back form of the vector fields $\boldsymbol{f}(s,t)$ and $\boldsymbol{m}(s,t)$ to the *reference* material configuration (basis $\{\boldsymbol{E_i}\}$) by means of the orthogonal transformation $\boldsymbol{\Lambda}(s,t)$. Accordingly we have the relations:

$$\boldsymbol{N}(S) = \boldsymbol{\Lambda}^T(s,t)\,\boldsymbol{f}(s,t) \tag{4.56}$$

$$\boldsymbol{M}(S) = \boldsymbol{\Lambda}^T(s,t)\,\boldsymbol{m}(s,t) \tag{4.57}$$

The geometrical meaning of $\boldsymbol{N}(S,t)$ and $\boldsymbol{M}(S,t)$ follows from the observation that

$$\boldsymbol{f} = \boldsymbol{\Lambda}\,\boldsymbol{N} = \boldsymbol{\Lambda}\,N_i\boldsymbol{E_i} = N_i\,\boldsymbol{\Lambda}\boldsymbol{E_i} = N_i\boldsymbol{t_i} \tag{4.58}$$

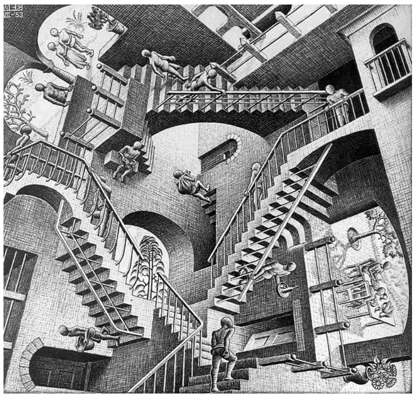
$$\boldsymbol{m} = \boldsymbol{\Lambda}\,\boldsymbol{M} = \boldsymbol{\Lambda}\,M_i\boldsymbol{E_i} = M_i\,\boldsymbol{\Lambda}\boldsymbol{E_i} = M_i\boldsymbol{t_i} \tag{4.59}$$

Thus, the components of the force and moment vectors $\boldsymbol{f}$ and $\boldsymbol{m}$ relative to the moving frame $\{\boldsymbol{t_i}\}$ equal those of $\boldsymbol{N}$ and $\boldsymbol{M}$ relative to the reference frame $\{\boldsymbol{E_i}\}$, i.e. they remain invariant under superposed rigid body motions.
The component form of the equations in the material description are obtained by substitution into (4.55).

[5]The more convenient test function corresponds to the kinematically admissible variation $\delta\boldsymbol{\eta} = (\delta\boldsymbol{u}, \delta\boldsymbol{\Psi}) \in \mathbb{R}^3 \times \mathcal{T}_{\boldsymbol{\Lambda}}^{\text{mat}}$ of the pair $(\boldsymbol{u}, \boldsymbol{\Lambda})$

Chapter 5

Geometrically exact beam theory: virtual works principle



The pure and simple truth is rarely pure and never simple.

O. Wilde

The purpose of this section is to formulate properly the three-dimensional principle of virtual works for the Reissner-Simo beam theory introduced in the previous Chapters. As noted by Makinen [36], the virtual work may be viewed as a linear form on the tangent field-bundle, which in turn is a tangent bundle of the placement manifold at fixed time. Basing on the framework introduced in Chapter 2, firstly a formal definition of virtual work on manifolds basing on the concepts of virtual displacement on manifold is given. Then is presented a detailed derivation of the nonlinear functional corresponding to the virtual work principle, both in spatial and material description. Finally, a simple constitutive law steaming from the existence of a strain energy density function is derived in the same fashion of Pimenta & Yojo [49].

5.1 Principle of virtual works

In order to develop a organic path toward the definition of the principle of virtual works, we introduce step by step the needed quantities.

Definition 5.1 (Virtual displacement)**.** *Let introduce the spatial position field defined by means of the position vector field $\boldsymbol{x}$, which individuates each point in the current region Ω, and the material position field $\boldsymbol{X}$, which individuates each point in the reference region Ω*
The virtual displacement $\delta\boldsymbol{x}$ at any generalized place vector $\boldsymbol{x} \in \mathbb{E}^n$ and a fixed time instant $t = t_0$, is defined as any possible change of the displacement field $\boldsymbol{x}$

$$\delta\boldsymbol{x} = \boldsymbol{x}_{\boldsymbol{\varepsilon}} - \boldsymbol{x} \tag{5.1}$$

where $\boldsymbol{x}_{\boldsymbol{\varepsilon}}$ stands for an infinitesimal perturbed position field.

The attribute possible refers to the kinematical requirement to be respectful of the geometric constraints on the manifold, and in particular $\delta\boldsymbol{x}$ is such that it assumes a prescribed value on the boundary region where displacements are assigned.

Definition 5.2 (Virtual work on point-manifold)**.** *Virtual work on the tangent point-bundle $\mathcal{T}_{\boldsymbol{x_0}}\mathcal{M}_{t_0}$ at the fixed time instant $t = t_0$ and the place vector $\boldsymbol{x}(t_0) = \boldsymbol{x}_0 \in \mathcal{M}_{t_0}$ is defined as a linear form by*

$$\delta W \triangleq \boldsymbol{f} \cdot \delta\boldsymbol{x} \tag{5.2}$$

where the virtual displacement $\delta\boldsymbol{x} \in \mathcal{T}_{\boldsymbol{x_0}}\mathcal{M}_{t_0}$, and the force vector $\boldsymbol{f} = \boldsymbol{f}(t_0, \boldsymbol{x_0}) \in \mathcal{T}^*_{\boldsymbol{x_0}}\mathcal{M}_{t_0}$ which belongs to the dual point-space[1]*.*

Forces may be classified into external, internal and additionally into inertial forces. Last group may be regarded as an effective force, because if an external active force is acting on a particle, which is otherwise free, then the inertial force may be regarded as the reaction force which take place to maintain force equilibrium in dynamical sense.

Definition 5.3 (Virtual work on field-manifold)**.** *The virtual work on the tangent field bundle at a fixed time instant $t = t_0$ and the place vector $\boldsymbol{x_0} = \boldsymbol{x}(t_0) \in \mathcal{T}_{\boldsymbol{x_0}}\mathcal{M}_{t_0}$ is defined as an integral over the domain of the body $\mathscr{B}$*

$$\delta W \triangleq \int_{\mathscr{B}} \boldsymbol{f} \cdot \delta\boldsymbol{x} \, \mathrm{d}V \tag{5.3}$$

where the virtual displacement filed $\delta\boldsymbol{x} \in \mathcal{T}_{\boldsymbol{x_0}}\mathcal{M}_{t_0}$ and the force filed $\boldsymbol{f} = \boldsymbol{f}(t_0, \boldsymbol{x}_0) \in \mathcal{T}^*_{\boldsymbol{x_0}}\mathcal{M}_{t_0}$.

Definition 5.4 (Principle of virtual works)**.** *The principle of virtual works states that at a dynamical equilibrium, the total virtual work with respect to any virtual displacement, at the fixed time instant $t = t_0$ and the place vector $\boldsymbol{x_0}$, vanishes, i.e.*

$$\delta W = 0 \qquad \forall \delta\boldsymbol{x} \in \mathcal{T}_{\boldsymbol{x_0}}\mathcal{M}_{t_0} \tag{5.4}$$

[1]For the fixed time $t = t_0$, the place vector $\boldsymbol{x_0} = \boldsymbol{x}(t_0)$ defines a tangent point-space at the point $\boldsymbol{x_0} \in \mathcal{M}_{t_0}$ such as $\mathcal{T}_{\boldsymbol{x_0}}\mathcal{M}_{t_0} = \delta\boldsymbol{x} \in \mathbb{E}^n | (\boldsymbol{x_0}, \delta\boldsymbol{x}) \in \mathcal{T}_{\boldsymbol{x_0}}\mathcal{M}_{t_0}$. Therefore we may denote for any virtual displacement vector $\delta\boldsymbol{x} \in \mathcal{T}_{\boldsymbol{x_0}}\mathcal{M}_{t_0}$, where the base point $\boldsymbol{x_0}$ is included in the notation as a subscript.

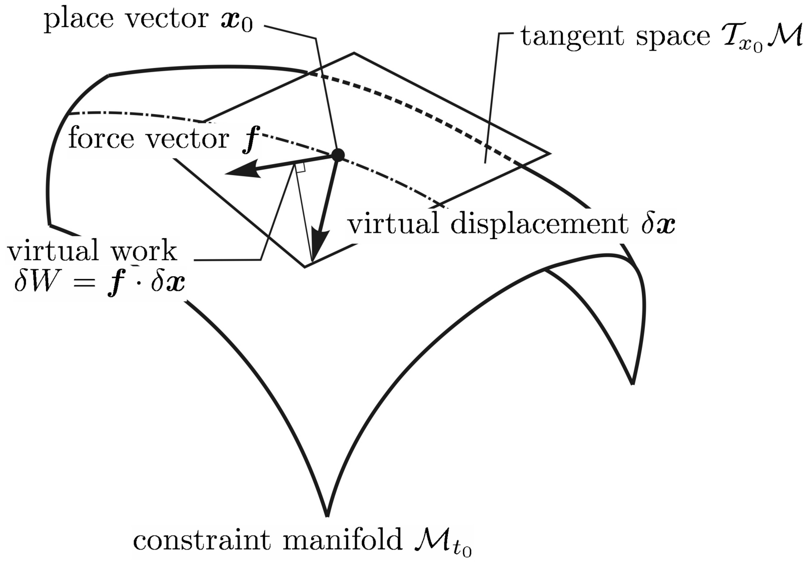


Figure 5.1: A geometric representation of the virtual work on the generic manifold.

5.1.1 Principle of virtual works in spatial description

In this section, we derive the dynamic balance equations of the linear and rotational momentum in a weak form for Reissner's beam. We will rather closely follow the paper [16] for showing that Reissner's beam formulation is also a stress resultant formulation, consistent with continuum mechanics at a resultant level.

Also this kind of equilibrium equations can be given in two forms, depending on which is the domain of integration: the current spatial configuration Ω or the material reference configuration Ω_0.

Initial boundary-value problem (IBVP). The finite element method requires the formulation of the balance laws in the form of variational principles. One of the most fundamental balance laws is the Cauchy's first equation of motion (i.e. balance of mechanical energy) (4.26)

$$\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\,\ddot{\mathbf{u}} \tag{5.5}$$

valid for every point $\boldsymbol{x} \in \Omega$ and for all times t.

We assume also that the boundary surface $\partial\Omega$ of a continuum body $\mathscr{B}$ occupying region Ω, is decomposed into disjoint parts so that:

$$\partial\Omega = \partial\Omega_u \cup \partial\Omega_\sigma \qquad \text{with} \qquad \partial\Omega_u \cap \partial\Omega_\sigma = \varnothing \tag{5.6}$$

We distinguish two classes of **boundary conditions**, namely the *Dirichlet boundary conditions* (or essential boundary conditions), which correspond to a prescribed displacement field (geometric restraints) $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ on the kinematic boundary, and the

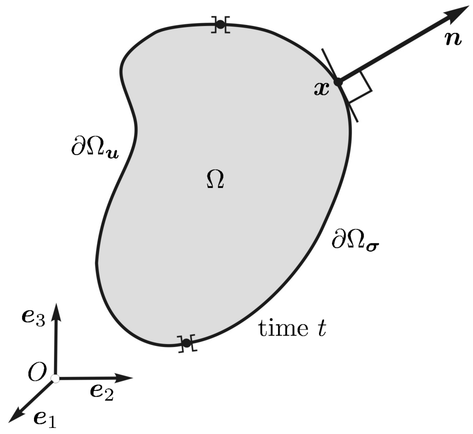


Figure 5.2: Partition of boundary surface $\partial\Omega$ in two-dimensional space at time t.

Neumann boundary conditions (or natural boundary condition), which are identified physically with the surface traction $\boldsymbol{t_n} = \boldsymbol{t_n}(\boldsymbol{x}, \boldsymbol{n}, t)$ on the loaded boundary (force restraints).
We write

$$\begin{aligned} \boldsymbol{u} &= \bar{\boldsymbol{u}} && \text{on} \quad \partial\Omega_u \\ \boldsymbol{t_n} &= \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}}_{\boldsymbol{n}} && \text{on} \quad \partial\Omega_\sigma \end{aligned} \tag{5.7}$$

where the $\bar{(\cdot)}$ denotes prescribed (given) functions on the boundaries $\partial\Omega_{(\cdot)} \subset \partial\Omega$ of a continuum body occupying the region Ω. The unit exterior vector normal to the boundary surface $\partial\Omega_\sigma$ is characterized by $\boldsymbol{n}$. The prescribed displacement field $\bar{\boldsymbol{u}}$ and the prescribed Cauchy traction vector $\bar{\boldsymbol{t}}_{\boldsymbol{n}}$ (force measured for unit current surface area) are specified respectively on a portion $\partial\Omega_u \subset \partial\Omega$, called kinematic boundary, and on the remainder $\partial\Omega_\sigma$ called static boundary.
Because the traction boundary conditions $\boldsymbol{t_n} = \bar{\boldsymbol{t}}_{\boldsymbol{n}}$ on $\partial\Omega_\sigma$ arise directly from the weak equilibrium, they are often called *natural* boundary conditions. Instead, the displacements boundary conditions $\boldsymbol{u} = \bar{\boldsymbol{u}}$ on $\partial\Omega_{\boldsymbol{u}}$ are called *essential* boundary conditions, since they have to be imposed out of the weak form.

The second-order differential equation (4.26) itself requires additional data in the form of **initial conditions**. For instance, the generalized displacement field $\boldsymbol{\eta}|_{t=0}$ and the velocity field $\dot{\boldsymbol{\eta}}|_{t=0}$ at the initial time $t = 0$ are specified as

$$\boldsymbol{\eta}(\boldsymbol{x}, y)|_{t=0} = \boldsymbol{\eta}_0(\boldsymbol{X}) \qquad \dot{\boldsymbol{\eta}}(\boldsymbol{x}, y)|_{t=0} = \dot{\boldsymbol{\eta}}_0(\boldsymbol{X}) \tag{5.8}$$

where $(\cdot)_0$ denotes a prescribed function in Ω_0. Since we agreed to consider a stress-free reference configuration at $t = 0$, the initial values $(\cdot)_0$ are assumed to be zero in our case.

In order to achieve **compatibility** of the kinematic boundary and initial conditions we require additionally on $\partial\Omega_u$ that

$$\bar{\boldsymbol{\eta}} = \boldsymbol{\eta}_0(\boldsymbol{X}) \qquad \bar{\dot{\boldsymbol{\eta}}} = \dot{\boldsymbol{\eta}}_0(\boldsymbol{X}) \tag{5.9}$$

Now, the problem is to find a motion that satisfies equation (4.26) with the prescribed boundary and initial conditions, and compatibility conditions. This leads to the formulation of the **strong form** (or differential form) of the initial boundary-value problem (IBVP). Given the body forces, and both boundary and initial conditions, find the generalized displacement field $\boldsymbol{\eta} = \{\boldsymbol{u}; \boldsymbol{\theta}\}$ so that

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \rho\,\ddot{\mathbf{u}} & \text{in} \quad \Omega \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}^T & \text{in} \quad \Omega \\ \boldsymbol{t_n} = \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\boldsymbol{t}}_{\boldsymbol{n}} & \text{on} \quad \partial\Omega_\sigma \\ \boldsymbol{u} = \bar{\boldsymbol{u}} & \text{on} \quad \partial\Omega_u \\ \boldsymbol{\eta}(\boldsymbol{x}, \boldsymbol{y})|_{t=0} = \boldsymbol{\eta}_0(\boldsymbol{x}) \\ \dot{\boldsymbol{\eta}}(\boldsymbol{x}, \boldsymbol{y})|_{t=0} = \dot{\boldsymbol{\eta}}_0(\boldsymbol{x}) \end{cases} \tag{5.10}$$

Note that the set of equations (5.10) generally defines a nonlinear initial boundary-value problem for the unknown displacement field $\mathbf{u}$. In addition, we need a constitutive equation for the stress $\boldsymbol{\sigma}$ which is, in general, a nonlinear function of the displacement field $\mathbf{u}$.

Principle of virtual works. Let consider here the weak (integral) form of the Cauchy's equilibrium equation (4.25), which leads to the formulation of principle of virtual works written with respect to the *current configuration* Ω. Also consider the virtual displacement $\delta\boldsymbol{u}(\boldsymbol{x})$ an arbitrary weighting vector function. Multiplying the differential Cauchy's equilibrium equations (4.26) by the weighting function $\delta\boldsymbol{x}$ and integrating over the current domain, we obtain

$$f(\boldsymbol{u}, \delta\boldsymbol{u}) = \int_\Omega \left[(\mathrm{div}\boldsymbol{\sigma} + b - \rho\,\ddot{\mathbf{u}}) \cdot \delta\boldsymbol{x} \right] \,\mathrm{d}v = 0 \qquad \forall\,\delta\boldsymbol{u} \in \delta\mathcal{U} \tag{5.11}$$

which is the integral (weak) form of the equation of motion written with respect to the current (spatial) configuration. The fundamental lemma of calculus of variations guaranties this weak equation to be equal to the strong one (for further details see [72]). Splitting the integral, the previous equation is rewritten as

$$\int_\Omega \mathrm{div}\boldsymbol{\sigma} \cdot \delta\boldsymbol{x} \,\mathrm{d}v + \int_\Omega \boldsymbol{b} \cdot \delta\boldsymbol{x} \,\mathrm{d}v - \int_\Omega \rho\,\ddot{\mathbf{u}} \cdot \delta\boldsymbol{x} \,\mathrm{d}v = 0 \qquad \forall\,\delta\boldsymbol{x} \in \delta\mathcal{U} \tag{5.12}$$

Consider the first integral $\int_\Omega \mathrm{div}\boldsymbol{\sigma} \cdot \delta\boldsymbol{x}\,\mathrm{d}v$. The scalar product between the divergence of a tensor and a vector can be expressed in term of the vector gradient by the rule (A.18). Hence we can rewrite the equality

$$\mathrm{div}\boldsymbol{\sigma} \cdot \delta\boldsymbol{x} = \mathrm{div}(\boldsymbol{\sigma}^T \delta\boldsymbol{x}) - \boldsymbol{\sigma} : \mathrm{grad}\delta\boldsymbol{x}$$

and because $\boldsymbol{\sigma}$ is a symmetric tensor ($\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$, see (4.29)), trivially we have

$$\mathrm{div}\boldsymbol{\sigma} \cdot \delta\boldsymbol{x} = \mathrm{div}(\boldsymbol{\sigma}\delta\boldsymbol{x}) - \boldsymbol{\sigma} : \mathrm{grad}\delta\boldsymbol{x}$$

and thus the considered integral becomes

$$\int_\Omega \mathrm{div}\boldsymbol{\sigma} \cdot \delta\boldsymbol{x}\,\mathrm{d}v = \int_\Omega \mathrm{div}(\boldsymbol{\sigma}\delta\boldsymbol{x})\,\mathrm{d}v - \int_\Omega \boldsymbol{\sigma} : \mathrm{grad}\delta\boldsymbol{x}\,\mathrm{d}v \tag{5.13}$$

In order to rearrange this expression, we examine first the second term $\int_\Omega \boldsymbol{\sigma} : \text{grad}\delta\boldsymbol{x}\text{d}v$. Since $\boldsymbol{\sigma}$ is symmetric, we can use the rule of double contraction between a tensor and a symmetric tensor (A.19) to write

$$\int_\Omega \boldsymbol{\sigma} : \text{grad}\delta\boldsymbol{x}\,\text{d}v = \int_\Omega \boldsymbol{\sigma} : \text{sym}[\text{grad}\delta\boldsymbol{x}]\,\text{d}v = \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{e}\,\text{d}v \tag{5.14}$$

where we recognized in sym[grad$\delta\boldsymbol{x}$] the *Euler-Almansi* strain tensor's virtual variation [2] $\delta\boldsymbol{e}$.

Consider now the other term $\int_\Omega \text{div}(\boldsymbol{\sigma}\delta\boldsymbol{x})\text{d}v$. Applying first the divergence theorem and then the symmetry of $\boldsymbol{\sigma}$, this integral can be given as

$$\int_\Omega \text{div}(\boldsymbol{\sigma}\delta\boldsymbol{x})\text{d}v = \int_{\partial\Omega} \mathbf{n}\cdot\boldsymbol{\sigma}\delta\boldsymbol{x}\,\text{d}s = \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\cdot\delta\boldsymbol{x}\,\text{d}s$$

Since $\delta\boldsymbol{x}$ vanishes on the part of the boundary surface $\partial\Omega_{\boldsymbol{u}}$ where displacements are prescribed, the integral over the whole boundary region reduces to an integral over only $\partial\Omega_\sigma$, i.e. the region where traction are assigned

$$\int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\cdot\delta\boldsymbol{x}\,\text{d}s = \int_{\partial\Omega_{\boldsymbol{\sigma}}} \boldsymbol{t}_\mathbf{n}\cdot\delta\boldsymbol{x}\,\text{d}s$$

Therefore, by substitution into previous equation we get

$$\int_\Omega \text{div}(\boldsymbol{\sigma}\delta\boldsymbol{x})\text{d}v = \int_{\partial\Omega_{\boldsymbol{\sigma}}} \boldsymbol{t}_\mathbf{n}\cdot\delta\boldsymbol{x}\,\text{d}s \tag{5.15}$$

Recollecting results of equations (5.14) and (5.15) and substituting them into (5.13), the expression of the term $\int_\Omega \text{div}\boldsymbol{\sigma}\cdot\delta\boldsymbol{x}\,\text{d}v$ takes the form

$$\int_\Omega \text{div}\boldsymbol{\sigma}\cdot\delta\boldsymbol{x}\,\text{d}v = \int_{\partial\Omega_{\boldsymbol{\sigma}}} \boldsymbol{t}_\mathbf{n}\cdot\delta\boldsymbol{x}\,\text{d}s - \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{e}\,\text{d}v \tag{5.16}$$

Substituting backwards this expression into equation (5.12) we obtain

$$\begin{aligned}\int_{\partial\Omega_{\boldsymbol{\sigma}}} \boldsymbol{t}_\mathbf{n}\cdot\delta\boldsymbol{x}\,\text{d}s - \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{e}\,\text{d}v + \int_\Omega \boldsymbol{b}\cdot\delta\boldsymbol{x}\,\text{d}v \\ - \int_\Omega \rho\,\ddot{\mathbf{u}}\cdot\delta\boldsymbol{x}\,\text{d}v = 0 \qquad \forall\,\delta\boldsymbol{x}\in\delta\mathcal{U}\end{aligned} \tag{5.17}$$

which changing sign and reordering terms finally provides the *principle of virtual works* written in *current configuration*

$$\boxed{\begin{aligned}\int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{e}\,\text{d}v - \int_\Omega \boldsymbol{b}\cdot\delta\boldsymbol{x}\,\text{d}v - \int_{\partial\Omega_{\boldsymbol{\sigma}}} \bar{\boldsymbol{t}}_\mathbf{n}\cdot\delta\boldsymbol{x}\,\text{d}s \\ + \int_\Omega \rho\,\ddot{\mathbf{u}}\cdot\delta\boldsymbol{x}\,\text{d}v = 0 \qquad \forall\,\delta\boldsymbol{x}\in\delta\mathcal{U}\end{aligned}} \tag{5.18}$$

[2] For demonstration that sym[grad$\delta\boldsymbol{x}$] $= \delta\boldsymbol{e}$ see [28] page 376.

with boundary conditions

$$\boldsymbol{u} = \bar{\boldsymbol{u}} \qquad \partial\Omega_{\boldsymbol{u}} \subset \partial\Omega \tag{5.19}$$

The principle of virtual works states that at the equilibrium configuration the virtual work $\boldsymbol{\sigma} : \delta\boldsymbol{e}$ done by fixed $\boldsymbol{\sigma}$ with the virtual variation $\delta\boldsymbol{e}$ on the whole volume, equals the sum of work done with virtual displacement [3] $\delta\boldsymbol{u}$ by the body forces $\mathbf{b}$, the inertia forces $\rho\,\ddot{\mathbf{u}}$ per unit of current volume, and surface tractions $\bar{\boldsymbol{t}}_{\mathbf{n}}$ on the boundary area $\partial\Omega_{\boldsymbol{\sigma}}$. Usually the terms in which the total virtual work is divided are three parts: external, inertial and inertial as indicated by the notation

$$\boxed{\delta W_{tot} = \delta W_{int} - \delta W_{ext} + \delta W_{inert} = 0} \tag{5.20}$$

where the terms above are given by the following expressions

$$\delta W_{int} = \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{e}\,\mathrm{d}v \tag{5.21}$$

$$\delta W_{ext} = \int_\Omega \boldsymbol{b}\cdot\delta\boldsymbol{x}\,\mathrm{d}v + \int_{\partial\Omega_{\boldsymbol{\sigma}}} \bar{\boldsymbol{t}}_{\mathbf{n}}\cdot\delta\boldsymbol{x}\,\mathrm{d}s \tag{5.22}$$

$$\delta W_{inert} = \int_\Omega \rho\,\ddot{\mathbf{u}}\cdot\delta\boldsymbol{x}\,\mathrm{d}v \tag{5.23}$$

where the first integral, δW_{int}, is called *internal (mechanical) virtual work*, the second one δW_{ext}, *external (mechanical) virtual work*, whereas the third term is called *inertial virtual work*. It's worth to note that stress, body forces, and traction vectors are all defined on *current* region Ω, which clearly is also the integral domain.
The equality, along with the initial conditions, stated by the principle of virtual works, characterizes the weak form (or variational form) of the initial boundary-value problem, natural counterpart of the strong form (5.10)

$$f(\boldsymbol{u},\delta\boldsymbol{x}) = \int_\Omega [\boldsymbol{\sigma} : \delta\boldsymbol{e} - (\boldsymbol{b} - \rho\,\ddot{\mathbf{u}})\cdot\delta\boldsymbol{x}]\,\mathrm{d}v - \int_{\partial\Omega_{\boldsymbol{\sigma}}} \boldsymbol{t}_{\mathbf{n}}\cdot\delta\boldsymbol{x}\,\mathrm{d}s = 0 \tag{5.24}$$

$$\int_\Omega \boldsymbol{u}(\boldsymbol{x},y)|_{t=0}\cdot\delta\boldsymbol{x}\,\mathrm{d}v = \int_\Omega \boldsymbol{u}_0(\boldsymbol{X})\cdot\delta\boldsymbol{x}\,\mathrm{d}v \tag{5.25}$$

$$\int_\Omega \dot{\boldsymbol{u}}(\boldsymbol{x},y)|_{t=0}\cdot\delta\boldsymbol{x}\,\mathrm{d}v = \int_\Omega \dot{\boldsymbol{u}}_0(\boldsymbol{X})\cdot\delta\boldsymbol{x}\,\mathrm{d}v \tag{5.26}$$

It is important to emphasize that the principle of virtual works does not necessitate the existence of a potential. Indeed, no statement in regard to a particular material is invoked. Therefore the principle of virtual works is general, in the sense that it is applicable to any material, including inelastic material.

[3] A virtual displacement is defined as a kinematical admissible perturbation (variation) of generalized displacement field, i.e. satisfying the kinematic boundary conditions. Strictly speaking an *admissible* configuration is a concept that embodies all others as particular case, indeed a *perturbed* configuration is an admissible variation from an admissible configuration, whereas an ensemble of perturbed configurations is used to establish *incremental* variations.

5.1.2 Principle of virtual works in material description

Initial boundary-value problem (IBVP). The equations of motion of continuum with boundary conditions, in terms of the first Piola-Kirchhoff stress tensor, can be written as

$$\begin{cases} \mathrm{DIV}\boldsymbol{P} + \mathbf{B} = \rho_0\,\ddot{\mathbf{U}} & \text{in} \quad \Omega_0 \\ \boldsymbol{P}\boldsymbol{F}^T = \boldsymbol{F}\boldsymbol{P}^T & \text{in} \quad \Omega_0 \\ \boldsymbol{t}_{\boldsymbol{N}} = \boldsymbol{P}\cdot\boldsymbol{N} = \bar{\boldsymbol{t}}_{\boldsymbol{N}} & \text{on} \quad \partial\Omega_{\sigma,0} \\ \boldsymbol{U} = \bar{\boldsymbol{U}} & \text{on} \quad \partial\Omega_{u,0} \\ \boldsymbol{U}(\boldsymbol{X},\boldsymbol{Y})|_{t=0} = \boldsymbol{U}_0(\boldsymbol{X}) & \\ \dot{\boldsymbol{U}}(\boldsymbol{X},\boldsymbol{Y})|_{t=0} = \dot{\boldsymbol{U}}_0(\boldsymbol{X}) & \end{cases} \tag{5.27}$$

where $\mathbf{B}$ is the body force vector, ρ_0 the density of the material body, $\boldsymbol{N}$ the normal vector of the traction boundary, $\bar{\boldsymbol{t}}_{\boldsymbol{N}}$ the given traction vector, and $\boldsymbol{U}$ the given displacement vector. Note that the base points are given in the material coordinate, but they occupy the tangent spaces of the spatial configuration.

Principle of virtual works. We consider here the principle of virtual work wherein the integral domain is the reference region Ω_0 of the continuum body, bounded by a reference boundary surface $\partial\Omega_0$. The computation is developed starting again from the equation of motion, but written in material reference

$$\mathrm{DIV}\boldsymbol{P} + \mathbf{B} = \rho_0\,\ddot{\mathbf{U}} \tag{5.28}$$

corresponding to (4.32). Here, $\mathbf{P}$, $\mathbf{B}$ and $\rho_0\ddot{\boldsymbol{U}}$ denote the first Piola-Kirchhoff stress tensor, the reference body force and the inertia force per unit reference volume, respectively. For the Dirichlet and Neumann boundary conditions, i.e $\boldsymbol{U} = \boldsymbol{U}(\boldsymbol{X}, t)$ and $\boldsymbol{T} = \boldsymbol{T}(\boldsymbol{X}, \boldsymbol{N}, t)$ we may write, by analogy with (5.7)

$$\boldsymbol{U} = \bar{\boldsymbol{U}} \qquad \text{on} \qquad \partial\Omega_{0,u} \tag{5.29}$$

$$\boldsymbol{T}_{\boldsymbol{N}} = \boldsymbol{P}\cdot\boldsymbol{N} = \bar{\boldsymbol{T}}_{\boldsymbol{N}} \qquad \text{on} \qquad \partial\Omega_{0,\sigma} \tag{5.30}$$

where the unit outwards vector normal to the boundary surface $\partial\Omega_{0,\sigma}$ is characterized by $\mathbf{N}$. The prescribed displacement field $\bar{\boldsymbol{U}}$ and the prescribed Piola-Kirchhoff traction vector $\bar{\boldsymbol{T}}_{\boldsymbol{N}}$ (force measured per unit of reference surface area) are specified on the disjoint parts $\partial\Omega_{0,u}$ and $\partial\Omega_{0,\sigma}$, respectively. The second-order differential equation (5.28) must be supplemented by initial conditions for the displacement field and the velocity field at the instant of time $t = 0$.
Using the aforementioned concepts, one may show [4] the principle of virtual works in the *material description* expressed in terms of the virtual displacement, i.e.

$$\boxed{\int_{\Omega_0} \boldsymbol{P} : \mathrm{Grad}\delta\boldsymbol{u}\;\mathrm{d}V - \int_{\Omega_0} \boldsymbol{B}\cdot\delta\boldsymbol{u}\,\mathrm{d}V - \int_{\partial\Omega_{0,\boldsymbol{\sigma}}} \bar{\boldsymbol{T}}_{\boldsymbol{N}}\cdot\delta\boldsymbol{u}\,\mathrm{d}S + \int_{\Omega_0} \rho_0\,\ddot{\mathbf{U}}\cdot\delta\boldsymbol{u}\;\mathrm{d}V = 0 \qquad \forall\,\delta\boldsymbol{u}\in\delta\mathcal{U}} \tag{5.31}$$

[4]For further details see Appendix B

with the virtual displacement field $\delta\boldsymbol{u}$ (here defined on the *reference configuration*) satisfying implicitly the essential boundary condition $\delta\boldsymbol{u} = 0$ on the part of the boundary surface $\partial\Omega_{0,u}$, where the displacement field $\bar{\boldsymbol{U}}$ is prescribed.

As before, the principle of virtual works states that at the equilibrium configuration the virtual work $\boldsymbol{P} : \mathrm{Grad}\delta\boldsymbol{u}$ done by fixed $\boldsymbol{P}$ with the virtual variation $\mathrm{Grad}\delta\boldsymbol{u}$ on the whole volume, equals the sum of work done with virtual displacement $\delta\boldsymbol{u}$ by the body forces $\boldsymbol{B}$, the inertia forces $\rho_0\,\ddot{\mathbf{U}}$ per unit of reference volume, and surface tractions $\bar{\boldsymbol{T}}_{\boldsymbol{N}}$ on the boundary area $\partial\Omega_{\sigma,0}$. Grouping the terms in (5.31) we obtain again the same expression of (5.20), where the terms are expressed in material form as

$$\delta W_{int} = \int_{\Omega_0} \boldsymbol{P} : \boldsymbol{\delta F}\,\mathrm{d}V \tag{5.32}$$

$$\delta W_{ext} = \int_{\Omega_0} \boldsymbol{B} \cdot \boldsymbol{\delta x}\,\mathrm{d}V + \int_{\partial\Omega_{\boldsymbol{\sigma},0}} \bar{\boldsymbol{T}}_{\mathbf{N}} \cdot \delta\boldsymbol{x}\,\mathrm{d}S \tag{5.33}$$

$$\delta W_{inert} = \int_{\Omega_0} \rho_0\,\ddot{\mathbf{U}} \cdot \delta\boldsymbol{x}\,\mathrm{d}v \tag{5.34}$$

Alternatively, the material form of the principle of virtual works sought, can be obtained simply by *pull-back* operation of relation (5.18) to the reference configuration. To do that, one can start to express the internal and external virtual work δW_{int} and δW_{ext} in equations (5.21) and (5.22) and the contribution $\int_\Omega \rho\,\ddot{\mathbf{u}} \cdot \delta\boldsymbol{x}\,\mathrm{d}v$ in terms of material variables.

Internal virtual work. The internal virtual work δW_{int} can be expressed in two equivalent forms, using the first Piola-Kirchhoff stress tensor $\mathbf{P}$ or the second Piola-Kirchhoff stress tensor $\mathbf{S}$. Here only the first approach is developed.

Let consider the internal virtual work in the current configuration δW_{int} (5.21) and rewrite it using $\boldsymbol{\sigma} : \delta\boldsymbol{e} = \boldsymbol{\sigma} : \mathrm{grad}\,\delta\boldsymbol{x}$. Recalling the relations between current and reference differential volume, respectively $\mathrm{d}v$ and $\mathrm{d}V$, and between spatial and material gradient [5] of $\delta\boldsymbol{x}$, respectively $\mathrm{grad}\,\delta\boldsymbol{x}(\boldsymbol{x})$ and $\mathrm{GRAD}\,\delta\boldsymbol{x}(\boldsymbol{X})$

$$\mathrm{d}v = J\,\mathrm{d}V \qquad \text{and} \qquad \mathrm{grad}\,\delta\boldsymbol{x} = \mathrm{GRAD}\,\delta\boldsymbol{x} \cdot \boldsymbol{F}^{-1} \tag{5.35}$$

where $J = \det\boldsymbol{F}$, the internal virtual work takes the form

$$\begin{aligned}\delta W_{int} &= \int_\Omega \boldsymbol{\sigma} : \delta\boldsymbol{e}\,\mathrm{d}v \\ &= \int_\Omega \boldsymbol{\sigma} : \mathrm{grad}\,\delta\boldsymbol{x}\,\mathrm{d}v \\ &= \int_{\Omega_0} \left[\boldsymbol{\sigma} : \mathrm{GRAD}\,\delta\boldsymbol{x}\,\boldsymbol{F}^{-1}J\right]\,\mathrm{d}V\end{aligned} \tag{5.36}$$

where the symmetry of the Cauchy stress tensor $\boldsymbol{\sigma}$ is to be used (see equation (4.29)). The equation states that the internal virtual work, integrated on the current configuration domain, can be given as an integral on the reference domain of the function in brackets.

[5] For further details see [20].

Using the property of double contraction between a tensor and a product of tensors (A.17), and then the relations between $\boldsymbol{\sigma}$ and the first Piola-Kirchhoff stress tensor $\mathbf{P}$ ($J\boldsymbol{\sigma}\boldsymbol{F}^{-T} = \boldsymbol{P}$) and between $\boldsymbol{\delta u}$ and the back-rotated variation $\boldsymbol{\delta F}^r$ (GRAD $\boldsymbol{\delta x} = \boldsymbol{\delta F}^r$) [6] the function in brackets can be rearranged in the form

$$\boldsymbol{\sigma} : \mathrm{GRAD}\, \boldsymbol{\delta x}\, \boldsymbol{F}^{-1} J = J\boldsymbol{\sigma}\boldsymbol{F}^{-T} : \mathrm{GRAD}\, \boldsymbol{\delta x} = \boldsymbol{P} : \boldsymbol{\delta F}^r \tag{5.37}$$

Finally, by substitution of this expression into (5.36), we obtain the *internal virtual work* written on *reference configuration* in terms of Piola-Kirchhoff stress tensor $\mathbf{P}$

$$\delta W_{int} = \int_{\Omega_0} \boldsymbol{P} : \boldsymbol{\delta F}^r \, \mathrm{d}V \tag{5.38}$$

It's worth to note how the stress tensor $\mathbf{P}$ turns out to be work conjugate with the virtual variation of deformation gradient $\boldsymbol{\delta F}^r$. Let now substitute equation (3.80) into the integrand of (5.38)

$$\boldsymbol{P} : \boldsymbol{\delta F}^r = \boldsymbol{P} : [(\boldsymbol{\delta\Gamma} + \boldsymbol{\delta K} \times \boldsymbol{R}) \otimes \boldsymbol{E}_3] \tag{5.39}$$

and recalling equation (4.34) the previous delivers

$$\boldsymbol{P} : \boldsymbol{\delta F}^r = (\boldsymbol{T}_i \otimes \boldsymbol{E}_i) : [(\boldsymbol{\delta\Gamma} + \boldsymbol{\delta K} \times \boldsymbol{R}) \otimes \boldsymbol{E}_3] \tag{5.40}$$

where sum is intended on repeated indices (Einstein's notation). With the help of (A) we may rewrite

$$\boldsymbol{P} : \boldsymbol{\delta F}^r = [(\boldsymbol{T}_i \otimes \boldsymbol{E}_i)\boldsymbol{E}_3] \cdot (\boldsymbol{\delta\Gamma} + \boldsymbol{\delta K} \times \boldsymbol{R}) \tag{5.41}$$

and again, applying the tensor product definition

$$\begin{aligned} \boldsymbol{P} : \boldsymbol{\delta F}^r &= \boldsymbol{T}_i(\boldsymbol{E}_i \cdot \boldsymbol{E}_3) \cdot (\boldsymbol{\delta\Gamma} + \boldsymbol{\delta K} \times \boldsymbol{R}) \\ &= \boldsymbol{T}_3 \cdot (\boldsymbol{\delta\Gamma} + \boldsymbol{\delta K} \times \boldsymbol{R}) \end{aligned} \tag{5.42}$$

From (5.38), decomposing the triple integral over the beam volume into the integral along the beam axis (linear domain $[0, L]$) and the integral over the generic cross-section (surface domain A), the internal virtual work follows as

$$\begin{aligned} \delta W_{int} &= \int_{\Omega_0} \boldsymbol{P} : \boldsymbol{\delta F}^r \, \mathrm{d}V \\ &= \int_{\Omega_0} \boldsymbol{T}_3 \cdot (\boldsymbol{\delta\Gamma} + \boldsymbol{\delta K} \times \boldsymbol{R}) \, \mathrm{d}V \\ &= \int_{\Omega_0} \boldsymbol{T}_3 \cdot \boldsymbol{\delta\Gamma} \, \mathrm{d}V + \int_{\Omega_0} \boldsymbol{T}_3 \cdot (\boldsymbol{\delta K} \times \boldsymbol{R}) \, \mathrm{d}V \\ &= \int_0^L \int_A \boldsymbol{T}_3(S,\xi) \cdot \boldsymbol{\delta\Gamma}(S) \, \mathrm{d}\xi \, \mathrm{d}S + \int_0^L \int_A \boldsymbol{\delta K} \cdot [\boldsymbol{R} \times \boldsymbol{T}_3(S,\xi)] \, \mathrm{d}\xi \, \mathrm{d}S \\ &= \int_0^L \boldsymbol{N} \, \boldsymbol{\delta\Gamma} \, \mathrm{d}S + \int_0^L \boldsymbol{M} \, \boldsymbol{\delta K} \, \mathrm{d}S \end{aligned} \tag{5.43}$$

[6] See [28] page 374 for this proof.

where the following cross-sectional stress resultants per unit length (4.35) and (4.36) were introduced, according to back-rotated expressions (4.56) and (4.57)

$$\boldsymbol{N}(S) \doteq \int_A \boldsymbol{T_3}(S,\xi)\,\mathrm{d}\xi \tag{5.44}$$

$$\boldsymbol{M}(S) \doteq \int_A \boldsymbol{R} \times \boldsymbol{T_3}(S,\xi)\,\mathrm{d}\xi \tag{5.45}$$

Collecting these generalized forces in the cross-section vector of material (back-rotated) stress resultants $\boldsymbol{\sigma}^r$, and with the help of the generalized strain measure $\boldsymbol{\delta\varepsilon}$, the material internal virtual work assumes the compact form

$$\boxed{\begin{aligned}\delta W_{int} &= \int_0^L (\boldsymbol{N}\cdot\boldsymbol{\delta\Gamma} + \boldsymbol{M}\cdot\boldsymbol{\delta K})\,\mathrm{d}S \\ &= \int_0^L \boldsymbol{\sigma}^r\cdot\boldsymbol{\delta\varepsilon}^r\,\mathrm{d}S\end{aligned}} \tag{5.46}$$

which is a functional of the fields $\boldsymbol{\sigma}^r$ and $\boldsymbol{\delta\varepsilon}$. It's noteworthy that the derived beam internal virtual work expression (5.46) is fully consistent with the beam kinematic hypothesis and the finite elasticity theory conceived.

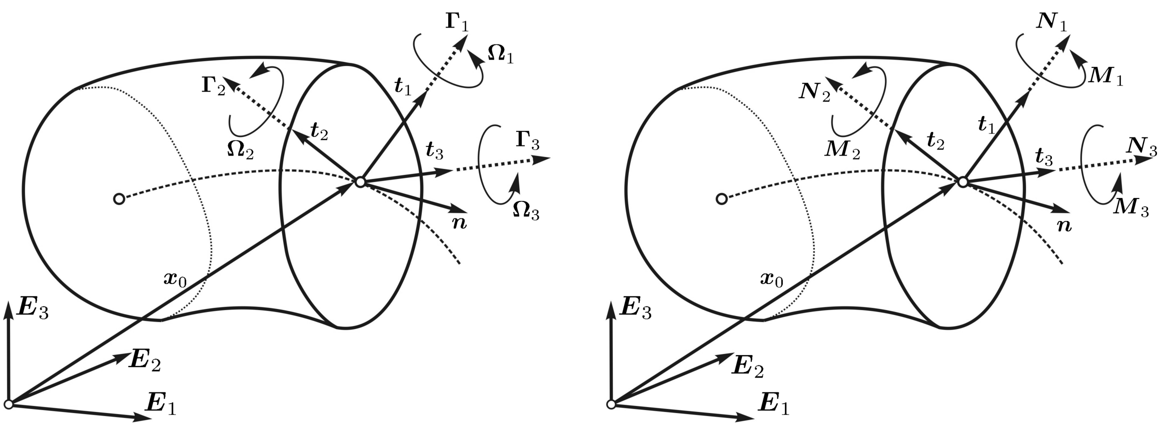


Figure 5.3: Geometric representation of static and kinematic quantities for beams. (a): reduced strains components. (b): stress resultant components.

Finally one may note how derivations (5.43) are obtained basically substituting a prescription of the three-dimensional deformation, enclosed in the kinematic assumptions, into the three-dimensional deformation gradient. This in turn leads to a natural separation of kinematics of the chosen reference curve (mostly the line of centroids) and the points lying in cross-sections (see [68]). Then, the three-dimensional principle of virtual works changes the integral order from three to one if the over-cross-section integration is applied (see [57]). Naturally, the stress quantities resulting from the cross section integrals, supply the one-dimensional statics.
recapitulating, in such an approach, the kinematics is constrained to comply with the assumed hypothesis and the conjugate statics from the assumed hypothesis via the principle of virtual works. Thereby the one-dimensional principle of virtual works inherently contains approximations due to the primary assumptions.

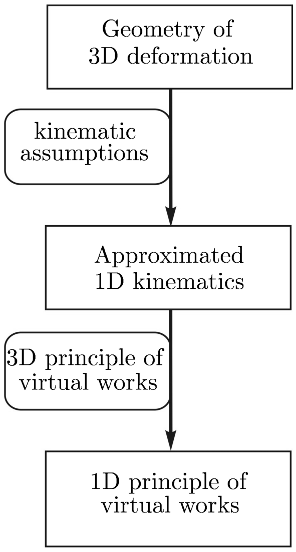


Figure 5.4: Derivation of the geometrically exact beam theory: classical approach with kinematic hypothesis according to [56] and [16].

External virtual work. Suppose that surface and body loadings are applied along the rod. The surface loading per unit of reference area is mapped from lateral contour $\boldsymbol{\Gamma}$ is denoted by $\bar{\boldsymbol{T}}_{\boldsymbol{N}}$, while the body loading per unit reference volume is mapped from $\boldsymbol{\Omega}_0$ and identified by $\boldsymbol{B}$.

Let consider the external virtual work written in current configuration, δW_{ext} (see equation (5.22)). The term associated with the body force, $\int_{\Omega} \boldsymbol{b} \cdot \delta \boldsymbol{u}\, \mathrm{d}v$, can be expressed in term of an integral over the reference domain Ω_0 using again the relation ($\mathrm{d}v = J\,\mathrm{d}V$) between current and reference differential volume

$$\int_{\Omega} \boldsymbol{b} \cdot \delta \boldsymbol{x}\, \mathrm{d}v = \int_{\Omega_0} \boldsymbol{B} \cdot \delta \boldsymbol{x}\, \mathrm{d}V \qquad \text{with} \qquad \boldsymbol{B} = J\, \boldsymbol{b} \tag{5.47}$$

where $\delta \boldsymbol{x}$ is the perturbed displacement.

To express the boundary work $\int_{\partial\Omega_{\boldsymbol{\sigma}}} \bar{\boldsymbol{t}}_{\mathbf{n}} \cdot \delta \boldsymbol{x}\, \mathrm{d}s$ as an integral over the reference domain, we recall that by definition of differential force $\mathrm{d}\boldsymbol{f}$ and making use of the Cauchy's theorem, we have

$$\mathrm{d}\boldsymbol{f} = \boldsymbol{t}_{\boldsymbol{n}}\, \mathrm{d}s = \boldsymbol{T}_{\boldsymbol{N}}\, \mathrm{d}S \qquad \text{with} \qquad \boldsymbol{t}_{\boldsymbol{n}} = \boldsymbol{\sigma} \cdot \boldsymbol{n} \quad \text{and} \quad \boldsymbol{T}_{\boldsymbol{N}} = \boldsymbol{P} \cdot \boldsymbol{N} \tag{5.48}$$

Specializing these relations for the boundary region where tractions are assigned, we get

$$\bar{\boldsymbol{t}}_{\boldsymbol{n}} \mathrm{d}s = \bar{\boldsymbol{T}}_{\boldsymbol{N}} \mathrm{d}S$$

and consequentially

$$\int_{\partial\Omega_{\bar{\boldsymbol{t}}_{\boldsymbol{n}}}} \bar{\boldsymbol{t}}_{\boldsymbol{n}} \cdot \delta \boldsymbol{x}\, \mathrm{d}s = \int_{\partial\Omega_{\bar{\boldsymbol{T}}_{\boldsymbol{N}}}} \bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot \delta \boldsymbol{x}\, \mathrm{d}S \tag{5.49}$$

Finally, substituting the results (5.47) and (5.49) into the expression of the external work (5.22), we finally get the external virtual work written on *reference configuration*

$$\delta W_{ext} = \int_{\Omega_0} \boldsymbol{B} \cdot \boldsymbol{\delta x} \, \mathrm{d}V + \int_{\partial\Omega_{\bar{\boldsymbol{T}}_{\boldsymbol{N}}}} \bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot \boldsymbol{\delta x} \, \mathrm{d}S \tag{5.50}$$

With the help of equation (3.49), and splitting the surface integral between the base surface A_b and the lateral surface A_l, the previous can be recast into

$$\begin{aligned}
\delta W_{ext} &= \int_V \boldsymbol{B} \cdot \boldsymbol{\delta x} \, \mathrm{d}V + \int_{A_l} \bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot \boldsymbol{\delta x} \, \mathrm{d}S + \int_{A_b} \bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot \boldsymbol{\delta x} \, \mathrm{d}S \\
&= \int_V \boldsymbol{B} \cdot \boldsymbol{\delta x} \, \mathrm{d}V + \int_{A_l} \bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot \boldsymbol{\delta x} \, \mathrm{d}S + \left[\bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot \boldsymbol{\delta x}\right]_0^L \\
&= \int_0^L \int_A \boldsymbol{B} \cdot (\boldsymbol{\delta u} + \boldsymbol{\delta \theta} \times \boldsymbol{r}) \, \mathrm{d}A \, \mathrm{d}S + \\
&\quad + \int_0^L \int_\Gamma \bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot (\boldsymbol{\delta u} + \boldsymbol{\delta \theta} \times \boldsymbol{r}) \, \mathrm{d}\Gamma \, \mathrm{d}S + \\
&\quad + \left[\bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot (\boldsymbol{\delta u} + \boldsymbol{\delta \theta} \times \boldsymbol{r})\right]_0^L \\
&= \int_0^L \left[\left(\int_A \boldsymbol{B} \, \mathrm{d}A + \int_\Gamma \bar{\boldsymbol{T}}_{\boldsymbol{N}} \, \mathrm{d}\Gamma \right) \cdot \delta \boldsymbol{u} + \right. \\
&\quad \left. + \left(\int_A \boldsymbol{r} \times \boldsymbol{B} \, \mathrm{d}A + \int_\Gamma \boldsymbol{r} \times \bar{\boldsymbol{T}}_{\boldsymbol{N}} \, \mathrm{d}\Gamma \right) \cdot \boldsymbol{\delta \theta} \right] \mathrm{d}S + \\
&\quad + \left[\bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot \boldsymbol{\delta u} + \boldsymbol{r} \times \bar{\boldsymbol{T}}_{\boldsymbol{N}} \cdot \boldsymbol{\delta \theta}\right]_0^L \\
&= \int_0^L \left[\bar{\boldsymbol{N}} \cdot \boldsymbol{\delta u} + \bar{\boldsymbol{M}} \cdot \boldsymbol{\delta \theta}\right] \mathrm{d}S + \left[\widehat{\boldsymbol{N}} \cdot \boldsymbol{\delta u} + \widehat{\boldsymbol{M}} \cdot \boldsymbol{\delta \theta}\right]_0^L
\end{aligned} \tag{5.51}$$

where was introduced equations (4.47) and (4.52) using $\bar{\boldsymbol{f}} = \bar{\boldsymbol{N}}$ and $\bar{\boldsymbol{m}} = \bar{\boldsymbol{M}}$ only for fake of clearness in the formalism. We recall that $\bar{\boldsymbol{N}}$ denotes the applied external forces acting on a part of the boundary of the beam, by unit length in the reference configuration, whereas $\bar{\boldsymbol{M}}$ represents the applied external moments acting on the beam by unit length of the reference configuration. similarly we have introduced

$$\widehat{\boldsymbol{N}} \doteq \bar{\boldsymbol{T}}_{\boldsymbol{N}} \tag{5.52}$$

$$\widehat{\boldsymbol{M}} \doteq \boldsymbol{r} \times \bar{\boldsymbol{T}}_{\boldsymbol{N}} \tag{5.53}$$

which can be interpreted as the external applied loads acting on the beam's ends (in the sense of the *static boundary conditions*). In fact, we note that in (5.51) the natural boundary conditions at the rod ends were directly accounted.
Introducing now equation (2.98) into equation (5.51), one has

$$\delta W_{ext} = \int_0^L \left[\bar{\boldsymbol{N}} \cdot \boldsymbol{\delta u} + \bar{\boldsymbol{\mu}}^r \cdot \boldsymbol{\delta \psi}\right] \mathrm{d}S + \left[\widehat{\boldsymbol{N}} \cdot \boldsymbol{\delta u} + \widehat{\boldsymbol{\mu}}^r \cdot \boldsymbol{\delta \psi}\right]_0^L \tag{5.54}$$

where the vectors emerging

$$\bar{\boldsymbol{\mu}}^r = \boldsymbol{T}^T \bar{\boldsymbol{M}} \tag{5.55}$$

$$\widehat{\boldsymbol{\mu}}^r = \boldsymbol{T}^T \widehat{\boldsymbol{M}} \tag{5.56}$$

are the distributed external pseudo-moments which are energetically conjugated with $\boldsymbol{\psi}$. Notice that the true work conjugate of $\boldsymbol{\psi}$ is not simply the moment resultants as usually happens on geometrically linear theories.
Introducing now the vector of external loading per unit of reference length, and the vector of external loading on the bases defined by

$$\bar{\boldsymbol{q}}^r = \begin{bmatrix} \bar{\boldsymbol{N}} \\ \bar{\boldsymbol{\mu}}^r \end{bmatrix} \qquad \widehat{\boldsymbol{q}}^r = \begin{bmatrix} \widehat{\boldsymbol{N}} \\ \widehat{\boldsymbol{\mu}}^r \end{bmatrix} \tag{5.57}$$

with the help of (3.25) the external virtual work can be written as

$$\boxed{\delta W_{ext} = \int_0^L \bar{\boldsymbol{q}}^r \cdot \boldsymbol{\delta\eta} \, \mathrm{d}S + [\widehat{\boldsymbol{q}}^r \cdot \boldsymbol{\delta\eta}]_0^L} \tag{5.58}$$

where $\boldsymbol{\delta\eta}$ stands for an infinitesimal perturbation of generalized displacement field.

Inertial virtual work. Inertia forces can be expressed either in spatial or material components, where the latter give their expression in the inertial frame oriented according to the direction of material axes at the time instant considered. Mutatis mutandis for the former. We remark that the denomination "spatial" or "material" are employed only to indicate the way in which quantities are handled.
However, we choose the material representation as it seems more "natural" and "convenient" to express the virtual work of inertia forces for dynamic problems.

To express the inertial virtual work we need for instance the time derivative of the spatial place vector $\boldsymbol{x}$ with respect to the inertial frame. This approach allows to bypass completely the use of a floating reference frame and avoid complications arising from accounting Coriolis' and centrifugal effects. Making use of equations (2.106) and (2.127) one obtains

$$\boldsymbol{x} = \boldsymbol{x}_0 + \boldsymbol{r} = \boldsymbol{x}_0 + \boldsymbol{\Lambda}\boldsymbol{R} \tag{5.59}$$

$$\dot{\boldsymbol{x}} = \dot{\boldsymbol{x}}_0 + \boldsymbol{\Lambda}\tilde{\boldsymbol{\Omega}}\boldsymbol{R} \tag{5.60}$$

$$\ddot{\boldsymbol{x}} = \ddot{\boldsymbol{x}}_0 + \boldsymbol{\Lambda}\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}\boldsymbol{R} + \boldsymbol{\Lambda}\tilde{\boldsymbol{A}}\boldsymbol{R} \tag{5.61}$$

where $\tilde{\boldsymbol{\Omega}}, \tilde{\boldsymbol{A}} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{spat}} SO(3)$. It's easy to proof that $\dot{\boldsymbol{x}}, \ddot{\boldsymbol{x}} \in \mathcal{T}_{\boldsymbol{\Lambda}}^{\mathrm{spat}}$.
Now, from equation (5.23), deciding to describe the acceleration with respect to the inertial fixed frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ such as the formal equivalence $\ddot{\mathbf{u}} = \ddot{\mathbf{x}}$ holds, the virtual work done by inertia forces becomes

$$\delta W_{inert} = \int_{\Omega_0} \rho_0 \, \ddot{\mathbf{x}} \cdot \boldsymbol{\delta x} \, \mathrm{d}V \tag{5.62}$$

and substituting the virtual displacement vector $\boldsymbol{\delta x}$ given in (3.50), and the acceleration vector $\ddot{\boldsymbol{x}}$ from equation (5.61), into the virtual work of inertia forces, and decomposing the volume integral into the line integral along the beam axes (domain

$[0, L]$) and the surface integral over the generic cross-section (domain A), we obtain

$$\begin{aligned}
\delta W_{inert} &= \int_V \rho_0 \, \ddot{\mathbf{x}} \cdot \boldsymbol{\delta x} \, \mathrm{d}V \\
&= \int_V \rho_0 \, (\ddot{\boldsymbol{x}}_0 + \boldsymbol{\Lambda}\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}\boldsymbol{R} + \boldsymbol{\Lambda}\tilde{\boldsymbol{A}}\boldsymbol{R}) \cdot (\boldsymbol{\delta u} + \boldsymbol{\Lambda}\delta\tilde{\boldsymbol{\Theta}}\boldsymbol{R}) \, \mathrm{d}V \\
&= \int_V \boldsymbol{\delta u} \cdot \rho_0 \, (\ddot{\boldsymbol{x}}_0 + \boldsymbol{\Lambda}\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}\boldsymbol{R} + \boldsymbol{\Lambda}\tilde{\boldsymbol{A}}\boldsymbol{R}) + \\
&\qquad + \rho_0(\ddot{\boldsymbol{x}}_0 + \boldsymbol{\Lambda}\tilde{\boldsymbol{\Omega}}\tilde{\boldsymbol{\Omega}}\boldsymbol{R} + \boldsymbol{\Lambda}\tilde{\boldsymbol{A}}\boldsymbol{R}) \cdot (\boldsymbol{\Lambda}\delta\tilde{\boldsymbol{\Theta}}\boldsymbol{R}) \, \mathrm{d}V \\
&= \int_0^L \boldsymbol{\delta u} \cdot A_\rho \ddot{\boldsymbol{x}}_0 \, \mathrm{d}S + \int_0^L \boldsymbol{\delta\Theta} \cdot (\boldsymbol{J}_\rho \boldsymbol{A} + \tilde{\boldsymbol{\Omega}}\boldsymbol{J}\boldsymbol{\Omega}) \, \mathrm{d}S
\end{aligned} \tag{5.63}$$

where we have been used the center line condition, for which the static moment vanishes if the beam reference curve is chosen as the geometric centroid line of the beam cross-section, that is

$$\int_A \tilde{\boldsymbol{R}} \, \mathrm{d}A = \int_A (X_1 \tilde{\boldsymbol{R}}_1 + X_2 \tilde{\boldsymbol{R}}_2) \, \mathrm{d}A = \tilde{\mathbf{0}} \tag{5.64}$$

in the first term and the identity (A.14). Moreover we have denoted the scalar mass density per unit length of the reference beam

$$A_\rho = \int_A \rho_0 \, \mathrm{d}A \tag{5.65}$$

and the inertial tensor $\boldsymbol{J}_\rho$ of the cross-section, also defined as the second mass moment density (second-order tensor) per unit length of the reference configuration given by (see appendix E)

$$\boldsymbol{J}_\rho = [J_{ij}] \, \boldsymbol{E}_i \otimes \boldsymbol{E}_j \tag{5.66}$$

where

$$[J_{ij}] = \int_A \begin{bmatrix} X_1^2 & 0 & 0 \\ 0 & X_2^2 & 0 \\ 0 & 0 & X_1^2 + X_2^2 \end{bmatrix} \rho_0 \, \mathrm{d}A \tag{5.67}$$

in the case in which the principal axes of the inertial tensor $\boldsymbol{J}_\rho$ are parallel to the basis vectors $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$. It can be viewed as the rotational mass (second order mass moment density) per unit length of the beam.
We note the correspondence between the virtual work expressed by equation (5.63), and the equations of motion for a rigid body. The formula (5.63) can be viewed as the lengthwise parametrized equations for the rigid body without external forces.
Finally, we derive the inertial virtual work form in terms of the total material rotation $\boldsymbol{\Psi}$ and its virtual variation $\boldsymbol{\delta\Psi}$ in order to achieve the total Lagrangian formulation. We need to express spin vectors such as the virtual incremental rotation vector $\boldsymbol{\delta\Theta}$ (see equation (2.92)), the angular velocity vector $\boldsymbol{\Omega}$ (see equation (2.117)), the angular

acceleration vector $\boldsymbol{A}$ (see equation (2.129)), and the curvature vector $\boldsymbol{K}$ (see equation (2.144)) in terms of the total rotation vector, giving

$$\boldsymbol{\delta\Theta} = \boldsymbol{T}^T \cdot \boldsymbol{\delta\Psi} \qquad \boldsymbol{\Omega} = \boldsymbol{T}^T \cdot \dot{\boldsymbol{\Psi}}$$
$$\boldsymbol{A} = \boldsymbol{T}^T \cdot \ddot{\boldsymbol{\Psi}} + \dot{\boldsymbol{T}}^T \cdot \dot{\boldsymbol{\Psi}} \qquad \boldsymbol{K} = \boldsymbol{T}^T \boldsymbol{\Psi}'$$

We note that $\boldsymbol{\delta\Psi}, \boldsymbol{\Psi}, \dot{\boldsymbol{\Psi}}, \ddot{\boldsymbol{\Psi}} \in \mathcal{T}_I^{\text{mat}}$, whereas spin vectors $\boldsymbol{\delta\Theta}, \boldsymbol{\Omega}, \boldsymbol{A} \in \mathcal{T}_\Lambda^{\text{mat}}$. Now, we can write the virtual work of inertia forces δW_{inert} (5.63) in terms of the total material vector, yielding (details can be consulted in [36] and reference therein)

$$\begin{aligned} \delta W_{inert} = & \int_0^L \boldsymbol{\delta u} \cdot A_\rho \ddot{\boldsymbol{x}}_0 \ \mathrm{d}S + \\ & + \int_0^L \boldsymbol{\delta\Psi} \cdot (\boldsymbol{T}\boldsymbol{J}\boldsymbol{T}^T \ddot{\boldsymbol{\Psi}} + \boldsymbol{T}\boldsymbol{J}\dot{\boldsymbol{T}}^T \dot{\boldsymbol{\Psi}} + \boldsymbol{T}(\boldsymbol{T}^T\dot{\boldsymbol{\Psi}})^\sim \boldsymbol{J}\boldsymbol{T}^T\dot{\boldsymbol{\Psi}}) \ \mathrm{d}S \\ = & \int_0^L \boldsymbol{G}_{inert} \cdot \boldsymbol{\delta\eta} \ \mathrm{d}S \end{aligned} \tag{5.68}$$

where terms are grouped in the inertia force vector $\boldsymbol{G}_{inert}$ and the virtual generalized displacement vector $\boldsymbol{\delta\eta}$. The notation $(\tilde{\cdot}) = (\cdot)^\sim$ has been used for the skew-symmetric tensor obtained from certain "large" arguments.

The summation of the virtual works (5.46), (5.58) and (5.63) according to the virtual work theorem (5.20) gives the final weak form $\delta W_{tot} = \delta W_{int} + \delta W_{ext} + \delta W_{inert} = 0$, $\forall \boldsymbol{\delta\eta}$ (in particular for $\boldsymbol{\delta\eta} = \boldsymbol{0}$ on the kinematic boundary points $\partial\Omega_u$). It coincides with the variational form of the internal power written in terms of reduced strains and stress resultants (see [56]). Note that the final form of the principle of virtual works is, up to now absolutely general, free from any restrictions or assumptions.

5.2 Constitutive equations

Having the equilibrium conditions expressed in a weak form, the stress strain constitutive relations should now be supplied. Constitutive equations provides an additional information about the material and geometrical properties of the body under consideration, and completes the formulation of the boundary value problem for the reduced three-dimensional beam theory.

One has to recognize that a constitutive law is, from its own nature, an experimental concept. It is based on material properties, which are measured in some specific conditions and are always approximate. In the case of present derived theory, not only material properties are involved in the relations, but also some geometrical features [7]. For example, note how the popular assumption about cross-sections of the beam remain plane during the deformation affects the constitutive law.

Not entering deeply into the complicated problem of the reduction of constitutive relations, but anyhow enabling the numerical calculations, let us confine our atten-

[7]The presence of geometry in constitutive relations is expressed via the area and various inertia moments.

tion to the *hyperelastic* [8], *isotropic* and *homogeneous* material that may undergoes arbitrarily large strains. We postulate the existence of a stored energy function called *specific strain energy function* per unit volume of the reference configuration depending on $\Psi = \Psi(\boldsymbol{\varepsilon}^r)$. In particular, an elastic material is said to be a *hyperelastic* or a *Green–elastic* material if exists a scalar valued function $\Psi_{int} = \Psi_{int}(\boldsymbol{\varepsilon}^r)$ such that

$$\boldsymbol{\sigma}^r \triangleq \frac{\partial \Psi_{int}}{\partial \boldsymbol{\varepsilon}^r} \tag{5.69}$$

which in turn can be recast in

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \partial \boldsymbol{\Psi} / \partial \boldsymbol{\Gamma} \\ \partial \boldsymbol{\Psi} / \partial \boldsymbol{K} \end{bmatrix} \tag{5.70}$$

Here $(\cdot)^r$ represents the rotated-back quantity. The gradient of $\boldsymbol{\sigma}^r$ defines an operator $\boldsymbol{D}^r$ called material elasticity tensor, given by

$$\boldsymbol{D}^r \triangleq \frac{\partial \boldsymbol{\sigma}^r}{\partial \boldsymbol{\varepsilon}^r} = \begin{bmatrix} \partial \boldsymbol{N} / \partial \boldsymbol{\Gamma} & \partial \boldsymbol{N} / \partial \boldsymbol{K} \\ \partial \boldsymbol{M} / \partial \boldsymbol{\Gamma} & \partial \boldsymbol{M} / \partial \boldsymbol{K} \end{bmatrix} \tag{5.71}$$

i.e. $\boldsymbol{D}^r$ contains the cross-section elastic tangent moduli. Recalling equation (5.69), $\boldsymbol{D}^r$ is also given by

$$\boldsymbol{D}^r = \frac{\partial^2 \Psi_{int}}{\partial {\boldsymbol{\varepsilon}^r}^2} = \begin{bmatrix} \frac{\partial^2 \Psi}{\partial \boldsymbol{\Gamma}^2} & \frac{\partial^2 \Psi}{\partial \boldsymbol{\Gamma} \partial \boldsymbol{K}} \\ \frac{\partial^2 \Psi}{\partial \boldsymbol{K} \partial \boldsymbol{\Gamma}} & \frac{\partial^2 \Psi}{\partial \boldsymbol{K}^2} \end{bmatrix} \tag{5.72}$$

Thus, $\boldsymbol{D}^r$ is symmetric. For linear elastic beams $\boldsymbol{D}^r$ is diagonal with constant coefficients, hence, for such rods, the quadratic (uncoupled) functional in the material variables $\Psi_{int} = 1/2(\boldsymbol{\varepsilon}^r \cdot \boldsymbol{D}^r \boldsymbol{\varepsilon}^r)$ reduces to

$$\boldsymbol{\sigma}^r = \boldsymbol{D}^r \boldsymbol{\varepsilon}^r \tag{5.73}$$

Assuming the rod axis coincides with the line of the cross-sectional centroids, and the cross-section axis lies parallel to the principal axis of inertia, the matrix $\boldsymbol{D}^r$ may be given in the beams's local system by

$$\boldsymbol{D}^r = \begin{bmatrix} EA & 0 & 0 & 0 & 0 & 0 \\ 0 & GA_1^* & 0 & 0 & 0 & 0 \\ 0 & 0 & GA_2^* & 0 & 0 & 0 \\ 0 & 0 & 0 & EJ_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & EJ_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & GJ_T \end{bmatrix} \tag{5.74}$$

in which A, A_1^*, A_2^*, J_1, J_2, J_T are the cross-sectional geometric properties. In particular, the cross-sectional geometric area A_1^* and A_2^* include shear factor χ, and the torsional moment J_{SV} refers to the classical De Saint-Venant torsion theory, instead

[8]When the work done by the stresses during a deformation process is dependent only on the initial state and the final configuration, the behavior of the material is said to be *path-independent*, and the material is termed *hyperelastic*.

of the polar moment $J_p = I_1 + I_2$. Consequently, we can recognize in EA the axial stiffness, GA_1^* and GA_2^* the shear stiffness with respect to the cross-section in principal directions, GJ_T the torsional stiffness, and EJ_1 and EJ_2 are the principal bending stiffness. Coefficients E and G are the material elastic moduli, Young's elastic modulus and shear modulus, respectively. It's worth to note these kind of constitutive equation, which is standard for an elastic beam in small displacements, completely neglects the effects of strain D_{ij} in directions $i \neq j$, i.e. the stress tensor components depend only on the strain components in the same direction and the Poisson coefficient ν doesn't appear in the law. All of this is in agreement with the kinematic hypothesis of lack of section deformability.

If an elastoplastic material law is to be considered, an incremental solution is required to handle the path-dependent evolution laws of the internal variables. Moreover, to account for the spread of yielding, numerical integration over the cross-section is also required. Nevertheless, plastic behavior in beams is often restricted to localized areas (e.g. plastic hinge zones in frame structures), which means that the cross-section integration only needs to be performed in those areas.

According to [53], since Reissner-Simo beam theory does not take into account the cross-section in-plane deformation, the integration of the elastoplastic constitutive equation requires the use of mixed stress-strain control, in which the out-of-plane strains imposed to the cross-section are complemented with null in-plane stresses.

EQUILIBRIUM EQUATIONS (Balance equations)

- Strong form (*differential*)
  i. Material: $\mathrm{DIV}\boldsymbol{P}+\mathbf{B}=\rho_0\,\ddot{\mathbf{U}}$ and $\boldsymbol{P}\boldsymbol{F}^T=\boldsymbol{F}\boldsymbol{P}^T$

  ii. Spatial: $\mathrm{div}\boldsymbol{\sigma}+\mathbf{b}=\rho\,\ddot{\mathbf{u}}$ and $\boldsymbol{\sigma}=\boldsymbol{\sigma}^T$
- Strong form (*differential*) in terms of resultants
  i. Spatial:

$$\begin{cases} \boldsymbol{f'}+\bar{\boldsymbol{f}}=A_\rho\ddot{\boldsymbol{u}} & \text{in} \quad \boldsymbol{\Omega} \\ \boldsymbol{m'}+\boldsymbol{x'_0}\times\boldsymbol{f}+\bar{\boldsymbol{m}}=\dot{\boldsymbol{\mu}} & \text{in} \quad \boldsymbol{\Omega} \\ \boldsymbol{f}=\bar{\boldsymbol{f}}_\Gamma & \text{on} \quad \partial\Omega_\sigma\equiv\Gamma_\sigma \\ \boldsymbol{m}=\bar{\boldsymbol{m}}_\Gamma & \text{on} \quad \partial\Omega_\sigma\equiv\Gamma_\sigma \\ \boldsymbol{u}=\bar{\boldsymbol{u}}_\Gamma & \text{on} \quad \partial\Omega_u\equiv\Gamma_u \end{cases}$$

  – Force resultant of internal forces over the cross-section:

$$\boldsymbol{f}(s,t)=\int_S \boldsymbol{T_3}(s,\boldsymbol{r})\,\mathrm{d}x$$

  – Torque resultant of internal forces over the cross-section:

$$\boldsymbol{m}(s,t)\doteq\int_S \boldsymbol{r}\times\boldsymbol{T_3}(s,\boldsymbol{r})\,\mathrm{d}$$

  – External force resultant over the boundary:

$$\bar{\boldsymbol{f}}(s,t)=\sum_{\Gamma=1}^{2}\int_{\partial\Omega_0}\boldsymbol{T_\Gamma}\cdot\boldsymbol{N_\Gamma}\,\mathrm{d}\Gamma+\int_{\Omega_0}\boldsymbol{B}\,\mathrm{d}x$$

  – External moment resultant over the boundary:

$$\bar{\boldsymbol{m}}(s,t)=\sum_{\Gamma=1}^{2}\int_{\partial\Omega_0}\boldsymbol{r}\times(\boldsymbol{T_\Gamma}\cdot\boldsymbol{N_\Gamma})\,\mathrm{d}\Gamma+\int_{\Omega_0}\boldsymbol{r}\times\mathbf{B}\,\mathrm{d}x$$

- Weak form (*integral*)
  i. Material:

$$\int_{\Omega_0}\boldsymbol{P}:\delta\boldsymbol{F}\,\mathrm{d}V-\int_{\Omega_0}\boldsymbol{B}\cdot\delta\boldsymbol{x}\,\mathrm{d}V-\int_{\partial\Omega_{0,\boldsymbol{\sigma}}}\bar{\boldsymbol{T}}_{\boldsymbol{N}}\cdot\delta\boldsymbol{x}\,\mathrm{d}S+ \\ +\int_{\Omega}\rho_0\,\ddot{\mathbf{U}}\cdot\delta\boldsymbol{x}\,\mathrm{d}V=0 \qquad \forall\,\delta\boldsymbol{x}\in\delta\mathcal{U}$$

  ii. Spatial:

$$\int_{\Omega}\boldsymbol{\sigma}:\delta\boldsymbol{e}\,\mathrm{d}v-\int_{\Omega}\boldsymbol{b}\cdot\delta\boldsymbol{x}\,\mathrm{d}v-\int_{\partial\Omega_{\boldsymbol{\sigma}}}\bar{\boldsymbol{t}}_{\mathbf{n}}\cdot\delta\boldsymbol{x}\,\mathrm{d}s+ \\ +\int_{\Omega}\rho\,\ddot{\mathbf{u}}\cdot\delta\boldsymbol{x}\,\mathrm{d}v=0 \qquad \forall\,\delta\boldsymbol{u}\in\delta\mathcal{U}$$

Part II

Finite Element Method

Chapter 6

Spatial discretization: Finite Element Method

Come forth into the light of things,
Let nature be your teacher.

W. Wordsworth

The main purpose of this part of the work is to present a Galerkin discretization of the virtual work functional described in previous Chapter. To this end, a brief introduction to the classical FEM spatial discretization is firstly made, along with an extensive discussion devoted to the isoparametric Galerkin approximation. Hence, the direct derivation of the finite element approximation of the out-of-balance forces is also presented for dynamic problems, starting from the principle of virtual works. Finally, a section devoted to the solution methods is included, providing a general overview of the common approach for solving the non-linear system of algebraic equations which gives rise form the discretization with FEM.

6.1 Finite element Galerkin spatial discretization

The description of the dynamics of a geometrically exact beam generates an initial boundary value problem (e.g. see (5.27)) whose weak form (see equation (5.31)) can be solved by several numerical approximation techniques, with the help of a

proper constitutive equation. The displacement-based finite element incremental solution procedure is often the only one capable of solving such a problem. Basically, a finite element solution is advantageous since, if we select a discrete number of nodes on the beam to compute the displacements (problem's unknowns), the nonlinear [1] differential equations which govern the dynamics of problem turn into a nonlinear algebraic system of equations, solvable applying a incremental/iterative numerical approach. The attribute *algebraic* here means that this system contains a *finite* number of equations and unknowns. Physically the algebraic system represents equilibrium of forces at the discrete level, more specifically, for discrete model coming from finite element method, the equilibrium of nodal forces. These are collectively known as *force residual equations*.
The manner in which the finite element method is treated within the context of classical Galerkin approach is described hereafter.

6.1.1 Spatial discretization

Following the classical approach, integrals over a given domain which arise from the application of finite element method can be regarded as a sum of integrals over a set of subdomain called element. Accordingly, we let the continuous body subdivided into N_{el} finite elements

$$\Omega \;\approx\; \Omega_h \;=\; \bigcup_{e=1}^{N_{el}} \Omega_e \tag{6.1}$$

where Ω_h is the approximation to the domain created by the set of elements, Ω_e is the domain of a typical element, and N_{el} is the total number of elements. Overlapping of finite elements is not allowed. The boundary of the region ∂B_h consist of curves or areas $\partial\Omega_e$ of the element Ω_e such as (see Figure 6.1)

$$\partial\Omega \;\approx\; \partial\Omega_h \;=\; \bigcup_{e=1}^{N_{el}} \partial\Omega_e \tag{6.2}$$

Integrals may now be summed over each elements and written as

$$\int_\Omega (\cdot)\,\mathrm{d}\Omega \;\approx\; \int_{\Omega_h} (\cdot)\,\mathrm{d}\Omega_h \;=\; \bigcup_{e=1}^{N_{el}} \int_{\Omega_e} (\cdot)\,\mathrm{d}\Omega \tag{6.3}$$

In this way calculations are performed on the element basis, and the final solution is assembled according to equation (6.3). The operator $\cup$ is chosen here instead of the $\sum$ symbol in order to denote that an assembly process takes place in which all element contributions have not only to be added up, but also the kinematical compatibility between the elements has to be fulfilled.

[1]The attribute *nonlinear* stands to recall the nonlinear nature of rotation manifold, which "contaminates" also the way in which balance equations are established with reference to the deformed configuration of beam. This effectively amounts to using a nonlinear theory.

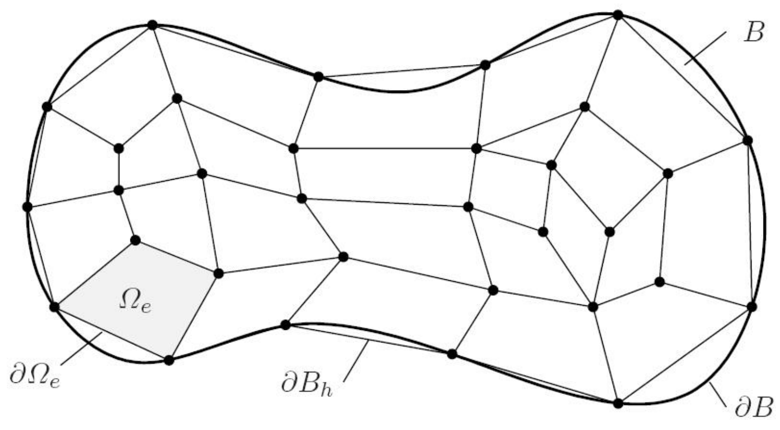


Figure 6.1: General discretization of domain Ω.

In the context of beam element, the domain $[0, L]$ is divided into $N_{el} = N - 1$ subintervals as

$$[0, L] = \bigcup_{e=1}^{N_{el}} I_e^{L_e} \qquad \text{with} \quad I_i^{L_e} \cap I_j^{L_e} = \emptyset,\ \forall i, j \in \{1, \ldots, N_{el}\} \tag{6.4}$$

where $I_e^{L_e} \subset [0, L_e]$ denotes a typical element with length $L_e > 0$, and N_{el} is the total number of elements.

6.1.2 Local parametrization of beam axes

Let consider a finite element discretization of the reference line of centroids $\mathscr{C}_0(\xi) : [0, L_e]$ consisting of N nodal points with position vector $\boldsymbol{X}_n$, where $n = 1, 2, \ldots, N$. For a given time instant t, the parametrization of $\mathscr{C}_0$ could be done with the mapping

$$\begin{aligned} \xi \in [0, L] \longmapsto \boldsymbol{X}(\xi, t) &= \sum_{i=1}^{N} N_i(\xi)\, \hat{\boldsymbol{X}}_i(t) \\ &= N_i(\xi)\, \hat{\boldsymbol{X}}_i(t) \end{aligned} \tag{6.5}$$

where Einstein's summation convention extending over nodes 1 to N was used. Here $N_i(\xi)$ denotes the standard global finite shape function, ξ is the natural coordinate, whereas $\hat{\boldsymbol{X}}_i$ is the cartesian coordinates of $i-$th node.
It should be emphasized that during the motion, nodes and elements are permanently attached to the material particles with which they were initially associated. Consequently, the subsequent motion is fully described in terms of the current position $\boldsymbol{x}(t)$ of the nodal particles as

$$\boxed{\boldsymbol{x}(\xi, t) = \sum_{i=1}^{N} N_i(\xi)\, \hat{\boldsymbol{x}}_i(t)} \tag{6.6}$$

Let N_{el} be the total number of elements in the discretization, each of them possessing n_e nodes, and let $ID(a,e)$ be the $(n_e \times N_{el})$ array that relates global and local element node numbers according to the usual mapping

$$A = ID(a,e) \qquad \text{for} \quad a = 1,\dots,n_e \quad \text{and} \quad e = 1,\dots,N_{el} \tag{6.7}$$

Denoting by $N_a^e(\xi)$ the n_e element shape functions with $\xi \in [0, L_e]$, the global shape functions are defined over each element by the relation

$$N^A(\xi)|_{\Omega_e} = N_a^e(\xi) \quad \text{for} \quad A = ID(a,e) \tag{6.8}$$

which ensures C^0 continuity of the discretization. The element shape functions are subjected to the usual completeness conditions

$$N_a^e(\xi_b) = \delta_a^b \quad \text{and} \quad \sum_{i=1}^{n_e} N_i^e(\xi) = 1 \tag{6.9}$$

where $\xi_b \in [0, L_e]$, $b = 1,\dots,n_e$ are the isoparametric coordinates of the nodes within the element.

6.1.3 Isoparametric Galerkin approximation

In this work we adopt an isoparametric interpolation by adopting the same shape functions used to describe the element geometry and the displacement field, which allows one to construct very easily any deformed configuration. Hence, in a pure displacement finite element model, the generalized displacement field solution $\boldsymbol{\eta} = (\boldsymbol{u}, \boldsymbol{\Psi})$, which is regarded as the primary field variable in the problem, is interpolated (approximated) within one finite element $e = 1,\dots,N_e$ basing on the nodal quantities

$$\boldsymbol{\eta}_{exact} \approx \boldsymbol{\eta}^e(\xi,t) = \boldsymbol{N}(\xi)\cdot\hat{\boldsymbol{\eta}}^e(t) \tag{6.10}$$

$$= \sum_{i=1}^{n_e} N_i(\xi)\hat{\eta}_i^e(t) \tag{6.11}$$

$$= \boldsymbol{N}_i(\xi)\hat{\boldsymbol{\eta}}_i^e(t) \tag{6.12}$$

where ξ denotes the independent variable in natural coordinate corresponding to the beam arc-length parameter, t is a generic time instant, n_e is still the number of nodes of the element, $\hat{\boldsymbol{\eta}}^e$ is the vector of unknown element nodal values which collects the vectors $\hat{\boldsymbol{u}}^e$ and $\hat{\boldsymbol{\Psi}}^e$ of the nodal values of the independent variables (three translational degrees of freedom $\{\hat{\boldsymbol{u}}_x, \hat{\boldsymbol{u}}_y, \hat{\boldsymbol{u}}_z\}$ and three rotational degree of freedom $\{\hat{\boldsymbol{\Psi}}_x, \hat{\boldsymbol{\Psi}}_y, \hat{\boldsymbol{\Psi}}_z\}$ for each node), and $\boldsymbol{N}$ is the interpolation matrix of element shape functions, with $N_i(\xi)$ the shape function associated to node i.

Equation (6.10) basically states how the independent model function is approximated by a linear combination of the corresponding nodal values $\boldsymbol{\eta}_i^e(t)$ via the shape functions $N_i(\xi)$.

Equation (6.10) can be spell out emphasizing the dependent variables $\boldsymbol{u}^e$ and $\boldsymbol{\Psi}^e$ and their isoparametric interpolation

$$\text{Displacement filed:} \qquad \boldsymbol{u}^e(\xi, t) = \sum_{i=1}^{n_e} N_i(\xi)\hat{u}_i^e(t) \tag{6.13}$$

$$\text{Rotation field:} \qquad \boldsymbol{\Psi}^e(\xi, t) = \sum_{i=1}^{n_e} N_i(\xi)\hat{\Psi}_i^e(t) \tag{6.14}$$

The interpolation (6.14) for the total rotation field is clearly acceptable because the nodal rotation vectors $\hat{\Psi}_i^e$ belong to the same rotational vector space. Following the same spatial discretization, the variation of generalized displacement field reads

$$\boldsymbol{\delta\eta}^e = \boldsymbol{N}(\xi)\,\boldsymbol{\delta\hat{\eta}}^e(t) \quad \longrightarrow \quad \begin{cases} \boldsymbol{\delta u}^e(\xi, t) = \sum_{i=1}^{n_e} N_i(\xi)\boldsymbol{\delta\hat{u}}_i^e(t) \\ \boldsymbol{\delta\Psi}^e(\xi, t) = \sum_{i=1}^{n_e} N_i(\xi)\boldsymbol{\delta\hat{\Psi}}_i^e(t) \end{cases} \tag{6.15}$$

In the finite element implementation of the presented beam model, we select the simplest set of linear interpolation schemes: the element reference configuration is approximated with an assembly of 2-node elements ($n_e = 2$), and a standard local element shape functions reads

$$N_1^e(\xi) = 1 - \frac{\xi}{L_e} \qquad N_2^e(\xi) = \frac{\xi}{L_e} \qquad \xi \in [0, L_e] \tag{6.16}$$

i.e. linear with respect to the element natural coordinate ξ. Note that the shape function $N_1^e(\xi)$ has the value 1 at node 1 and 0 at node 2; conversely, shape function $N_2^e(\xi)$ has the value 0 at node 1 and 1 at node 2 as depicted in Figure 6.2. It follows from the fact that element displacement interpolations are based on physical node values. Thereby, the interpolation formula (6.10) can be expressed as

$$\boldsymbol{\eta}^e(\xi) = \boldsymbol{N}^e(\xi) \cdot \boldsymbol{\hat{\eta}}^e \tag{6.17}$$

$$= \boldsymbol{N}_1^e(\xi)\boldsymbol{\hat{\eta}}_1^e + \boldsymbol{N}_2^e(\xi)\boldsymbol{\hat{\eta}}_2^e \tag{6.18}$$

$$= \begin{bmatrix} \boldsymbol{N}_1^e & \boldsymbol{N}_2^e \end{bmatrix} \begin{bmatrix} \boldsymbol{\hat{\eta}}_1^e \\ \boldsymbol{\hat{\eta}}_2^e \end{bmatrix} \tag{6.19}$$

An advantage of the developed formulation is that all interpolation operations can be performed in the local element system, while the global response is constructed by means of assemblage operations. This is done using a connectivity matrix which relates the local nodal degrees-of-freedom with the global nodal degrees-of-freedom.

Computation of derivatives. To obtain strains, and associated variations, the derivatives of the displacement field have to be computed with respect to the coordinates of the initial or current configuration. Within an element Ω_e for a formulation with respect to the initial configuration by

$$\frac{\partial \boldsymbol{\eta}^e}{\partial \eta} = \sum_{i=1}^{n_e} \frac{\partial N_i(\xi)}{\partial \xi} \boldsymbol{\hat{\eta}}_i^e \tag{6.20}$$

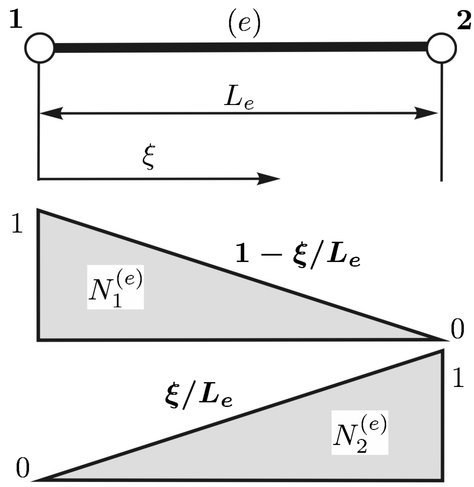


Figure 6.2: Two-node beam element with linear shape functions.

6.1.4 Uniformly reduced integration

Generally the computation of integrals for the finite element arrays is performed using numerical integration (i.e. quadrature). It is known that exactly integrated standard Galerkin finite element models, like the one introduced, presents some difficulties in integrations involving stress resultants. The source of these difficulties is well known and can be traced back to an overconstrained approximation for the transverse shear field. For this reason the phenomenon is called *shear locking*.

In the context of linear isoparametric interpolation function, in order to alleviate locking problems, integrations along the coordinate ξ of the pure displacement weak form are performed numerically by means of uniformly reduced numerical quadrature, adopting only one Gauss point. This is a commonly used procedure in shear deformable beam elements.

Although at first sight this seems a rather numerical trick, however, it can be justified by certain class of convergent mixed models [2] well established in literature and successfully applied in practice.

6.2 FE approximation of the out-of-balance forces

The discrete approximation to the weak form of momentum balance gives rise to the so-called *force residual equation*, or residual equation for short. Its explicit derivation is presented below starting from the expressions of virtual works principle.

Internal force vector. The finite element approximation of the internal component of the virtual work principle $\delta W_{int}(\boldsymbol{\eta})$, with $\boldsymbol{\eta} = (\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_N)^T = [(\boldsymbol{u}_1, \boldsymbol{\Psi}_1), \ldots, (\boldsymbol{u}_N, \boldsymbol{\Psi}_N)]^T$ the vector containing nodal values of the configuration variables $(\boldsymbol{u}, \boldsymbol{\Psi})_{(e)}$, could be obtained starting to substitute the virtual strain (3.79) into the equation (5.46)

[2]Mixed finite element methods are characterized by simultaneous interpolation of displacement, strains and stress fields.

$$\begin{aligned}\delta W_{int} &= \int_0^L \boldsymbol{\delta\varepsilon}^{rT} \boldsymbol{\sigma}^r \, \mathrm{d}S \\ &= \int_0^L (\boldsymbol{S\Delta\delta\eta})^T \boldsymbol{\sigma}^r \, \mathrm{d}S \end{aligned} \tag{6.21}$$

and in turn, substituting equation (6.15) in its global form

$$\begin{aligned}\delta W_{int} &= \int_0^L \boldsymbol{\delta\eta}^T \boldsymbol{\Delta}^T \boldsymbol{S}^T \boldsymbol{\sigma}^r \, \mathrm{d}S \\ &= \int_0^L (\boldsymbol{N\delta\hat{\eta}})^T \boldsymbol{\Delta}^T \boldsymbol{S}^T \boldsymbol{\sigma}^r \, \mathrm{d}S \\ &= \boldsymbol{\delta\hat{\eta}}^T \int_0^L \boldsymbol{N}^T \boldsymbol{\Delta}^T \boldsymbol{S}^T \boldsymbol{\sigma}^r \, \mathrm{d}S \end{aligned} \tag{6.22}$$

where the stress vector $\boldsymbol{\sigma}^r$ is easily obtained by means of equation (5.73) is a linear elastic constitutive law is adopted. Since the principle of virtual work is valid for any virtual displacement field, the term $\boldsymbol{\delta\hat{\eta}}$ could be dropped giving rise the final form of the internal force vector

$$\begin{aligned}\boldsymbol{f}_{int} &= \int_0^L \boldsymbol{N}^T \boldsymbol{\Delta}^T \boldsymbol{S}^T \boldsymbol{\sigma}^r \, \mathrm{d}S \\ &= \sum_{i=1}^{N_e} \int_0^{L_e} \boldsymbol{N}^T \boldsymbol{\Delta}^T \boldsymbol{S}^T \boldsymbol{\sigma}^r \, \mathrm{d}S \\ &= \sum_{i=1}^{N_e} \boldsymbol{f}_{int,i}^{(e)} \end{aligned} \tag{6.23}$$

where $\boldsymbol{f}_{int}^{(e)}$ denotes the internal force vector related to the typical element I_e^h. From a numerical standpoint, for a 2-node three-dimensional beam element vector $\boldsymbol{f}_{int}^{(e)}$ has dimension 12×1, as 6 degrees of freedom are assumed at each node. The relative integral is calculated by means of a reduced 1-node quadrature rule such as the following

$$\begin{aligned}\boldsymbol{f}_{int}^{(e)} &= \int_0^{L_e} \boldsymbol{N}^T \boldsymbol{\Delta}^T \boldsymbol{S}^T \boldsymbol{\sigma}^r \, \mathrm{d}S \\ &= \boldsymbol{N}^T(G)\, \boldsymbol{\Delta}^T(G)\, \boldsymbol{S}^T(G)\, \boldsymbol{\sigma}^r(G)\, L_e \end{aligned} \tag{6.24}$$

where all the quantities inside the integral are evaluated at the unique quadrature point G.

External force vector. In the same way as for the internal force vector, the finite element approximation of the external component of the virtual work principle, given in equation (5.58), once substituting equation (6.15) in its global form, reads

$$\begin{aligned}\delta W_{ext} &= \int_0^L \bar{\boldsymbol{q}}^r \cdot \boldsymbol{\delta\eta}\ \mathrm{d}S + [\hat{\boldsymbol{q}}^r \cdot \boldsymbol{\delta\eta}]_0^L \\ &= \int_0^L \bar{\boldsymbol{q}}^r \cdot (\boldsymbol{N}\boldsymbol{\delta}\hat{\boldsymbol{\eta}})\ \mathrm{d}S + [\hat{\boldsymbol{q}}^r \cdot (\boldsymbol{N}\boldsymbol{\delta}\hat{\boldsymbol{\eta}})]_0^L \\ &= \boldsymbol{\delta}\hat{\boldsymbol{\eta}}^T \int_0^L \boldsymbol{N}^T\bar{\boldsymbol{q}}^r\ \mathrm{d}S + \boldsymbol{\delta}\hat{\boldsymbol{\eta}}^T \left[\boldsymbol{N}^T\,\hat{\boldsymbol{q}}^r\right]_0^L \end{aligned} \tag{6.25}$$

As before, since the principle of virtual work is valid for any virtual displacement field, the term $\boldsymbol{\delta}\hat{\boldsymbol{\eta}}$ could be dropped giving rise the final form of the external force vector

$$\begin{aligned}\boldsymbol{f}_{ext} &= \int_0^L \boldsymbol{N}^T\bar{\boldsymbol{q}}^r\ \mathrm{d}S + \left[\boldsymbol{N}^T\,\hat{\boldsymbol{q}}^r\right]_0^L \\ &= \sum_{i=1}^{N_e} \int_0^{L_e} \boldsymbol{N}^T\bar{\boldsymbol{q}}^r\ \mathrm{d}S + \left[\boldsymbol{N}^T\,\hat{\boldsymbol{q}}^r\right]_0^{L_e} \\ &= \sum_{i=1}^{N_e} \boldsymbol{f}_{ext,i}^{(e)} \end{aligned} \tag{6.26}$$

where $\boldsymbol{f}_{ext}^{(e)}$ denotes the external force vector related to the typical element I_e^h. Consistently to what done for the internal force vector, for a 2-node three-dimensional beam element vector $\boldsymbol{f}_{ext}^{(e)}$ has dimension 12×1, as 6 degrees of freedom are assumed at each node.

Inertial force vector. The FE discretization of the inertial contribution to the out of balance force vector can be calculated starting from equation (5.68) reported for convenience below

$$\delta W_{inert} = \int_0^L \boldsymbol{G}_{inert} \cdot \boldsymbol{\delta\eta}\ \mathrm{d}S \tag{6.27}$$

The virtual generalized displacement vector $\boldsymbol{\delta\eta}$ can be substituted by equation (6.15) in its global form, whereas variables $\ddot{\boldsymbol{x}}_0$, $\dot{\boldsymbol{\Psi}}$, $\ddot{\boldsymbol{\Psi}}$ which appear in the term $\boldsymbol{G}_{inert}$ can be isoparametrically discretized as

$$\ddot{\boldsymbol{x}}_0 = \boldsymbol{N}_i\,\hat{\ddot{\boldsymbol{x}}}_i \tag{6.28}$$

$$\dot{\boldsymbol{\Psi}} = \boldsymbol{N}_i\,\hat{\dot{\boldsymbol{\Psi}}}_i \tag{6.29}$$

$$\ddot{\boldsymbol{\Psi}} = \boldsymbol{N}_i\,\hat{\ddot{\boldsymbol{\Psi}}}_i \tag{6.30}$$

because relation (6.12) holds also at the global level. Applying the same consideration of above, substited equation (6.15), equation (6.27) becomes

$$\begin{aligned}\delta W_{inert} &= \int_0^L (\boldsymbol{N}\boldsymbol{\delta}\hat{\boldsymbol{\eta}})^T\boldsymbol{G}_{inert}\ \mathrm{d}S \\ &= \boldsymbol{\delta}\hat{\boldsymbol{\eta}}^T \int_0^L \boldsymbol{N}^T\boldsymbol{G}_{inert}\ \mathrm{d}S \end{aligned} \tag{6.31}$$

and finally, the inertial force vector reads

$$\begin{aligned} \boldsymbol{f}_{inert} &= \int_0^L \boldsymbol{N}^T \boldsymbol{G}_{inert} \, \mathrm{d}S \\ &= \sum_{i=1}^{N_e} \int_0^{L_e} \boldsymbol{N}^T \boldsymbol{G}_{inert} \, \mathrm{d}S \\ &= \sum_{i=1}^{N_e} \boldsymbol{f}_{inert,i}^{(e)} \end{aligned} \tag{6.32}$$

where term $\boldsymbol{G}_{inert}$ was not explicited for sake of simplicity.

Force residual equation. Substituting equations (6.23), (6.26) and (6.32) into the expression (5.20) of the principle of virtual works, we obtain the compact force residual equation

$$\boldsymbol{r}(\boldsymbol{\eta}(t)) = \boldsymbol{r}[\boldsymbol{u}(t), \boldsymbol{\Psi}(t)] = \boldsymbol{f}_{int} - \boldsymbol{f}_{ext} + \boldsymbol{f}_{inert} = \boldsymbol{0} \tag{6.33}$$

where $\boldsymbol{\eta}$ is the generalized displacement vector, $\boldsymbol{u}$ is the displacement vector containing the degrees of freedom that characterize the configuration of the structures, $\boldsymbol{\Psi}$ similarly denotes the total rotation vector and $\boldsymbol{r}$ is the residual vector that collects the out-of-balance forces conjugated to $\boldsymbol{u}$ and $\boldsymbol{\Psi}$. It is worth to note that all quantities in equation (6.33) are time-dependent with parameter t.

An alternative version of equation (6.33) that displays more physical meaning, is the force balance form

$$\boldsymbol{f}_{int}(\boldsymbol{u}, \boldsymbol{\Psi}) + \boldsymbol{f}_{inert}(\boldsymbol{u}, \boldsymbol{\Psi}) = \boldsymbol{f}_{ext} \tag{6.34}$$

where $\boldsymbol{f}_{int}$ denotes the configuration-dependent internal forces exerted by the structure, whereas $\boldsymbol{f}_{inert}$ represents the time-dependent inertial forces generated by the spatial movement of the structure, and $\boldsymbol{f}_{ext}$ are the control-dependent external (or applied) forces, which in general may also be configuration-dependent.

The two versions (6.33) and (6.34) are equivalent such that read if (6.33) is verified, i.e. residual forces vector vanishes, the sum of the internal forces $\boldsymbol{f}_{int}$ and inertial forces $\boldsymbol{f}_{inert}$ balance the applied forces $\boldsymbol{f}_{ext}$, i.e.

$$\boldsymbol{r} = \boldsymbol{0} \quad \Longrightarrow \quad \boldsymbol{f}_{int} + \boldsymbol{f}_{inert} = \boldsymbol{f}_{ext} \tag{6.35}$$

which can be iteratively solved by any numerical methods.

We remark that the residual equation derived above for nonlinear dynamical systems can be restricted to static problems simply by eliminating the inertial force term.

6.3 Solution methods

In previous Chapters we have covered the governing equations of geometrically nonlinear structural analysis, and in the last sections the discretization of those equations by finite element methods. The result is a set of parametrized nonlinear algebraic equations called *residual force equations*, which solution provides the equilibrium response (static or dynamic) of the structure. Here we want to give some insight on the solution methods suitable for digital computation.

Basically, all solution procedures of practical importance are strongly rooted in the idea of "advancing the solution" by continuation. The basic approach is to follow the equilibrium response as the control state parameters vary by small amounts. The process involves a hierarchical breakdown into stages, incremental steps and iterative steps as depicted in Figure 6.3. The middle level, incrementation, is always present, while staging may be missing if there is only one control parameter, and finally iteration may be missing if there is no correction process.

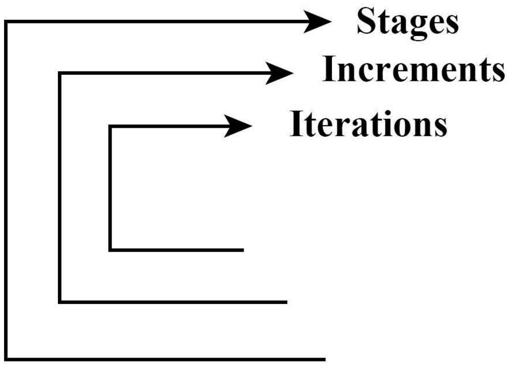


Figure 6.3: Nasty hierarchy in nonlinear solution methods: stages, increments and iterations.

Therefore, processing a complex nonlinear problem generally involves performing a series of analysis stages. Multiple control parameters are not varied independently in each stage, and may therefore be characterized by a single stage control parameter. To advance the solution, the stage is broken down into *incremental steps*, or *increments* for short. Incremental steps are identified by the subscript n: for example, the state vector after the $n-$th increment is $\boldsymbol{\eta}_n$, and the state vector before any increment (at stage start) is $\boldsymbol{\eta}_0$. Over each incremental step the state vector $\boldsymbol{\eta}$ undergoes finite changes denoted by $\Delta\boldsymbol{\eta}_n$.

In nonlinear problems we are interesting in "tracing the response". For a typical stage this reads to perform a sequence of incremental steps with control parameter λ, to find the equilibrium states $\{\boldsymbol{\eta}_n, \lambda_n\}$ is sufficient number to provide the response $\boldsymbol{\eta} = \boldsymbol{\eta}(\lambda)$ of the structure. If the control parameter is associated with a loading amplitude, the response path is known as the fundamental equilibrium path, and the incremental procedure is called *load control*. The physical analogy would be a test machine in which the operator increases the load to specific values. On the other hand, if the analysis is concerned with values of loading parameter λ close to zero, such as the case of buckling analysis, a more computationally efficient way to perform the incremental procedure consists to use a displacement control stepping, where the control parameter assumes now the role of governing displacement. This approach is widely adopted when the structural behavior is not uniquely determined for a given range of loading, e.g. in the case of snap-through response.

The purpose of the iteration level is to eliminate, or reduce, the so called drifting

error, which plagues incremental step at the beginning. Iteration steps are identified by the superscript k: for example $\{\boldsymbol{\eta}_n^k, \lambda_n^k\}$ may denote the solution after $k-$th iteration of the $n-$th step, whereas $\{\boldsymbol{\eta}_n^0, \lambda_n^0\}$ is the predicted solution before starting the iteration process. Solutions accepted after each increment following completion of the corrective process, are often of interest to users because they represent approximations to equilibrium states. They are therefore saved as they are computed. On the other hand, intermediate results of iterative process are rarely of interest, unless one is studying the convergence process.
The most popular class of iterative methods pertains to the Newton-Raphson method, and its numerous variants. These are collectively called newton-like methods, and only require access to the past converged solution. In the present work, only the incremental procedure (load control) is implemented, while the whole iteration routine is performed using the Matlab built in function *fsolve*. This choice seems to be efficient and robust for computations and convenient for quick coding.

6.3.1 Implicit time-stepping schemes

In computational dynamics, besides computation of displacements and rotations, one also needs to obtain the values of velocities and accelerations at the chosen instants in the time interval of interest. The standard Newmark family algorithms for nonlinear elastodynamics is used to that end. It is important to note that any method in this family leads to one-step scheme.

Newmark scheme for finite rotation. For a nonlinear dynamics problem with translational degrees of freedom only (e.g. the present case with all rotations constrained), the standard implementation of the Newmark algorithm can be used. Namely, velocities and accelerations at time t_{n+1} are computed as

$$\dot{\boldsymbol{u}}_{n+1} = \frac{\gamma}{\beta\,\Delta t}(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) + \frac{\beta-\gamma}{\beta}\dot{\boldsymbol{u}}_n + \frac{\beta - 0.5\gamma}{\beta}\ddot{\boldsymbol{u}}_n \tag{6.36}$$

$$\ddot{\boldsymbol{u}}_{n+1} = \frac{1}{\beta\,\Delta t^2}(\boldsymbol{u}_{n+1} - \boldsymbol{u}_n) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{u}}_n - \frac{0.5-\beta}{\beta}\ddot{\boldsymbol{u}}_n \tag{6.37}$$

where $\Delta t = t_{n+1} - t_n$ is a typical time step, and $\beta \in [0, 1/2]$, $\gamma \in [0, 1]$ are the classical (scalar) Newmark parameters. Replacing these approximations into the residual forces equation (8.1), we obtain a system of nonlinear equations in incremental displacements as

$$\boldsymbol{r}(\boldsymbol{u}(t+1)) = \mathbf{0} \tag{6.38}$$

Typical choice for $\beta = 1/4$ and $\gamma = 1/2$ leads to a second-order accuracy unconditionally stable scheme. The corresponding algorithm is yet referred to as trapezoidal rule, or average acceleration method, the reason for which becomes clear analyzing the alternative form (called acceleration form of the Newmark approximations)

$$\boldsymbol{u}(n+1) = \Delta t\dot{\boldsymbol{u}}(n) + \Delta t^2[(0.5-\beta)\ddot{\boldsymbol{u}}(n) + \beta\ddot{\boldsymbol{u}}(n+1)] \tag{6.39}$$

$$\dot{\boldsymbol{u}}(n+1) = \dot{\boldsymbol{u}}(n) + \Delta t[(1-\gamma)\ddot{\boldsymbol{u}}(n) + \gamma\ddot{\boldsymbol{u}}(n+1)] \tag{6.40}$$

The just described standard implementation of the Newmark scheme can directly be used for problems dealing with finite rotations, only if the *total* rotation vector is chosen for parametrizing the finite rotations (see e.g. [34]). In that case, from the known values of rotation vector and its time derivative at time t_n, $\mathbf{\Psi}_n = \mathbf{\Psi}(t_n)$, $\dot{\mathbf{\Psi}}_n = \dot{\mathbf{\Psi}}(t_n)$, $\ddot{\mathbf{\Psi}}_n = \ddot{\mathbf{\Psi}}(t_n)$, via Newmark approximations we obtain

$$\dot{\mathbf{\Psi}}_{n+1} = \frac{\gamma}{\beta\,\Delta t}(\mathbf{\Psi}_{n+1} - \mathbf{\Psi}_n) + \frac{\beta-\gamma}{\beta}\dot{\mathbf{\Psi}}_n + \frac{\beta-0.5\gamma}{\beta}\ddot{\mathbf{\Psi}}_n \tag{6.41}$$

$$\ddot{\mathbf{\Psi}}_{n+1} = \frac{1}{\beta\,\Delta t^2}(\mathbf{\Psi}_{n+1} - \mathbf{\Psi}_n) - \frac{1}{\beta\Delta t}\dot{\mathbf{\Psi}}_n - \frac{0.5-\beta}{\beta}\ddot{\mathbf{\Psi}}_n \tag{6.42}$$

Hence, we can condense the previous two time-stepping algorithm into the following using the generalized displacement vector $\boldsymbol{\eta}$

$$\dot{\boldsymbol{\eta}}_{n+1} = \frac{\gamma}{\beta\,\Delta t}(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n) + \frac{\beta-\gamma}{\beta}\dot{\boldsymbol{\eta}}_n + \frac{\beta-0.5\gamma}{\beta}\ddot{\boldsymbol{\eta}}_n \tag{6.43}$$

$$\ddot{\boldsymbol{\eta}}_{n+1} = \frac{1}{\beta\,\Delta t^2}(\boldsymbol{\eta}_{n+1} - \boldsymbol{\eta}_n) - \frac{1}{\beta\Delta t}\dot{\boldsymbol{\eta}}_n - \frac{0.5-\beta}{\beta}\ddot{\boldsymbol{\eta}}_n \tag{6.44}$$

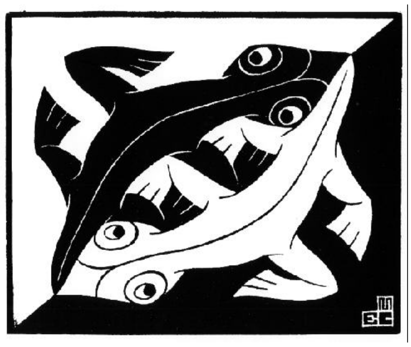
with the residual forces equations which reads

$$\boldsymbol{r}(\boldsymbol{\eta}(t+1)) = \boldsymbol{r}(\boldsymbol{u}(t+1), \mathbf{\Psi}(t+1)) = \mathbf{0} \tag{6.45}$$

Unfortunately, there are three serious drawbacks to this simple application of the Newmark scheme to finite rotations. Firstly, a fundamental deficiency of this parametrization inherits the ill-conditioning problem which arises for rotations in the neighborhood of 2π and its multiples, as already established occuring in statics. In second place, even though Newmark's scheme has been widely applied to the study of the dynamic response of structures, Makinen [35] states that it only constitutes an approximated version of the corrected formula, which are given in his work for the spatial and material descriptions. The main reasons are that material descriptions of the angular velocity and angular acceleration vectors involved in the updating procedures belong to different tangent spaces at different times. Lastly, Newmark's family of implicit schemes fails to preserve certain conservation laws of the motion, such as the total energy and momentum of nonlinear Hamiltonian systems, introducing numerical (fictitious) dissipation (see e.g. [15]). A further improvement in the development of robust time-stepping schemes is provided by the energy-momentum conserving algorithms (see [54], [3] and [48]), successfully applied to the nonlinear dynamic problems of beams, shells and rigid bodies. As pointed out in [44], for solving structures which dissipate most of the energy throughout inelastic mechanisms (as likely the case for seismic resistant structures), no great advantages are obtained by means of using sophisticated formulations for time-stepping algorithms, and hence the approximated version of the Newmark algorithm on rotational manifold (as originally proposed in [59]) seems to be accurate enough for practical purposes.

Chapter 7

Numerical results



Any fool can write code that a computer can understand. Good programmers write code that humans can understand.

M. Fowler

In this chapter we consider a series of numerical simulations that illustrate the performance of the displacement-based formulation described in the previous Chapter. In the first two examples attention is focused on the plane problem, i.e. where the rotation field is easily described by means of a single rotation angle. Even though the three-dimensional rotations are not yet fully involved, these examples permit to test the capability of reproducing large displacements in both static and dynamic case. The last example, on the other hand, is concerned with fully three-dimensional dynamic deformation. Throughout all the examples discussed below, the analysis is limited to the linear elastic constitutive model defined in (5.74).

7.1 Analysis of plane problems

7.1.1 Pure bending of a cantilever beam subject to end moment

A cantilevered straight beam subjected to a concentrated free end moment M is usually the first problem tackle to test the accuracy of the described element under extreme inextensional bending and large deformation (see Figure 7.1). This problem has been analyzed by a number of researchers including Bathe and Bolourchi (1979) [6], Simo and Vu-Quoc (1986) [57] among others. The finite element mesh here used consists of 12 elements with linear interpolation shape functions N_i and a one-point (uniformly reduced) quadrature is employed to integrate the internal force vector. The selected properties of the cantilever are: length $L = 10$ mm, axial stiffness $EA = 10^4$, shear stiffness $GA_1 = GA_2 = 10^4$, bending stiffness $EJ_1 = EJ_2 = 10^2$ and torsional stiffness $GJ_T = 10^2$, which corresponds to a circular cross-section of the beam with radius $R = 0.2$ mm, and the beam material properties with Young's modulus $E = 79577$ MPa and Poisson's ratio $\nu = 0$.

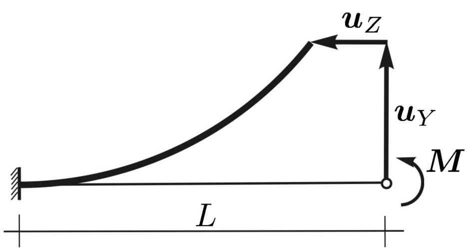


Figure 7.1: Pure bending of a cantilever beam subjected to end moment.

It is straightforward to see that for a prismatic elastic beam subjected to a tip moment, the only non-trivial deformation component is the flexural one on the plane YZ. Moreover, according to the classical Euler formula, this bending deformation is constant along the beam. Thus follow that the exact solution of the deformed shape must be a part of a circular curve, and following Figure 7.2, the analytic solution for the free-end displacement components can be obtained as [1]

$$\theta_x = \frac{ML}{EJ} \tag{7.1}$$

$$u_z = L - r\,\sin\theta = L - \frac{L}{\theta}\sin\theta = L - \frac{L}{\theta}\tan\frac{\theta}{2}(1+\cos\theta) \tag{7.2}$$

$$u_y = r - r\,\cos\theta = r(1-\cos\theta) = \frac{L}{\theta}\frac{1-\cos\theta}{\sin\theta}\sin\theta = \frac{L}{\theta}\tan\frac{\theta}{2}\sin\theta \tag{7.3}$$

where we have taken advantage of expressing angles in radiants in such a way that simply $L = r\cdot\theta^{rad}$ holds.

In the study, the bending moment applied at the end is increased from 0 to 20π, which forces the beam to deform into a full closed circle such as depicted in Figure 7.3. In the same picture is shown the shape of the deflected beam through solution increments.

Figure 7.4 presents a comparisons between the analytic solution and that obtained with finite element (12 elements) in terms of load-displacement curve at tip. Curves coming from the classical solution of Euler can be traced with reasonable accuracy up to the deformation corresponding to a full circle, and no significant discrepancies are

[1] We recall the trigonometric identity

$$\tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta} = \frac{1-\cos\theta}{\sin\theta}$$

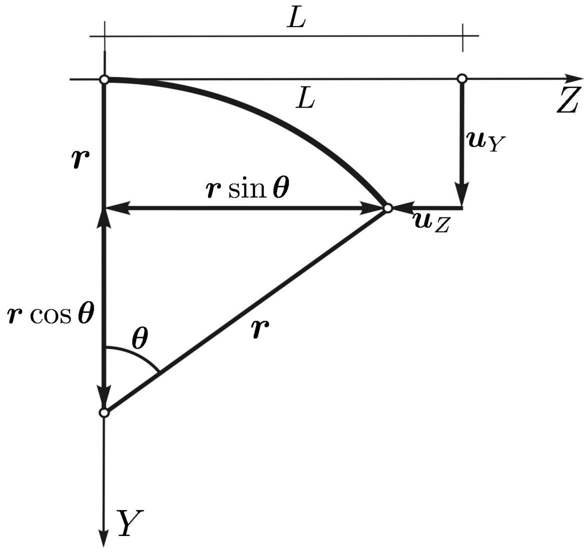


Figure 7.2: Roll up of a beam: analytical displacement filed.

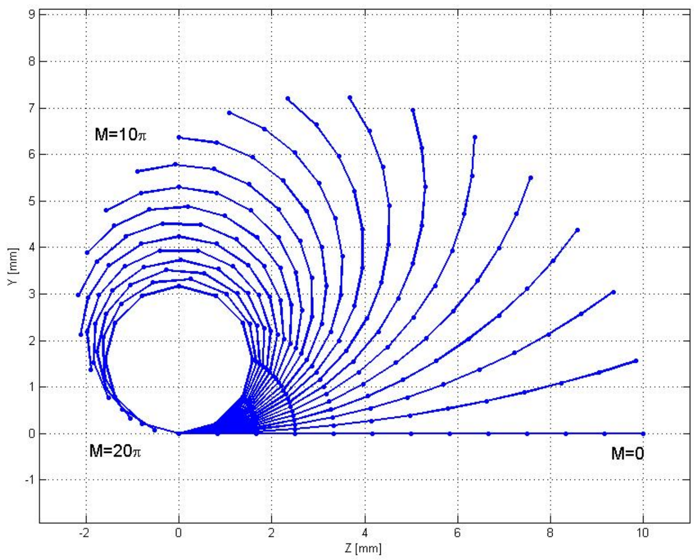


Figure 7.3: Shape of the beam throughout the deflection sequence.

distinguishable through displacement components. It is worthwhile to mention that the same equilibrium path is obtained even with just 3 elements.

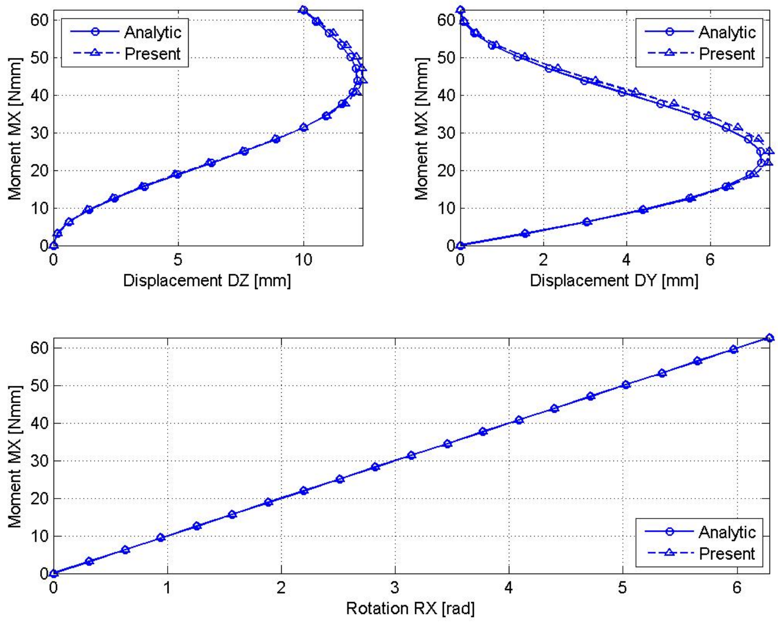


Figure 7.4: Cantilever beam subjected to end moment: end node beam displacement in Z direction versus end applied moment M_X (top left); end node beam displacement in Y direction versus end applied moment M_X (top right); end node beam rotation RX versus end applied moment M_X (bottom).

7.1.2 Cantilever beam subject to triangular end force pulse

This example aims at studying the nonlinear transient dynamics of a narrow rectangular cantilever beam subjected to a transverse concentrated force applied at the free end. This force is assumed time-variant according to the pattern shown in Figure 7.5.

The finite element uniform mesh consists of 4 elements with linear interpolation functions N_i, with one-point quadrature scheme for computing internal forces, whereas 3 Gauss points numerical quadrature is adopted for calculating inertial forces. The selected properties of the cantilever are: length $L = 2400$, section width and section height, respectively $b = 11.64$ and $h = 100$, material density, Young modulus and shear modulus respectively $\rho = 10^{-3}$, $E = 210 \cdot 10^3$ and $G = 80 \cdot 10^3$. The computations are carried out using Newmark scheme with parameters $\beta = 1/2$ and $\gamma = 1/2$ and with the constant time step $\Delta t = 0.1$s. The cross-sectional constitutive law is based on the gross shear areas, and the Saint-Venant torsional constant.

In Figure 7.6 are depicted the time histories of the loading force as well as displacement, velocity and acceleration of the tip. We may note how undamped free oscillations take place after a short transient interval, in which external loading is applied to the structure.

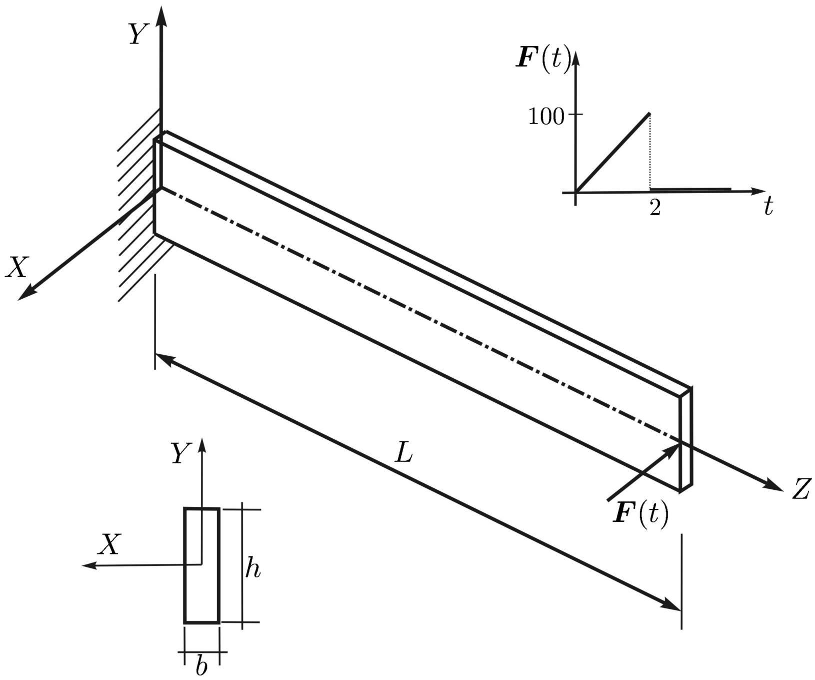


Figure 7.5: Cantilever beam under vertical load.

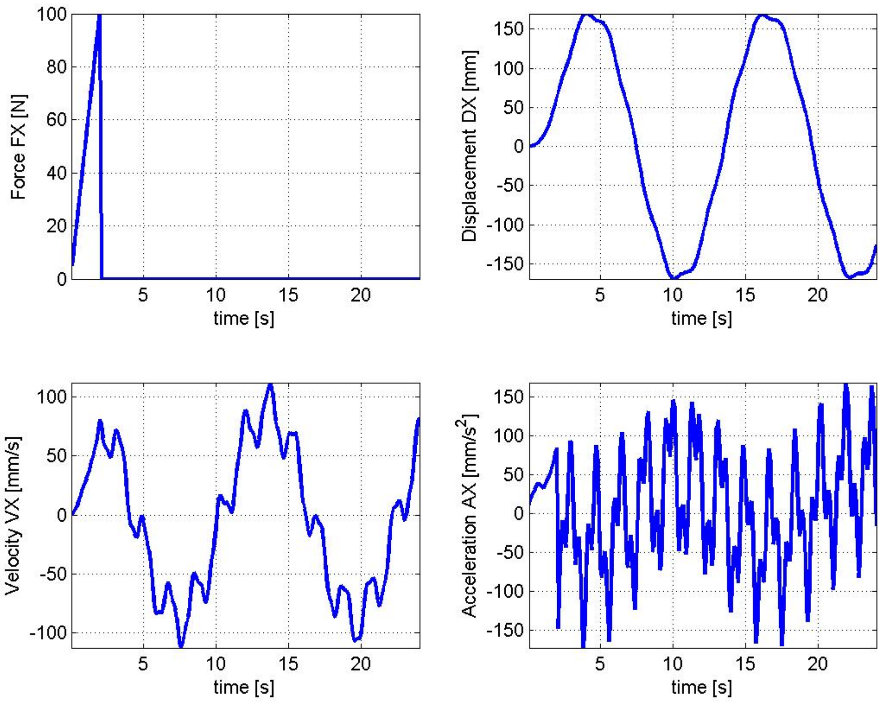


Figure 7.6: Cantilever beam under triangular end force pulse: the tip loading, displacement, velocity and acceleration time history.

Figure 7.7 focus on the interval [0 2]*sec* of the time history. We may note the nervous behavior of acceleration plot, mostly induced by the inertial effects of the left-end-side of the beam. Same "swing" effects are present in the displacement plot of Figure 7.6, basically related to the asyncronous free oscillations of parts of the beams which influence each other. This phenomenon, perfectly compatible with reality, becomes more evident for more flexible beams as expected.

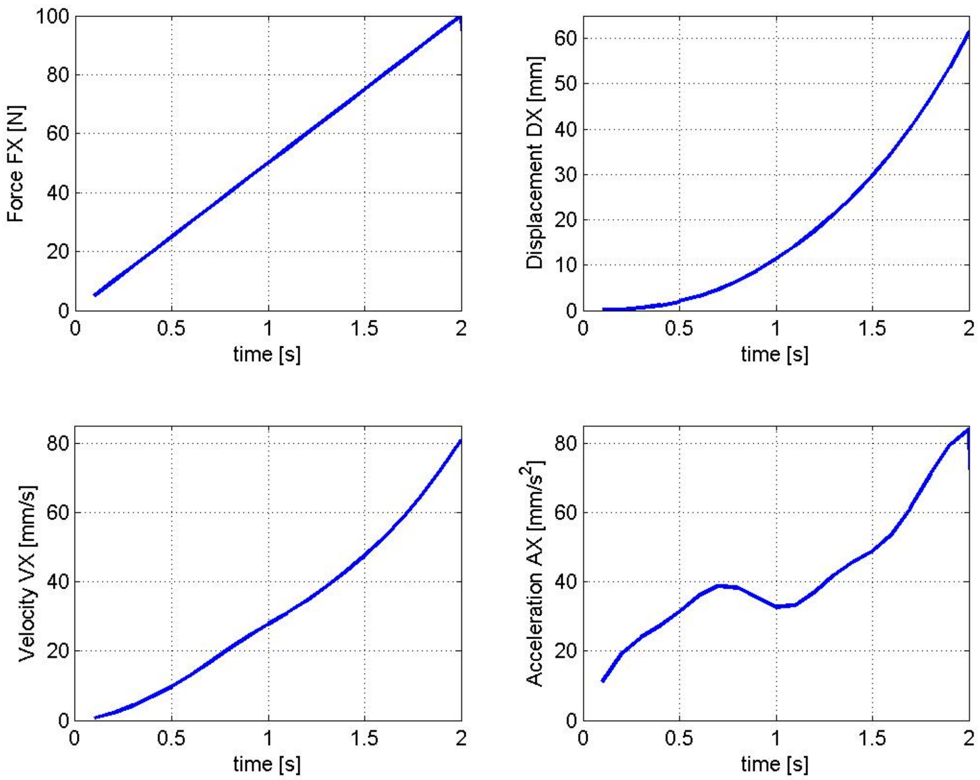


Figure 7.7: Cantilever beam under triangular end force pulse: detail of interval [0 2]sec.

7.2 Analysis of spatial problems

The complete verification of the finite rotation theory developed is possible just for problems in 3-D space. The example presented here exhibit an overall good performance in the dynamical case.

7.2.1 L-shape cantilever beam under triangular end force pulse

The right-angle cantilever beam depicted in Figure 7.8 is dynamically loaded by an out-of-plane concentrated force $F_x = 50N$ at the elbow. The shape and duration of the applied load is shown in the same figure. The total duration of the analysis is $t_u = 18$ s, which includes the period of time when the load is being applied and the following undamped free vibration of the system. The computations are carried out using Newmark scheme with parameters $\beta = 1/4$ and $\gamma = 1/2$, whereas the constant time step is assumed equal to $\Delta t = 0.1$ s. The mechanical properties of the cantilever are: arm length $L = 10$, translational stiffness $EA = GA_1^* = GA_2^* = 10^6$, rotational stiffness $EJ_1 = EJ_2 = GJ_t = 10^3$, mass density per unit length $A_\rho = 1$, and inertial tensor $J_\rho = \text{diag}[10\ 10\ 20]$.

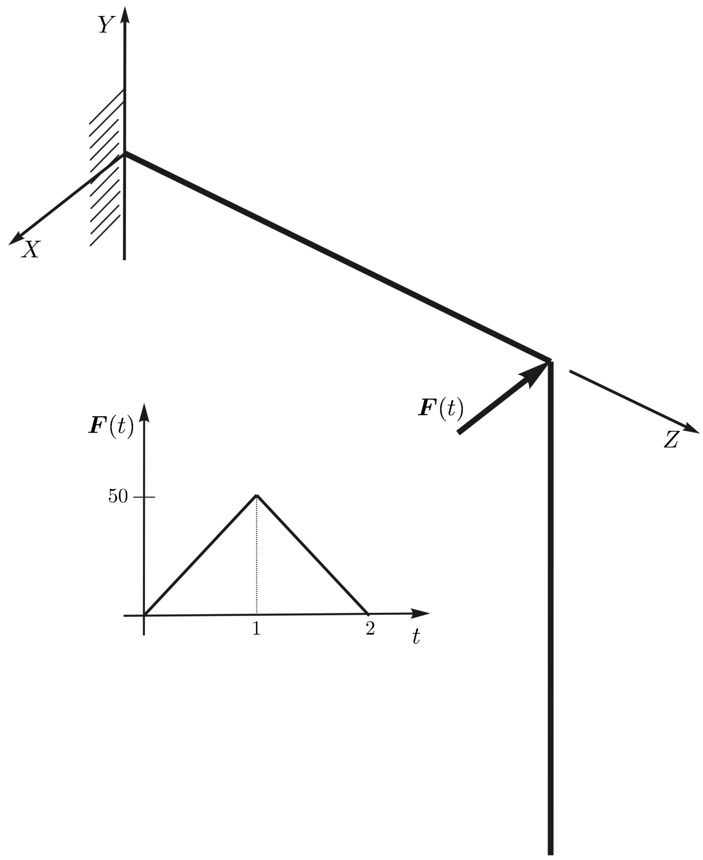


Figure 7.8: L-shape cantilever beam: perspective view. Geometry and loading data.

In Figure 7.9 and the following Figure 7.10 are depicted the time histories of the loading force as well as displacement, velocity and acceleration of the tip and elbow. We may note how undamped free oscillations take place after a short transient interval, in which external loading is applied to the structure. It is interesting to note in Figure 7.11 that the motion of the system involves large displacements with magnitude of the same order as the dimensions of the initial geometry.

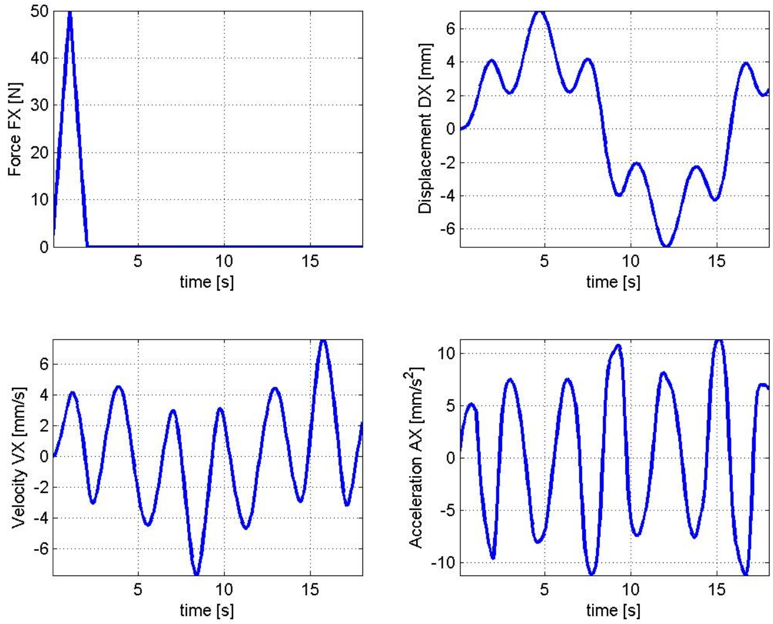


Figure 7.9: Right-angle cantilever beam subject to out-of-plane triangular end force pulse: loading, displacement, velocity and acceleration time history at the elbow point.

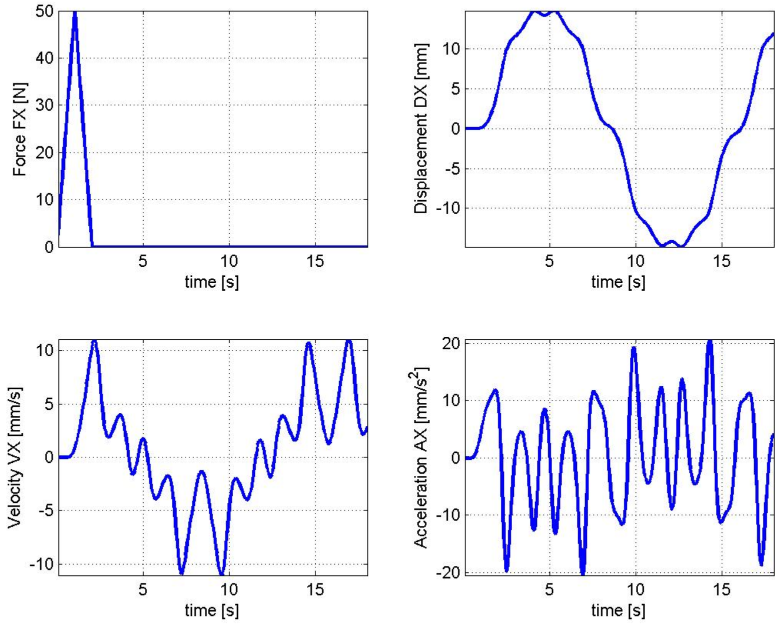


Figure 7.10: Right-angle cantilever beam subject to out-of-plane triangular end force pulse: time history of elbow loading; displacement, velocity and acceleration time histories at the tip point.

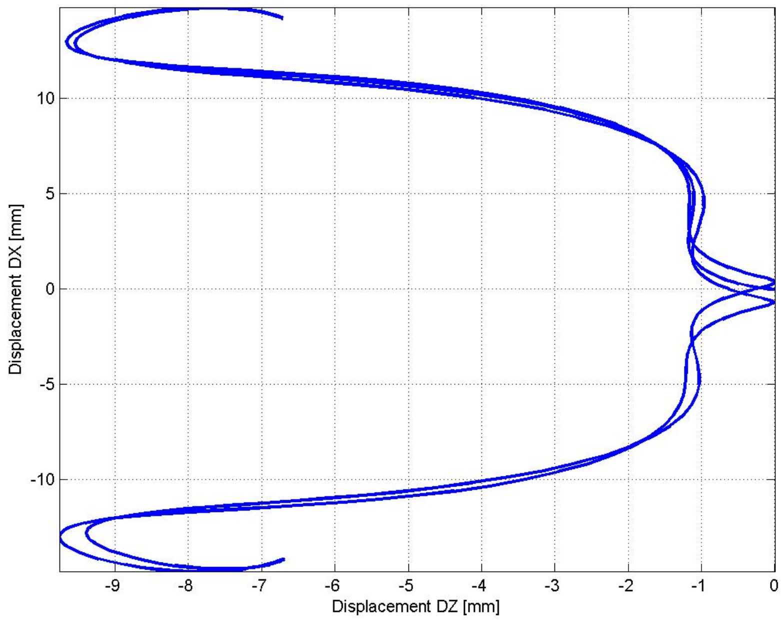
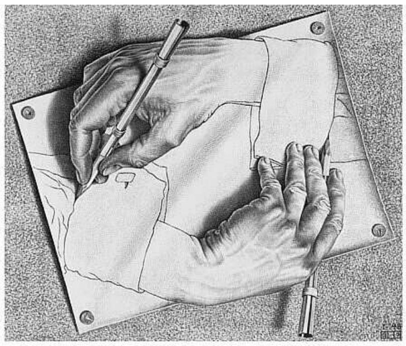


Figure 7.11: Right-angle cantilever beam subject to out-of-plane triangular end force pulse: tip displacement time history in the $X - Y$ plane.

Chapter 8

Conclusions and further research



Everything that is really great and inspiring is created by the individual who can labor in freedom.

A. Einstein

In this chapter conclusions about the results obtained in the formulation and the numerical implementation of a beam model able to consider the fully geometric nonlinearity are discussed. Last section is dedicated to the statement of new line of research related to the different topics covered in this thesis.

8.1 Conclusions

In this section we summarize the results achieved.

(I) *Theoretical objectives*

- **(I.1)** In Chapter 2 a thorough theoretical study of finite rotation has been exposed. Special care was reserved to consolidate the concepts of differentiable manifold and its specialization or the rotation group, as well as the description of manifold tangent space and its utility in the contest of rotations.
- **(I.2)** A detailed (though not exhaustive) introduction on the possible parametrization of the rotation manifold is delivered, emphasizing the Euler-Rodrigues vector parametrization. In addition, a configurational approach for describing large rotations in three-dimensional space is given, and finally an entire

section was dedicated to derivatives (both spatial and time) of rotation tensor.

- **(I.3)** In Chapter 3 the continuum based theory of beams capable of undergoing arbitrarily large displacements and rotations under the Reissner-Simo hypothesis has been presented. In the present work an initially straight and unstressed beam is considered as the reference configuration. A detailed description of the kinematic assumptions is carried out in the framework of the configurational description of the mechanics, along with the mathematical expression of the beam kinematics.
- **(I.4)** From the expression of the deformation gradient tensor the explicit forms of the translational and rotational strain measure acting on each material point of the cross-section are derived, both in material and spatial coordinates.
- **(I.5)** The linearization of the kinematical quantities necessary for expressing the principle of virtual work, is carried out basing on the concept of Gateaux directional derivative.
- **(I.6)** In Chapter 4 the equations of motion are deduced from the local form of the linear and angular balance momentum and integrating over the beam's volume. An appropriated (weak) form for the numerical implementations is obtained for the nonlinear functional corresponding to virtual work principle. A discussion about the deduction of reduced constitutive relations considering hyperlastic materials was fully covered.

(II) *Numerical objectives*

- **(II.1)** Chapter 6 describes the spatial discretization used in the Galerkin finite element approximation of the virtual work equation. The resulting FE approach yields a system of nonlinear algebraic equations well suited for the application of an iterative solution method. The developed displacement-based finite element is based on isoparametric interpolations of both the displacement and total rotation field.
- **(II.2)** In Chapter 6 the time discretization of the residual force equation is performed according to the Newmark's method.
- **(II.3-4)** The resulting displacement-based FE model is implemented using Matlab. The numerical validation of the presented formulation, in the static and dynamic cases, is performed throughout a set of examples considering the classical linear elastic constitutive laws. In the plots reported it is possible to appreciate a good agreement with results existing in literature.

8.2 Further lines of research

Several lines of research opens from the results of the present work. A list of possible directions are discussed in the following:

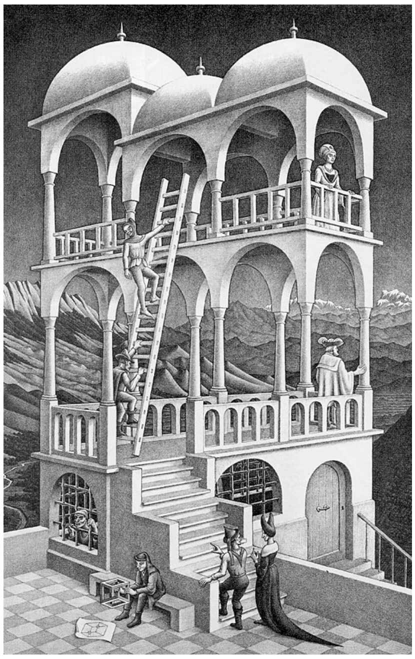
(i) *Finite deformation models with enhanced kinematical assumptions.* Several works have been devoted to the development of richer kinematics assumptions incorporated in geometrically exact beam models. See e.g. [64] and [18] for the inclusion of warping phenomena in elastic materials, [24] for anisotropic materials, [55], [25] and [26] for the case of plasticity with warping, among others. A possible contribution in further works can be given by the extension of the reviewed works for including more refined kinematics in a consistent mechanical framework.

(ii) *Improvement of constitutive law.* The basic equations of the geometrically exact beam theory here derived, contain no assumptions about the magnitude of deformations (even though initial assumptions about the kinematics are made), which could be arbitrarily large. Instead, limitations are introduced when using a linear elastic constitutive law. Hence, future efforts could be directed towards the deduction of 1D elastic constitutive relations from the 3D material and geometric properties, which permits to better conjugate the exactness of kinematics. Improvements are seeded to solve specific problems, e.g. multi-layer beams, strength-degradation materials, and so on.

(iii) *The extension of the present result to shell elements.* Another type of structural element widely applied in several areas of engineering, and also in earthquake engineering, is the shell element. Geometrically exact models for shells (see e.g. [11], [60], [61], [62] for the general theory; [10], [12], [63] for the case of variable thickness; the inclusion of inelasticity can be reviewed in [65]; a shell's formulation using drilling degrees of freedom can be consulted in [29], [30], [66]; the development of time-stepping schemes in [13], [14], [67], among a really large list of works) share with the present beam model the fact that both formulations produce a nonlinear configuration manifold involving the rotation manifold (or the two-sphere in the case of shells). Particularly, the so called shell formulation with drilling rotations has the same number of degrees of freedom as the beam model and, therefore, are well suited to be combined in a computer code able to simulate the behavior of one and two dimensional structural elements. A typical examples of such structures are the shear wall buildings and web stiffeners in steel bridges among many others. An interesting possibility is given by the fact of extending the formulation for composite materials to shell elements, and combine them with beams for studying structural problems. This extension could be useful even for introducing the effects of local failures (such as e.g. FRP delamination) in the global mechanical response of composite layered structures.

Appendix A

Mathematical recall



There is no formula that can deliver all truth, all harmony, all simplicity.

J. Barrow

In this brief appendix we give some mathematical recall that are essential for a plenty comprehension of the theoretical derivations obtained along the chapters.

Some properties of vector operations. The following properties of cross and scalar product are often used. Given vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbb{R}^3$ and scalars $\alpha, \beta \in \mathbb{R}$

- cross product between parallel vectors: $\boldsymbol{a} \times (\alpha \boldsymbol{a}) = \mathbf{0}$;
- cross product anticommutativity: $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$;
- mixed product identities: $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{c} \cdot (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{b} \cdot (\boldsymbol{c} \times \boldsymbol{a})$;
- double cross product identity: $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$;
- tensor product definition: $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}) = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$

Some properties of tensor operations.

- In general the dot product of second-order tensors is not commutative

$$\boldsymbol{AB} \neq \boldsymbol{BA} \qquad \text{and} \qquad \boldsymbol{Au} \neq \boldsymbol{uA} \tag{A.1}$$

- The transpose of a sum is the sum of the transposes

$$(\alpha \boldsymbol{A} + \beta \boldsymbol{B})^T = \alpha \boldsymbol{A}^T + \beta \boldsymbol{B}^T \tag{A.2}$$

- The transpose of a product is the product of the transposes in the opposite order

$$(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T \tag{A.3}$$

- The inverse of a product is the product of the inverses in the opposite order

$$(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1} \boldsymbol{A}^{-1} \tag{A.4}$$

- The transpose of a tensor product is the tensor product of the vectors in the opposite order

$$(\boldsymbol{u} \otimes \boldsymbol{v})^T = \boldsymbol{v} \otimes \boldsymbol{u} \tag{A.5}$$

- The associative property over tensor product is not valid

$$\boldsymbol{A}(\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{Au}) \otimes \boldsymbol{v} \tag{A.6}$$

- The property of double contraction holds as

$$\boldsymbol{A} : (\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{Av}) \cdot \boldsymbol{u} \tag{A.7}$$

- Given a rotation tensor $\boldsymbol{\Lambda} \in SO(3)$, the invariance of scalar product holds:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{\Lambda a} \cdot \boldsymbol{\Lambda b} \tag{A.8}$$

- Given a rotation tensor $\boldsymbol{\Lambda} \in SO(3)$, the distributivity of cross product [1] with respect to product with $\boldsymbol{\Lambda}$ holds

$$\boldsymbol{\Lambda}(\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{\Lambda a} \times \boldsymbol{\Lambda b} \tag{A.9}$$

- The *Lie algebra* operating on the axial vector $\boldsymbol{\Psi}$ and its skew-symmetric tensor $\tilde{\boldsymbol{\Psi}} \in so(3)$ can be identified with the vector product $\times$ on $\mathbb{R}^3$ by the formula

$$\tilde{\boldsymbol{\Psi}} \boldsymbol{h} = \boldsymbol{\Psi} \times \boldsymbol{h} \qquad \forall \boldsymbol{h} \in \mathbb{R}^3 \tag{A.10}$$

- Given axial vectors $\boldsymbol{\Psi}, \boldsymbol{h} \in \mathbb{R}^3$ and their skew-symmetric tensors $\tilde{\boldsymbol{\Psi}}, \tilde{\boldsymbol{h}} \in so(3)$, according to (A.10), the following relations holds for them

$$\tilde{\boldsymbol{\Psi}} \boldsymbol{h} = \boldsymbol{\Psi} \times \boldsymbol{h} = -\boldsymbol{h} \times \boldsymbol{\Psi} = -\tilde{\boldsymbol{h}} \boldsymbol{\Psi} = \tilde{\boldsymbol{h}}^T \boldsymbol{\Psi} \tag{A.11}$$

[1] In general the cross product obeys to this identity: $(\boldsymbol{Aa}) \times (\boldsymbol{Ab}) = (\det \boldsymbol{A}) \boldsymbol{A}^{-T} (\boldsymbol{a} \times \boldsymbol{b})$. If the tensor $\boldsymbol{A}$ is characterized by $\boldsymbol{AA}^T = \boldsymbol{I}$ and $\det \boldsymbol{A} = +1$, equation (A.9) holds as consequence.

- Given axial vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{h} \in \mathbb{R}^3$ and their skew-symmetric tensors $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}} \in so(3)$, according to (A.10) and using the double cross product identity for vectors, the following relations holds

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}}\boldsymbol{h} = \tilde{\boldsymbol{A}}(\boldsymbol{B} \times \boldsymbol{h}) = \boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{h}) = (\boldsymbol{A} \cdot \boldsymbol{h})\boldsymbol{B} - (\boldsymbol{A} \cdot \boldsymbol{B})\boldsymbol{h} \tag{A.12}$$

- The *Lie brackets* for a given set of two skew-symmetric tensors $\tilde{\boldsymbol{\Psi}}, \tilde{\boldsymbol{W}} \in so(3)$ and their axial vectors respectively $\boldsymbol{\Psi}, \boldsymbol{W} \in (R)^3$ holds

$$[\tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{W}} - \tilde{\boldsymbol{W}}\tilde{\boldsymbol{\Psi}}]\boldsymbol{h} = (\boldsymbol{\Psi} \times \boldsymbol{W}) \times \boldsymbol{h} \qquad \forall \boldsymbol{h} \in \mathbb{R}^3 \tag{A.13}$$

- The generalized *Lie brackets* reads

$$(\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}})\tilde{\boldsymbol{A}}\boldsymbol{B} = (\tilde{\boldsymbol{A}}\boldsymbol{b}) \times (\tilde{\boldsymbol{A}}\boldsymbol{B}) = \boldsymbol{0} \tag{A.14}$$

  such as

$$\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}\boldsymbol{B} = \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}}\tilde{\boldsymbol{A}}\boldsymbol{B} \tag{A.15}$$

- Given a skew tensor $\tilde{\boldsymbol{\Psi}}$ and its axial vector $\boldsymbol{\Psi}$, the following identity holds

$$\tilde{\boldsymbol{\Psi}}^2\boldsymbol{h} = \tilde{\boldsymbol{\Psi}}(\tilde{\boldsymbol{\Psi}}\boldsymbol{h}) = [\boldsymbol{\Psi} \otimes \boldsymbol{\Psi} - \Psi^2\boldsymbol{I}]\boldsymbol{h} \qquad \forall \boldsymbol{h} \in \mathbb{R}^3 \tag{A.16}$$

  where $\Psi = ||\boldsymbol{\Psi}|| = \sqrt{\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}}$.

- Given three arbitrary tensor $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$, the following property holds

$$\boldsymbol{A} : (\boldsymbol{B}\boldsymbol{C}) = (\boldsymbol{B}^T\boldsymbol{A}) : \boldsymbol{C} = (\boldsymbol{A}\boldsymbol{C}^T) : \boldsymbol{B} \tag{A.17}$$

- Given an arbitrary tensor $\mathbf{A}$ and a vector $\mathbf{u}$, the following property holds

$$\text{div}(\boldsymbol{A}^T\boldsymbol{u}) = \boldsymbol{A} : \text{grad}\boldsymbol{u} + \boldsymbol{u} \cdot \text{div}\boldsymbol{B} \tag{A.18}$$

- Given an arbitrary tensor $\mathbf{A}$ and a *symmetric* tensor $\mathbf{S}$, the following property holds

$$\boldsymbol{S} : \boldsymbol{A} = \boldsymbol{S} : \text{sym}[\boldsymbol{A}] \tag{A.19}$$

Fréchet derivative. Let assume $\mathcal{H}_1$ and $\mathcal{H}_2$ two two Hilbert spaces equipped with a norm $||\cdot||$, and let $\boldsymbol{\mathcal{F}}$ a functional $\boldsymbol{\mathcal{F}} : \mathcal{H}_1 \to \mathcal{H}_2$ defined on an open set $\mathcal{E} \subset \mathcal{H}_1$. Such functional is *Fréchet* differentiable at a point $\boldsymbol{x} \in \mathcal{H}_1$ in the direction $\boldsymbol{u}$ if exists a linear and limited functional $\delta\boldsymbol{\mathcal{F}}(\boldsymbol{x}) : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\forall \varepsilon > 0,\ \exists\, \delta > 0\ :\ ||\boldsymbol{\mathcal{F}}(\boldsymbol{x}+\boldsymbol{u}) - \boldsymbol{\mathcal{F}}(\boldsymbol{x}) - \delta\boldsymbol{\mathcal{F}}(\boldsymbol{x}) \cdot \boldsymbol{u}||_Y \leq \varepsilon ||\boldsymbol{u}||_{\mathcal{H}_1}$$
$$\forall \boldsymbol{u} \in \mathcal{H}_1 \quad \text{with} \quad ||\boldsymbol{u}||_{\mathcal{H}_1} \leq \delta$$

One refers to $\delta\boldsymbol{\mathcal{F}}(\boldsymbol{x}) \cdot \boldsymbol{u}$ as the Fréchet (strong) derivative of the functional $\boldsymbol{\mathcal{F}}$ at $\boldsymbol{x}$ in the direction $\boldsymbol{u}$.

Gateaux derivative. Given the functional $\boldsymbol{\mathcal{F}}:\mathcal{H}_1\to\mathcal{H}_2$ continuously differentiable, one defines the *Gateaux* (weak) derivative of $\boldsymbol{\mathcal{F}}$ at a point $\boldsymbol{x}$ in the direction of vector $\boldsymbol{u}$ as the following limits

$$\boxed{D\,\boldsymbol{\mathcal{F}}(\boldsymbol{x},\boldsymbol{u}) = D[\boldsymbol{\mathcal{F}}(\boldsymbol{x})]\cdot\boldsymbol{u} \triangleq \lim_{t\to 0}\frac{\boldsymbol{\mathcal{F}}(\boldsymbol{x}+t\boldsymbol{u})-\boldsymbol{\mathcal{F}}(\boldsymbol{x})}{t} = \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\mathcal{F}}(\boldsymbol{x},t\boldsymbol{u})\bigg|_{t=0}} \qquad (A.20)$$

with the scalar parameter $t\in\mathbb{R}$ used as limiting parameter in the derivative. It's worth to note that this definition generalizes the notion of the directional derivative of functions in Euclidean space. We recall that if a functional admits strong (Fréchet) derivative $\boldsymbol{\delta}\boldsymbol{\mathcal{F}}(\boldsymbol{x})\cdot\boldsymbol{u}$, it coincides with weak (Gateaux) derivative $D\boldsymbol{\mathcal{F}}(\boldsymbol{x})\cdot\boldsymbol{u}$.

Variation operators. The variation operator $\boldsymbol{\delta}$ is defined as the special case of Fréchet differential. Let introduce the concept with an example. Assume the functional $\boldsymbol{\mathcal{F}}$ and being the fixed parameter (time) $t=t_0$. The variation operator can be expressed as

$$\boldsymbol{\delta}\boldsymbol{\mathcal{F}}(t_0,\boldsymbol{x},\boldsymbol{v}) = D_{\boldsymbol{x}}\boldsymbol{\mathcal{F}}(t_0,\boldsymbol{x},\boldsymbol{v})\cdot\delta\boldsymbol{x} + D_{\boldsymbol{v}}\boldsymbol{\mathcal{F}}(t_0,\boldsymbol{x},\boldsymbol{v})\cdot\delta\boldsymbol{v} \qquad (A.21)$$

where $\boldsymbol{x}$ is place field, $\delta\boldsymbol{x}$ is a virtual displacement field, $\boldsymbol{v}$ is a velocity field, and $\delta\boldsymbol{v}$ is a virtual velocity field. Moreover, $D_{\boldsymbol{x}}, D_{\boldsymbol{v}}$ are Fréchet partial derivatives with respect to displacement and velocity, respectively.
The variation operator $\boldsymbol{\delta}$ depends linearly on the virtual displacement and the virtual velocity. Note a minor notational difference between the virtual quantity δ and variation operators $\boldsymbol{\delta}$. That is to distinguish the geometrical nature of virtual quantity, and the operational meaning of variation. In general, the variation of "something" and the virtual "something" are not equal, e.g. a virtual work may exists although there does not exist a work function at all, and neither the work variation.

Generally, the variation operator and the time derivative operator do not commute. In fact, let consider the constraint equation $\dot{\boldsymbol{x}}+t\dot{\boldsymbol{y}}=\boldsymbol{0}$, where t represents time variable. Its variation is $\boldsymbol{\delta}\dot{\boldsymbol{x}}+t\boldsymbol{\delta}\dot{\boldsymbol{y}}=\boldsymbol{0}$. On the other hand, the virtual displacement of the constraint equation is $\delta\boldsymbol{x}+t\delta\boldsymbol{y}=\boldsymbol{0}\Longleftrightarrow\boldsymbol{\delta}\boldsymbol{x}+t\boldsymbol{\delta}\boldsymbol{y}=\boldsymbol{0}$, whose time derivative is respectively

$$\frac{\mathrm{d}\boldsymbol{\delta}\boldsymbol{x}}{\mathrm{d}t}+\frac{\mathrm{d}\boldsymbol{\delta}\boldsymbol{y}}{\mathrm{d}t}+\boldsymbol{\delta}\boldsymbol{y}=\boldsymbol{0} \qquad (A.22)$$

This is clearly different from $\boldsymbol{\delta}\dot{\boldsymbol{x}}+t\boldsymbol{\delta}\dot{\boldsymbol{y}}=\boldsymbol{0}$, which is the variation of the original constraint equation (see [36]).

Appendix B

A family of trigonometric functions

Make everything as simple as possible, but not simpler.

A. Einstein

In order to alleviate the notation of $\mathbf{\Lambda}(\mathbf{\Psi})$ and $\boldsymbol{T}(\mathbf{\Psi})$ and their derivative quantities, it is useful to introduce a family of scalar quantities involving trigonometric functions for the Euler parameterization, which stem from a single power series expansion (see [53]). In this way, a more clear and rational presentation of the whole procedure is achieved. Let introduce the trigonometric functions

$$\begin{aligned} a_0(\Psi) &= \cos\Psi \\ a_1(\Psi) &= \frac{\sin\Psi}{\Psi} \\ a_2(\Psi) &= \frac{1-\cos\Psi}{\Psi^2} = \frac{1-a_0(\Psi)}{\Psi^2} \\ a_3(\Psi) &= \frac{\Psi-\sin\Psi}{\Psi^3} = \frac{1-a_1(\Psi)}{\Psi^2} \end{aligned}$$

where $\Psi = ||\mathbf{\Psi}||$. With this notation in hand, it results very useful to give the linearization of trigonometric functions $a_i(\Psi)$ with respect to $\delta\Psi$, which formally is written as

$$\delta(a_i(\Psi)) = \frac{\mathrm{d}a_i(\Psi)}{\mathrm{d}\Psi}\delta\Psi \tag{B.1}$$

with no matter of ambiguity since vector $\boldsymbol{\Psi}$ belongs to a linear space. Now, observing that by definition $\Psi^2 = \boldsymbol{\Psi} \cdot \boldsymbol{\Psi}$, linearizing both right-hand and left-hand sides we get

$$2\Psi\delta\Psi = \boldsymbol{\Psi} \cdot \delta\boldsymbol{\Psi} + \delta\boldsymbol{\Psi} \cdot \boldsymbol{\Psi} \quad \longrightarrow \quad \delta\Psi = \frac{\boldsymbol{\Psi} \cdot \delta\boldsymbol{\Psi}}{\Psi} \tag{B.2}$$

which substituted into (B.1) leads to

$$\begin{aligned} \delta(a_i(\Psi)) &= b_i(\Psi)(\boldsymbol{\Psi} \cdot \delta\boldsymbol{\Psi}) \\ b_i(\Psi) &= \frac{1}{\Psi}\frac{\mathrm{d}a_i(\Psi)}{\mathrm{d}\Psi} \end{aligned}$$

Each function $b_i(\Psi)$ is given explicitly by

$$\begin{aligned} b_0(\Psi) &= -\frac{\sin\Psi}{\Psi} \\ b_1(\Psi) &= \frac{\Psi\cos\Psi - \sin\Psi}{\Psi^3} \\ b_2(\Psi) &= \frac{\Psi\sin\Psi - 2 + 2\cos\Psi}{\Psi^4} = \frac{a_1 - 2a_2}{\Psi} \\ b_3(\Psi) &= \frac{3\sin\Psi - 2\Psi - \Psi\cos\Psi}{\Psi^5} = \frac{a_2 - 3a_3}{\Psi^2} \end{aligned}$$

in a fully analogous way to what done above, $c_i(\Psi)$ functions can be defined as

$$\begin{aligned} \delta(b_i(\Psi)) &= c_i(\Psi)(\boldsymbol{\Psi} \cdot \delta\boldsymbol{\Psi}) \\ c_i(\Psi) &= \frac{1}{\Psi}\frac{\mathrm{d}b_i(\Psi)}{\mathrm{d}\Psi} \end{aligned}$$

and are explicitly given by

$$\begin{aligned} c_0(\Psi) &= \frac{\sin\Psi - \Psi\cos\Psi}{\Psi^3} \\ c_1(\Psi) &= \frac{3\sin\Psi - \Psi^2\sin\Psi - 3\Psi\cos\Psi}{\Psi^5} \\ c_2(\Psi) &= \frac{8 - 8\cos\Psi - 5\Psi\sin\Psi + \Psi^2\cos\Psi}{\Psi^6} \\ c_3(\Psi) &= \frac{8\Psi + 7\Psi\cos\Psi + \Psi^2\sin\Psi - 15\sin\Psi}{\Psi^7} \end{aligned}$$

An extensive treatment of this subject can be found in [53].

Appendix C

Compound rotation about fixed axis

'Per me si va ne la città dolente,
per me si va ne l'etterno dolore,
per me si va tra la perduta gente.
Giustizia mosse il mio alto fattore;
fecemi la divina podestate,
la somma sapïenza e 'l primo amore.
Dinanzi a me non fuor cose create
se non etterne, e io etterno duro.
Lasciate ogni speranza, voi
ch'intrate'.

D. Alighieri, Inferno – Canto III

Here we provide a demonstration (see e.g. [1]) of the updated formula for rotation field parametrized according Rodrigues (see (2.65)), useful to characterize the sequential application of multiple rotations in Cayley-Rodrigues parametrization. Let a compound rotation consisting of two consecutive individual rotations, indicate with subscript (1) and (2), and consider the rotation of a rigid body motion as depicted in Figure C.1. Noting that M is the midpoint of $\overrightarrow{PP_1} = \boldsymbol{p}_\Delta$ we have

$$\overrightarrow{AM} = \frac{\boldsymbol{p} + \boldsymbol{p_1}}{2} \quad \rightarrow \quad \boldsymbol{p} + \boldsymbol{p_1} = 2\,\overrightarrow{AM} \tag{C.1}$$

where $\boldsymbol{p}_\Delta$ is the incremental position vector, defined according to the additional formula

$$\boldsymbol{p} + \boldsymbol{p}_\Delta = \boldsymbol{p_1} \quad \rightarrow \quad \boldsymbol{p_1} - \boldsymbol{p} = \boldsymbol{p}_\Delta \tag{C.2}$$

Since $\overrightarrow{CM}$ is perpendicular to $\overrightarrow{PP_1}$, we have

$$PM = \frac{1}{2} p_\Delta = CM\ \tan\theta/2 \tag{C.3}$$

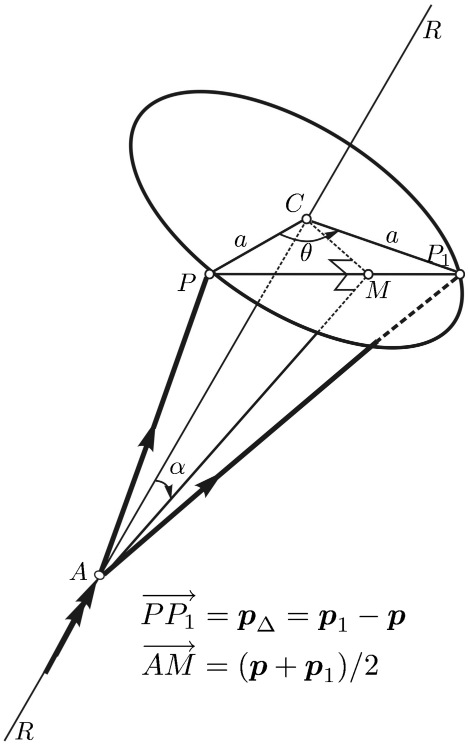


Figure C.1: On the vectorial-geometrical proof of $\boldsymbol{p}_1 - \boldsymbol{p} = \boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{p}_1)$ or $\boldsymbol{p}_1 - \boldsymbol{p} = \boldsymbol{R}[\boldsymbol{p} + \boldsymbol{p}_1]$.

Furthermore, we observe in Figure C.1 that $\boldsymbol{p}_\Delta$ stands contemporaneously normal on $\boldsymbol{\omega}$ and $2\overrightarrow{AM} = \boldsymbol{p} + \boldsymbol{p}_1$ and is hence concurrent with $\boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{p}_1)$. Noting (C.3) and the angle α between $\boldsymbol{\omega}$ and $\overrightarrow{AM}$, its magnitude is

$$\begin{aligned} |\boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{p}_1)| &= \sin\alpha \cdot 2\tan\frac{\theta}{2} \cdot 2\,AM \\ &= \frac{CM}{AM} \cdot 2\tan\frac{\theta}{2} \cdot 2\,AM \\ &= 2\,CM \cdot 2\tan\frac{\theta}{2} = |\boldsymbol{p}_\Delta| \end{aligned} \tag{C.4}$$

Thus we obtain the simple relation

$$2\boldsymbol{p}_\Delta = \boldsymbol{\omega} \times (\boldsymbol{p} + \boldsymbol{p}_1) = 2(\boldsymbol{p}_1 - \boldsymbol{p}) \tag{C.5}$$

Bearing in mind the definition of $\boldsymbol{R}$ (see (2.70)), equation (C.5) can also be rewritten in the more convenient matrix form

$$\boxed{2\boldsymbol{p}_1 - \boldsymbol{p} = \boldsymbol{R}\,[\boldsymbol{p} + \boldsymbol{p}_1]} \tag{C.6}$$

The reader should note again the vectorial equivalence of the operators $\boldsymbol{\omega} \times \boldsymbol{p}$ and $\boldsymbol{R} \times \boldsymbol{p}$ (see also (2.71)). In the subsequent analysis we also require the relation

$$\boldsymbol{\omega}^T \cdot \boldsymbol{p}_1 = \boldsymbol{\omega}^T \cdot \boldsymbol{p} \tag{C.7}$$

which follows immediately from the observation that the angles enclosed by $\boldsymbol{\omega}$, $\boldsymbol{p}$ and $\boldsymbol{\omega}$, $\boldsymbol{p}_1$ are equal and that furthermore $|\boldsymbol{p}_1| = |\boldsymbol{p}|$.

We are now in the position to examine a compound rotation exerting over two parameter vectors $\boldsymbol{\omega}_1$, $\boldsymbol{\omega}$ and assume first that the axis are fixed in space. Applying first a rotation $\boldsymbol{\omega}_1$, equation (C.6) becomes

$$2(\boldsymbol{p}_1 - \boldsymbol{p}) = \boldsymbol{R}_1 [\boldsymbol{p} + \boldsymbol{p}_1] \tag{C.8}$$

The subsequent rotation $\boldsymbol{\omega}_2$ transports $\boldsymbol{p}_1$ to $\boldsymbol{p}_2$ for which we have

$$2(\boldsymbol{p}_2 - \boldsymbol{p}_1) = \boldsymbol{R}_2 [\boldsymbol{p}_1 + \boldsymbol{p}_2] \tag{C.9}$$

Being our aim to establish a single compound rotation vector $\boldsymbol{\omega}_I$ and its associated matrix $\boldsymbol{R}_I$, which transfers the vector $\boldsymbol{p}$ in one go to $\boldsymbol{p}_2$. Following (C.5) and (C.6) the necessary condition reads

$$2(\boldsymbol{p}_2 - \boldsymbol{p}) = \omega_I^r \times [\boldsymbol{p}_2 + \boldsymbol{p}] \quad \text{or} \quad 2(\boldsymbol{p}_2 - \boldsymbol{p}) = \boldsymbol{R}_I^r [\boldsymbol{p}_2 + \boldsymbol{p}] \tag{C.10}$$

where the superscript "r" refers to axis fixed in space. We adopt in the analysis the more convenient matrix notation and obtain by addition of (C.8) and (C.9)

$$2(\boldsymbol{p}_2 - \boldsymbol{p}) = \boldsymbol{R}_1[\boldsymbol{p} + \boldsymbol{p}_1] + \boldsymbol{R}_2[\boldsymbol{p}_1 + \boldsymbol{p}_2]$$

adding and subtracting $\boldsymbol{R}_1\boldsymbol{p}_2$ and $\boldsymbol{R}_2\boldsymbol{p}$, it may be rearranged into

$$2(\boldsymbol{p}_2 - \boldsymbol{p}) = [\boldsymbol{R}_1 + \boldsymbol{R}_2][\boldsymbol{p}_2 + \boldsymbol{p}] - \boldsymbol{R}_1[\boldsymbol{p}_2 - \boldsymbol{p}_1] + \boldsymbol{R}_2[\boldsymbol{p}_1 - \boldsymbol{p}]$$

Using now (C.8) and (C.9) in the last two terms we obtain

$$\begin{aligned}
2(\boldsymbol{p}_2 - \boldsymbol{p}) &= [\boldsymbol{R}_1 + \boldsymbol{R}_2][\boldsymbol{p}_2 + \boldsymbol{p}] - \boldsymbol{R}_1\frac{\boldsymbol{R}_2}{2}[\boldsymbol{p}_2 + \boldsymbol{p}_1] + \boldsymbol{R}_2\frac{\boldsymbol{R}_1}{2}[\boldsymbol{p}_1 + \boldsymbol{p}] \\
&= \left[\boldsymbol{R}_1 + \boldsymbol{R}_2 - [\boldsymbol{R}_1\frac{\boldsymbol{R}_2}{2} - \boldsymbol{R}_2\frac{\boldsymbol{R}_1}{2}]\right][\boldsymbol{p}_2 + \boldsymbol{p}] \\
&\quad - \boldsymbol{R}_1\frac{\boldsymbol{R}_2}{2}[\boldsymbol{p}_1 - \boldsymbol{p}] - \boldsymbol{R}_2\frac{\boldsymbol{R}_1}{2}[\boldsymbol{p}_2 - \boldsymbol{p}_1]
\end{aligned} \tag{C.11}$$

By means of equivalence (2.71) and recalling the double cross product identity, for the last two terms we can observe

$$\begin{aligned}
-\boldsymbol{R}_1\frac{\boldsymbol{R}_2}{2}(\boldsymbol{p}_1 - \boldsymbol{p}) &= -\boldsymbol{R}_1\left[\frac{\boldsymbol{\omega}_2}{2} \times (\boldsymbol{p}_1 - \boldsymbol{p})\right] \\
&= -\boldsymbol{\omega}_1 \times \left[\frac{\boldsymbol{\omega}_2}{2} \times (\boldsymbol{p}_1 - \boldsymbol{p})\right] \\
&= -[\boldsymbol{\omega}_1^T(\boldsymbol{p}_1 - \boldsymbol{p})]\frac{\boldsymbol{\omega}_2}{2} + \left(\boldsymbol{\omega}_1^T \cdot \frac{\boldsymbol{\omega}_2}{2}\right)(\boldsymbol{p}_1 - \boldsymbol{p}) \\
&= \left(\boldsymbol{\omega}_1^T \cdot \frac{\boldsymbol{\omega}_2}{2}\right)(\boldsymbol{p}_1 - \boldsymbol{p})
\end{aligned} \tag{C.12}$$

$$\begin{aligned}
-\boldsymbol{R}_2\frac{\boldsymbol{R}_1}{2}(\boldsymbol{p}_2 - \boldsymbol{p}_1) &= -\boldsymbol{R}_2\left[\frac{\boldsymbol{\omega}_1}{2} \times (\boldsymbol{p}_2 - \boldsymbol{p}_1)\right] \\
&= -\boldsymbol{\omega}_2 \times \left[\frac{\boldsymbol{\omega}_1}{2} \times (\boldsymbol{p}_2 - \boldsymbol{p}_1)\right] \\
&= -[\boldsymbol{\omega}_2^T(\boldsymbol{p}_2 - \boldsymbol{p}_1)]\frac{\boldsymbol{\omega}_1}{2} + \left(\boldsymbol{\omega}_2^T \cdot \frac{\boldsymbol{\omega}_1}{2}\right)(\boldsymbol{p}_2 - \boldsymbol{p}_1) \\
&= \left(\frac{\boldsymbol{\omega}_1{}^T}{2} \cdot \boldsymbol{\omega}_2\right)(\boldsymbol{p}_2 - \boldsymbol{p}_1)
\end{aligned} \tag{C.13}$$

where we used the result (C.7) as

$$\boldsymbol{\omega}_1^T \cdot \boldsymbol{p}_1 = \boldsymbol{\omega}_2^T \cdot \boldsymbol{p} \qquad \text{and} \qquad \boldsymbol{\omega}_2^T \cdot \boldsymbol{p}_2 = \boldsymbol{\omega}_2^T \cdot \boldsymbol{p}_1 \tag{C.14}$$

Substituting (C.12) and (C.13) into (C.11) we find

$$\begin{aligned} 2(\boldsymbol{p}_2 - \boldsymbol{p}) &= \left[\boldsymbol{R}_1 + \boldsymbol{R}_2 - \left(\boldsymbol{R}_1\frac{\boldsymbol{R}_2}{2} - \boldsymbol{R}_2\frac{\boldsymbol{R}_1}{2}\right)\right][\boldsymbol{p}_2 + \boldsymbol{p}] + \frac{1}{2}(\boldsymbol{\omega}_1^T \cdot \boldsymbol{\omega_2})(\boldsymbol{p}_2 - \boldsymbol{p}) \\ &= \frac{4}{4 - \boldsymbol{\omega}_1^T \cdot \boldsymbol{\omega_2}}\left[\boldsymbol{R}_1 + \boldsymbol{R}_2 - \left(\boldsymbol{R}_1\frac{\boldsymbol{R}_2}{2} - \boldsymbol{R}_2\frac{\boldsymbol{R}_1}{2}\right)\right][\boldsymbol{p}_2 + \boldsymbol{p}] \end{aligned} \tag{C.15}$$

Comparison with the second equation in (C.10) yields the desired expression for the equivalent auxiliary matrix

$$\boldsymbol{R}_I^r = \frac{4}{4 - \boldsymbol{\omega}_1^T \cdot \boldsymbol{\omega_2}}\left[\boldsymbol{R}_1 + \boldsymbol{R}_2 - \left(\boldsymbol{R}_1\frac{\boldsymbol{R}_2}{2} - \boldsymbol{R}_2\frac{\boldsymbol{R}_1}{2}\right)\right] \tag{C.16}$$

Finally, remembering the equivalence of $\boldsymbol{R}$ and $[\boldsymbol{\omega}\times]$ operations and their extension which turns in $[\boldsymbol{R}_1\frac{\boldsymbol{R}_2}{2} - \boldsymbol{R}_2\frac{\boldsymbol{R}_1}{2}]\boldsymbol{p} = \frac{1}{2}(\boldsymbol{\omega}_1 \times \boldsymbol{\omega}_2) \times \boldsymbol{p}$ we also deduce the compound rotation vector $\boldsymbol{\omega}_I^r$ in the form

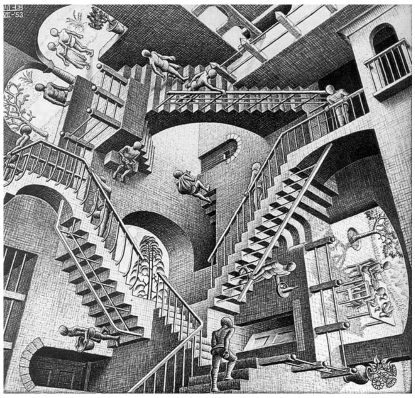
$$\boldsymbol{\omega}_I^r = \frac{4}{4 - \boldsymbol{\omega}_1^T \cdot \boldsymbol{\omega_2}}\left[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 - \frac{1}{2}\boldsymbol{\omega}_1 \times \boldsymbol{\omega_2}\right] \tag{C.17}$$

which using the cross product anticommutativity becomes

$$\boxed{\boldsymbol{\omega}_I^r = \frac{4}{4 - \boldsymbol{\omega}_1^T \cdot \boldsymbol{\omega_2}}\left[\boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 + \frac{1}{2}\boldsymbol{\omega_2} \times \boldsymbol{\omega}_1\right]} \tag{C.18}$$

Appendix D

Spin-like vectors by total rotation vector



A mathematician is a machine for turning coffee into theorems.

P. Erdös

In this appendix we report the entire algebraic deduction of the angular velocity vector $\boldsymbol{\omega}$ expressed in terms of total rotation vector time derivative, and by analogy the expression of curvature vector in terms of total rotation vector spatial derivative.

Angular velocity vectors by total rotation vector. The spatial skew-symmetric tensor $\tilde{\boldsymbol{\omega}}$ associated with the arbitrary rotational motion $\boldsymbol{\Lambda}$ is defined by $\tilde{\boldsymbol{\omega}} = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T$, with its axial vector $\boldsymbol{\omega} = \text{axial}[\tilde{\boldsymbol{\omega}}]$ being called the angular velocity vector or spin vector. Here we want to relate spin-like variables with the appropriate variation of the rotation vector.
In order to alleviate the notation, here we exploit the families of trigonometric functions introduced in Appendix B (according e.g. to what presented in [53]). Therefore (2.47) easily becomes

$$\boldsymbol{\Lambda} = \boldsymbol{I} + a_1(\Psi)\tilde{\boldsymbol{\Psi}} + a_2(\Psi)\tilde{\boldsymbol{\Psi}}^2 \tag{D.1}$$

$$\boldsymbol{\Lambda}^T = \boldsymbol{I} - a_1(\Psi)\tilde{\boldsymbol{\Psi}} + a_2(\Psi)\tilde{\boldsymbol{\Psi}}^2 \tag{D.2}$$

whereas its time derivative

$$\begin{aligned}\dot{\boldsymbol{\Lambda}} &= (a_1\tilde{\boldsymbol{\Psi}})^{\cdot} + (a_2\tilde{\boldsymbol{\Psi}}^2)^{\cdot} \\ &= \dot{a}_1\tilde{\boldsymbol{\Psi}} + a_1\dot{\tilde{\boldsymbol{\Psi}}} + \dot{a}_2\tilde{\boldsymbol{\Psi}}^2 + a_2(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}})\end{aligned} \tag{D.3}$$

where, for sake of simplicity, we neglect the dependence on rotation of trigonometric functions. Hence, by substitution, the material angular velocity tensor takes form

$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \dot{\boldsymbol{\Lambda}}\cdot\boldsymbol{\Lambda}^T \\ &= \left[\dot{a}_1\tilde{\boldsymbol{\Psi}} + a_1\dot{\tilde{\boldsymbol{\Psi}}} + \dot{a}_2\tilde{\boldsymbol{\Psi}}^2 + a_2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\right)\right]\cdot(\boldsymbol{I} - a_1\tilde{\boldsymbol{\Psi}} + a_2\tilde{\boldsymbol{\Psi}}^2) \\ &= \dot{a}_1\tilde{\boldsymbol{\Psi}} + a_1\dot{\tilde{\boldsymbol{\Psi}}} + \dot{a}_2\tilde{\boldsymbol{\Psi}}^2 + a_2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\right) + \\ &\quad - \dot{a}_1a_1\tilde{\boldsymbol{\Psi}}^2 - a_1^2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} - a_1\dot{a}_2\tilde{\boldsymbol{\Psi}}^3 - a_1a_2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2 + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}\right) + \\ &\quad + \dot{a}_1a_2\tilde{\boldsymbol{\Psi}}^3 + a_1a_2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2 + \dot{a}_2a_2\tilde{\boldsymbol{\Psi}}^4 + a_2^2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^3 + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2\right) + \\ &= \dot{a}_1\tilde{\boldsymbol{\Psi}} - \dot{a}_1a_1\tilde{\boldsymbol{\Psi}}^2 + \dot{a}_1a_2\tilde{\boldsymbol{\Psi}}^3 + a_1\dot{\tilde{\boldsymbol{\Psi}}} - a_1^2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + a_1a_2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2 + \\ &\quad + \dot{a}_2\tilde{\boldsymbol{\Psi}}^2 - a_1\dot{a}_2\tilde{\boldsymbol{\Psi}}^3 + \dot{a}_2a_2\tilde{\boldsymbol{\Psi}}^4 + a_2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\right) + \\ &\quad - a_1a_2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\right)\tilde{\boldsymbol{\Psi}} + a_2^2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\right)\tilde{\boldsymbol{\Psi}}^2\end{aligned} \tag{D.4}$$

Noting that

$$-a_1a_2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\right)\tilde{\boldsymbol{\Psi}} = -a_1a_2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2 - a_1a_2\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} \tag{D.5}$$

$$a_2^2\left(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\right)\tilde{\boldsymbol{\Psi}}^2 = a_2^2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^3 + a_2^2\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2 \tag{D.6}$$

and recalling properties (2.60)

$$\begin{aligned}\tilde{\boldsymbol{\Psi}}^3 &= -\psi^2\,\tilde{\boldsymbol{\Psi}} \\ \tilde{\boldsymbol{\Psi}}^4 &= -\psi^2\,\tilde{\boldsymbol{\Psi}}^2\end{aligned} \tag{D.7}$$

one obtain

$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= \dot{a}_1\tilde{\boldsymbol{\Psi}} - \dot{a}_1a_1\tilde{\boldsymbol{\Psi}}^2 - \Psi^2\dot{a}_1a_1\tilde{\boldsymbol{\Psi}} + \\ &\quad + a_1\dot{\tilde{\boldsymbol{\Psi}}} - a_1^2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + a_1a_2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2 + \\ &\quad + \dot{a}_2\tilde{\boldsymbol{\Psi}}^2 + \Psi^2a_1\dot{a}_2\tilde{\boldsymbol{\Psi}} - \Psi^2a_2\dot{a}_2\tilde{\boldsymbol{\Psi}}^2 + \\ &\quad + a_2(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}) - a_1a_2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2 - a_1a_2\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \\ &\quad - \Psi^2a_2^2\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + a_2^2\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}^2\end{aligned} \tag{D.8}$$

Putting out some term one get

$$\begin{aligned}\tilde{\boldsymbol{\omega}} &= a_1\tilde{\boldsymbol{\Psi}} + a_2\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}} + (a_2 - a_1^2 - \Psi^2a_2^2)\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \\ &\quad + (\dot{a}_1 - a_1a_2\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}} + \Psi^2a_1\dot{a}_2 - \Psi^2\dot{a}_1a_2)\tilde{\boldsymbol{\Psi}} + \\ &\quad + (\dot{a}_2 - \dot{a}_1a_1 - \Psi^2\dot{a}_2a_2 + a_2^2\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}})\tilde{\boldsymbol{\Psi}}^2\end{aligned} \tag{D.9}$$

Now using property (2.20) for two skew-symmetric tensors $\tilde{\boldsymbol{A}}$ and $\tilde{\boldsymbol{B}}$ with their relative axial vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, such that $\tilde{\boldsymbol{A}}\,\tilde{\boldsymbol{B}} = \boldsymbol{b} \otimes \boldsymbol{a} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{I}$, we recognize the following identity

$$
\begin{aligned}
(\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}})\tilde{\boldsymbol{\Psi}} &= \left[\dot{\boldsymbol{\Psi}} \otimes \boldsymbol{\Psi} - (\boldsymbol{\Psi} \cdot \dot{\boldsymbol{\Psi}})\boldsymbol{I}\right] \tilde{\boldsymbol{\Psi}} \\
&= (\dot{\boldsymbol{\Psi}} \otimes \boldsymbol{\Psi})\tilde{\boldsymbol{\Psi}} - (\dot{\boldsymbol{\Psi}} \cdot \dot{\boldsymbol{\Psi}})\tilde{\boldsymbol{\Psi}} \\
&= \dot{\boldsymbol{\Psi}} \otimes (\tilde{\boldsymbol{\Psi}}^T \boldsymbol{\Psi}) - (\boldsymbol{\Psi}\dot{\boldsymbol{\Psi}})\tilde{\boldsymbol{\Psi}} \\
&= -\dot{\boldsymbol{\Psi}} \otimes (\tilde{\boldsymbol{\Psi}}\boldsymbol{\Psi}) - (\boldsymbol{\Psi}\dot{\boldsymbol{\Psi}})\tilde{\boldsymbol{\Psi}} \\
&= -\dot{\boldsymbol{\Psi}} \otimes (\boldsymbol{\Psi} \times \boldsymbol{\Psi}) - (\boldsymbol{\Psi}\dot{\boldsymbol{\Psi}})\tilde{\boldsymbol{\Psi}} \\
&= -(\boldsymbol{\Psi}\dot{\boldsymbol{\Psi}})\tilde{\boldsymbol{\Psi}}
\end{aligned} \tag{D.10}
$$

where the property of cross product between parallel vectors is used. Sticking equation (D.10) into equation (D.9) we get

$$
\begin{aligned}
\tilde{\boldsymbol{\omega}} = a_1\tilde{\boldsymbol{\Psi}} &+ a_2\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}} + (a_2 - a_1^2 - \Psi^2 a_2^2)\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \\
&+ \left[\dot{a}_1 + a_1 a_2(\boldsymbol{\Psi} \cdot \dot{\boldsymbol{\Psi}}) + \Psi^2 a_1 \dot{a}_2 - \Psi^2 \dot{a}_1 a_2\right] \tilde{\boldsymbol{\Psi}} + \\
&+ \left[\dot{a}_2 - \dot{a}_1 a_1 - \Psi^2 \dot{a}_2 a_2 - a_2^2(\boldsymbol{\Psi} \cdot \dot{\boldsymbol{\Psi}})\right] \tilde{\boldsymbol{\Psi}}^2
\end{aligned} \tag{D.11}
$$

After some manipulations, the following identities are proved

$$
\begin{aligned}
a_2 - a_1^2 - \Psi^2 a_2^2 &= \frac{1 - \cos\Psi}{\Psi^2} - \frac{\sin^2\Psi}{\Psi^2} - \Psi^2\frac{(1 - \cos\Psi)^2}{\Psi^4} \\
&= \frac{\Psi^2(1 - \cos\Psi) - \Psi^2\sin^2\Psi - \Psi^2(1 - \cos\Psi)^2}{\Psi^4} \\
&= \frac{(1 - \cos\Psi) - \sin^2\Psi - (1 - \cos\Psi)^2}{\Psi^2} \\
&= \frac{1 - \cos\Psi - \sin^2\Psi - 1 - \cos\Psi^2 + 2\cos\Psi}{\Psi^2} \\
&= \frac{-1 - \cos\Psi + 2\cos\Psi}{\Psi^2} \\
&= \frac{-1 + \cos\Psi}{\Psi^2} \\
&= -\frac{1 - \cos\Psi}{\Psi^2} = -a_2
\end{aligned} \tag{D.12}
$$

Further useful identity comes from

$$
\Psi = ||\boldsymbol{\Psi}|| = \sqrt{\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}} \;\Rightarrow\; \dot{\Psi} = \frac{2\,\boldsymbol{\Psi} \cdot \dot{\boldsymbol{\Psi}}}{2\sqrt{\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}}} = \frac{\boldsymbol{\Psi}}{\Psi} \cdot \dot{\boldsymbol{\Psi}} \tag{D.13}
$$

such that the following identities holds

$$\begin{aligned}
\dot{a}_1 + & a_1 a_2(\boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}}) + \Psi^2 a_1\dot{a}_2 - \Psi^2\dot{a}_1 a_2 = \\
&= \frac{\cos\Psi - a_1}{\Psi}\dot{\Psi} + \frac{\sin\Psi}{\Psi}\frac{1-\cos\Psi}{\Psi^2}\Psi\dot{\Psi} + \Psi^2\frac{\sin\Psi}{\Psi}\frac{a_1 - 2a_2}{\Psi}\dot{\Psi} + \\
&\quad - \Psi^2\frac{\cos\Psi - a_1}{\Psi}\dot{\Psi}\frac{1-\cos\Psi}{\Psi^2} \\
&= \frac{\dot{\Psi}}{\Psi}[\cos\Psi - a_1 + a_1(1-\cos\Psi) + a_1\Psi\sin\Psi - 2a_2\Psi\sin\Psi \\
&\quad - (\cos\Psi - a_1)(1-\cos\Psi)] \\
&= \frac{\dot{\Psi}}{\Psi}[\cos\Psi - a_1 + a_1 - a_1\cos\Psi + \sin^2\Psi - 2\frac{1-\cos\Psi}{\Psi^2}\Psi\sin\Psi + \\
&\quad - (\cos\Psi - \cos^2\Psi - a_1 + a_1\cos\Psi)] \\
&= \frac{\dot{\Psi}}{\Psi}[a_1 - a_1\cos\Psi + \sin^2\Psi - 2a_2 + 2a_1\cos\Psi + \cos^2\Psi - a_1\cos\Psi] \\
&= \frac{\dot{\Psi}}{\Psi}(-a_1 + 1) \\
&= \dot{\Psi}\Psi\frac{1-a_1}{\Psi^2} \\
&= \dot{\boldsymbol{\Psi}}\cdot\boldsymbol{\Psi}\, a_3
\end{aligned} \tag{D.14}$$

and also

$$\begin{aligned}
\dot{a}_2 - & \dot{a}_1 a_1 - \Psi^2\dot{a}_2 a_2 - a_2^2(\boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}}) = \\
&= \dot{a}_2 - \dot{a}_1 a_1 - \Psi^2\dot{a}_2 a_2 - a_2^2(\Psi\cdot\dot{\Psi}) \\
&= \frac{a_1 - 2a_2}{\Psi}\dot{\Psi} - a_1\frac{\cos\Psi - a_1}{\Psi}\dot{\Psi} - \Psi^2\frac{a_1 - 2a_2}{\Psi}\dot{\Psi}a_2 - a_2^2\Psi\dot{\Psi} \\
&= \frac{a_1 - 2a_2 - a_1\cos\Psi + a_1^2 - \Psi^2 a_1 a_2 + 2a_2^2\Psi^2 - a_2^2\Psi^2}{\Psi}\dot{\Psi} \\
&= \left[\frac{\sin\Psi}{\Psi} - 2\frac{1-\cos\Psi}{\Psi^2} - \frac{\sin\Psi}{\Psi}\cos\Psi + \frac{\sin^2\Psi}{\Psi^2} + \right. \\
&\quad \left. -\Psi^2\frac{\sin\Psi}{\Psi}\frac{1-\cos\Psi}{\Psi^2} + 2\frac{(1-\cos\Psi)^2}{\Psi^2} - \frac{(1-\cos\Psi)^2}{\Psi^2}\right]\frac{\dot{\Psi}}{\Psi} \\
&= \frac{\Psi\sin\Psi - 2 + 2\cos\Psi - \Psi\sin\Psi\cos\Psi + \sin^2\Psi - \Psi\sin\Psi(1-\cos\Psi) + (1-\cos\Psi)^2}{\Psi^2}\frac{\dot{\Psi}}{\Psi} \\
&= 0
\end{aligned} \tag{D.15}$$

and finally equation (D.12) simplifies in

$$\tilde{\boldsymbol{\omega}} = a_1\dot{\tilde{\boldsymbol{\Psi}}} + a_2(\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}} - \dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}) + a_3(\dot{\boldsymbol{\Psi}}\cdot\boldsymbol{\Psi})\tilde{\boldsymbol{\Psi}} \tag{D.16}$$

Now, recalling that $\boldsymbol{\omega} = \text{axial}[\tilde{\boldsymbol{\omega}}]$ equation (D.16) turns into

$$\boldsymbol{\omega} = \text{axial}\left[a_1\dot{\tilde{\boldsymbol{\Psi}}} + a_2(\tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}} - \dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}}) + a_3(\dot{\boldsymbol{\Psi}}\cdot\boldsymbol{\Psi})\tilde{\boldsymbol{\Psi}}\right] \tag{D.17}$$

where focusing on the second term, the parallel use of Lie algebra (2.14) and Lie brackets (2.20) to the generic skew-symmetric tensors $\tilde{\boldsymbol{A}}, \tilde{\boldsymbol{B}}, \tilde{\boldsymbol{C}}$ and their relative axial vectors $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ provides the following correspondences

$$\begin{aligned}
&\text{Lie algebra:} \quad \tilde{\boldsymbol{C}}\,\boldsymbol{h} = \boldsymbol{c} \times \boldsymbol{h} \\
&\text{Lie brackets: } (\tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}})\boldsymbol{h} = (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{h} \\
&\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{A}}\tilde{\boldsymbol{B}} - \tilde{\boldsymbol{B}}\tilde{\boldsymbol{A}} \;\longleftrightarrow\; \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = \tilde{\boldsymbol{A}}\,\boldsymbol{b}
\end{aligned} \tag{D.18}$$

such that

$$\tilde{\boldsymbol{C}} = \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}} - \dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} \;\longleftrightarrow\; \boldsymbol{c} = \boldsymbol{\Psi} \times \dot{\boldsymbol{\Psi}} = \tilde{\boldsymbol{\Psi}}\,\dot{\boldsymbol{\Psi}} \tag{D.19}$$

In addition, bearing in mind tensor product definition $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{c} = (\boldsymbol{b} \cdot \boldsymbol{C})\boldsymbol{a}$, and the identity $\boldsymbol{\Psi} \otimes \boldsymbol{\Psi} = \tilde{\boldsymbol{\Psi}}^2 + (\boldsymbol{\Psi} \cdot \boldsymbol{\Psi})\boldsymbol{I}$ (see (2.22)), after some manipulations equation (D.17) becomes

$$\begin{aligned}
\boldsymbol{\omega} = \text{axial}[\tilde{\boldsymbol{\omega}}] &= a_1\dot{\boldsymbol{\Psi}} + a_2\tilde{\boldsymbol{\Psi}}\dot{\boldsymbol{\Psi}} + a_3(\dot{\boldsymbol{\Psi}} \cdot \boldsymbol{\Psi})\boldsymbol{\Psi} \\
&= a_1\dot{\boldsymbol{\Psi}} + a_2\tilde{\boldsymbol{\Psi}}\dot{\boldsymbol{\Psi}} + a_3(\boldsymbol{\Psi} \otimes \boldsymbol{\Psi})\dot{\boldsymbol{\Psi}} \\
&= a_1\dot{\boldsymbol{\Psi}} + a_2\tilde{\boldsymbol{\Psi}}\dot{\boldsymbol{\Psi}} + a_3\left[\tilde{\boldsymbol{\Psi}}^2 + (\boldsymbol{\Psi} \cdot \boldsymbol{\Psi})\,\boldsymbol{I}\right]\dot{\boldsymbol{\Psi}} \\
&= \left\{a_1 + a_2\tilde{\boldsymbol{\Psi}} + a_3\left[\tilde{\boldsymbol{\Psi}}^2 + (\boldsymbol{\Psi} \cdot \boldsymbol{\Psi})\boldsymbol{I}\right]\right\}\dot{\boldsymbol{\Psi}} \\
&= \left\{[a_1 + a_3(\boldsymbol{\Psi} \cdot \boldsymbol{\Psi})]\,\boldsymbol{I} + a_2\tilde{\boldsymbol{\Psi}} + a_3\tilde{\boldsymbol{\Psi}}^2\right\}\dot{\boldsymbol{\Psi}} \\
&= \left[\boldsymbol{I} + a_2\tilde{\boldsymbol{\Psi}} + a_3\tilde{\boldsymbol{\Psi}}^2\right]\dot{\boldsymbol{\Psi}}
\end{aligned} \tag{D.20}$$

and therefore the final expression of angular velocity vector $\boldsymbol{\omega}$ is

$$\boxed{\boldsymbol{\omega} = \boldsymbol{T}\,\dot{\boldsymbol{\Psi}}} \tag{D.21}$$

where is setting

$$\boldsymbol{T} = \boldsymbol{I} + a_2\tilde{\boldsymbol{\Psi}} + a_3\tilde{\boldsymbol{\Psi}}^2 \tag{D.22}$$

Curvature vector by total rotation vector. The spatial skew-symmetric tensor $\tilde{\boldsymbol{\kappa}}$ associated with the arbitrary rotational motion $\boldsymbol{\Lambda}$ is defined by $\tilde{\boldsymbol{\kappa}} = \boldsymbol{\Lambda}'\boldsymbol{\Lambda}^T$, with its axial vector $\boldsymbol{\kappa} = \text{axial}\,[\tilde{\boldsymbol{\kappa}}]$ being called the curvature vector. Since no conceptual differences appear in this relation with respect to what presented above, one can proof that this spin-like variable can be related to the variation of rotation vector via an expression analogous to equation (D.21)

$$\boxed{\boldsymbol{\kappa} = \boldsymbol{T}\,\boldsymbol{\Psi}'} \tag{D.23}$$

where $\boldsymbol{T}$ is still given by equation (D.22).

Derivative of transformation tensor. It's useful to note that by direct differentiation of equation (D.22) one get the the following spatial derivative

$$\begin{aligned}
\boldsymbol{T}' &= a_2'\tilde{\boldsymbol{\Psi}} + a_2\tilde{\boldsymbol{\Psi}}' + a_3'\tilde{\boldsymbol{\Psi}}^2 + a_3(\tilde{\boldsymbol{\Psi}}^2)' \\
&= b_2(\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}')\tilde{\boldsymbol{\Psi}} + a_2\tilde{\boldsymbol{\Psi}}' + b_3(\boldsymbol{\Psi} \cdot \boldsymbol{\Psi}')\tilde{\boldsymbol{\Psi}}^2 + a_3(\tilde{\boldsymbol{\Psi}}'\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\tilde{\boldsymbol{\Psi}}')
\end{aligned} \tag{D.24}$$

Using the expansions in Taylor's series of the trigonometric coefficient up to the second order and making the necessary simplifications, we can easily calculate the limit value of the tensor $\boldsymbol{T}'$ as

$$\lim_{\boldsymbol{\Psi}\to\boldsymbol{0}} \boldsymbol{T}'(\boldsymbol{\Psi},\boldsymbol{\Psi}') = -\frac{1}{2}\tilde{\boldsymbol{\Psi}}' \tag{D.25}$$

Analogous expression could be find for the time derivative of the same tensor

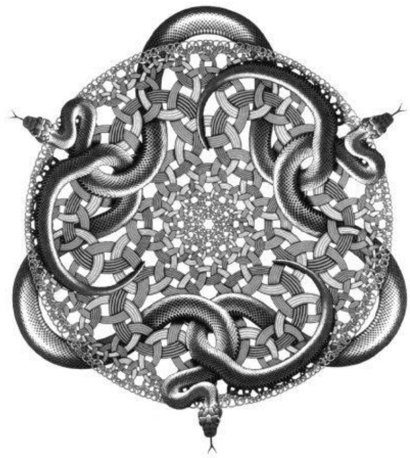
$$\begin{aligned}\dot{\boldsymbol{T}} &= \dot{a}_2\tilde{\boldsymbol{\Psi}} + a_2\dot{\tilde{\boldsymbol{\Psi}}} + \dot{a}_3\tilde{\boldsymbol{\Psi}}^2 + a_3(\dot{\tilde{\boldsymbol{\Psi}}})^2 \\ &= b_2(\boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}})\tilde{\boldsymbol{\Psi}} + a_2\dot{\tilde{\boldsymbol{\Psi}}} + b_3(\boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}})\tilde{\boldsymbol{\Psi}}^2 + a_3(\dot{\tilde{\boldsymbol{\Psi}}}\tilde{\boldsymbol{\Psi}} + \tilde{\boldsymbol{\Psi}}\dot{\tilde{\boldsymbol{\Psi}}})\end{aligned} \tag{D.26}$$

and as mentioned before, using Taylor's series we can compute the limit

$$\lim_{\boldsymbol{\Psi}\to\boldsymbol{0}} \dot{\boldsymbol{T}}(\boldsymbol{\Psi},\dot{\boldsymbol{\Psi}}) = -\frac{1}{2}\dot{\tilde{\boldsymbol{\Psi}}} \tag{D.27}$$

Appendix E

Alternative expression of angular momentum



I investigated the contours of an island, but what I discovered were the boundaries of the ocean.

L. Wittgenstein

In this appendix we recall some concepts derived from mechanics which regard the derivation of angular momentum and its time derivative in terms of inertial moment and spin vector $\boldsymbol{\omega}$.

Kinematics of rigid body. We introduce a reference frame of right-ended, rectangular coordinate axis, at a fixed origin O with orthonormal basis X_1, X_2, X_3, and a moving frame attached to a rigid body with origin G and orthonormal basis vector $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$. The generic point P of the body is defined by the vector

$$\overrightarrow{OP} = \overrightarrow{OG} + \overrightarrow{GP} = \overrightarrow{OG} + \xi_1 \boldsymbol{e_1} + \xi_2 \boldsymbol{e_2} + \xi_3 \boldsymbol{e_3} \tag{E.1}$$

where, in general, during the body motion the positions of point P and Q and the moving reference system are function of time, whereas are time-independent the spatial coordinates ξ_1, ξ_2, ξ_3, of point P. Taking the time derivative on both sides we get

$$\boldsymbol{v}_P = \boldsymbol{v}_G + \xi_1 \frac{d\boldsymbol{e_1}}{dt} + \xi_2 \frac{d\boldsymbol{e_2}}{dt} + \xi_3 \frac{d\boldsymbol{e_3}}{dt} \tag{E.2}$$

and expressing the derivative of spatial vector basis by means of Poisson formula we finally obtain

$$\begin{aligned}\boldsymbol{v}_P &= \boldsymbol{v}_G + \boldsymbol{\omega} \times (\xi_1 \boldsymbol{e_1} + \xi_2 \boldsymbol{e_2} + \xi_3 \boldsymbol{e_3}) \\ &= \boldsymbol{v}_G + \boldsymbol{\omega} \times \overrightarrow{GP}\end{aligned} \tag{E.3}$$

which is called fundamental equation of rigid motion. It represents the relation, at time t, between the velocities of two arbitrary points, P and G, of a moving rigid body.

Angular momentum of a rigid body. Let assume O a given point of the rigid body $\mathscr{B}$ assumed as a pole for the calculation of the angular momentum resultant. Assuming ρ the body density in the reference configuration. Recalling the (E.3) we have

$$\begin{aligned}\boldsymbol{L}_O &= \int_\Omega \rho\, \overrightarrow{OP} \times \boldsymbol{v}_P \, dV \\ &= \int_\Omega \rho\, \overrightarrow{OP} \times (\boldsymbol{v}_O + \boldsymbol{\omega} \times \overrightarrow{OP}) \, dV \\ &= \int_\Omega \rho\, \overrightarrow{OP} \times \boldsymbol{v}_O \, dV + \int_\Omega \rho\, \overrightarrow{OP} \times (\boldsymbol{\omega} \times \overrightarrow{OP}) dV \\ &= m\, \overrightarrow{OG} \times \boldsymbol{v}_O + \boldsymbol{I_O}\, \boldsymbol{\omega}\end{aligned} \tag{E.4}$$

where we used the relation

$$\begin{aligned}\overrightarrow{OP} \times (\boldsymbol{\omega} \times \overrightarrow{OP}) &= (\overrightarrow{OP})^2 \boldsymbol{\omega} - (\overrightarrow{OP} \cdot \boldsymbol{\omega}) \cdot \overrightarrow{OP} \\ &= ((\overrightarrow{OP})^2 \boldsymbol{I} - \overrightarrow{OP} \otimes \overrightarrow{OP}) \cdot \boldsymbol{\omega}\end{aligned} \tag{E.5}$$

and defining the inertia tensor $\mathbf{I_0}$ relative to the point O as

$$\boldsymbol{I_O} \doteq \int_\Omega \rho\, ((\overrightarrow{OP})^2 \boldsymbol{I} - \overrightarrow{OP} \otimes \overrightarrow{OP}) \, dV \tag{E.6}$$

For a rigid body, the balance equation of angular momentum with respect to a spatial pole O assumes an interesting form. By means of (E.4), the time derivative of angular momentum becomes

$$\begin{aligned}\frac{d\boldsymbol{L}}{dt} &= \frac{d}{dt}(\boldsymbol{I_O} \cdot \boldsymbol{\omega}) + m\, \frac{d\overrightarrow{OG}}{dt} \times \boldsymbol{v_O} + m\, \overrightarrow{OG} \times \frac{d\boldsymbol{v_O}}{dt} \\ &= \frac{d}{dt}(\boldsymbol{I_O} \cdot \boldsymbol{\omega}) + m\, (\boldsymbol{v_G} - \boldsymbol{v_O}) \times \boldsymbol{v_O} + m\, \overrightarrow{OG} \times \boldsymbol{a_O}\end{aligned} \tag{E.7}$$

To determine the value of first contribution $d(\boldsymbol{I_O}\, \boldsymbol{\omega})/dt$ one can observe that inertial operator is time-dependent with respect to inertial absolute reference system, therefore it is convenient express the absolute time derivative relative to the moving frame such that the inertial operator do not depend on time. Noting that $\boldsymbol{v_O} \times \boldsymbol{v_O} = \mathbf{0}$ we have

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= \frac{d_r(\boldsymbol{I_O}\,\boldsymbol{\omega})}{dt} + \boldsymbol{\omega} \times \boldsymbol{I_O}\,\boldsymbol{\omega} + m\boldsymbol{v_G} \times \boldsymbol{v_O} + m\overrightarrow{OG} \times \boldsymbol{a_O} \\ &= \boldsymbol{I_O}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I_O}\,\boldsymbol{\omega} + m\boldsymbol{v_G} \times \boldsymbol{v_O} + m\overrightarrow{OG} \times \boldsymbol{a_O} \end{aligned} \tag{E.8}$$

which easily reduces if we choose a fixed pole O (case $\boldsymbol{a_O} = 0$) and the pole coincide with the center of mass G (case $\boldsymbol{v_G} = \boldsymbol{v_O}$), obtaining

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} &= \boldsymbol{I_O}\,\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \boldsymbol{I_O}\,\boldsymbol{\omega} \\ &= \boldsymbol{I_O}\,\boldsymbol{\alpha} + \boldsymbol{\omega} \times \boldsymbol{L} \end{aligned} \tag{E.9}$$

where $\boldsymbol{\alpha} = d\boldsymbol{\omega}/dt$ is the angular acceleration.

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