Introduction to stochastic finite elements and application to the response of mechanical systems with uncertain parameters

Informal presentation of research status

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Quadrature of square integrable functions in $\mathbb{R}^N$

Discretization of random fields

Series expansion methods

Karhunen-Loève expansion

Galerkin expansion method

Orthogonal series expansion method

Optimal linear estimation method

Expansion optimal linear estimation method

Operators with uncertain parameters

Stochastic collocation approximation

Finite elements for beams using a residual formulation

Beams with uncertain mechanical properties

Stochastic Galerkin approximation using double orthogonal polynomials

Solution of hyperbolic problems with uncertain parameters

Response of a simple oscillator

Response to a stationary process

Response to a non-stationary separable process

Objectives

Principal objective:

The principal objective is to develop robust numerical methods to compute the statistical response of mechanical systems with uncertain state parameters and uncertain loading

Objectives

Specific objectives:

- Determination of the efficiency versus the numerical trade off between different methods, and with this in mind to elaborate a guideline to choose a numerical method taking into consideration the different dimensions of the problem.
- Develop different Covariance functions that represent in a realistical way the stochastic properties of the sate parameters of different mechanical systems, as well as the external loading parameters.
- Given the numerical method to be used, to develop straightforward reliability methods to compute probability failures of any type of systems modeled by differential equations.

Probability spaces and measures

- **trial**
- **outcome**, ω_i
- **sample space**, $\Omega = \{\omega_1, \ldots, \omega_N\}$

Remark

It must be pointed out that since ω_i are just elements that belong to the sample space, $\omega_i \in \Omega$, then $\omega_i \cap \omega_j = \varnothing \ \forall i \neq j$, where $\varnothing$ is the empty set.

There is also the possibility of obtaining more than 1 outcome for each trial that is performed, so it is necessary to define mathematically this possilbility in order to group all of these outcomes into a single set.

Definition

An **event** A is defined as a subset of the sample space, $A \subset \Omega$.

An event is allowed to have more than one statement, but in any case it always must satisfy that the statements must be connected at all times with logical statements such as **and**, **or**, etc.

Example

Let us observe the random phenomena of coin flipping. There can either be 2 outcomes, heads (H) or tails (T). So with no loss of generality, we can define ω_1 as H and ω_2 as T, then $\Omega = \{H, T\}$, which is the sample space.

Example

For a given year, the world cup will only host 3 teams; Italia, Brazil and France. The sample space of the possible champion is $\Omega = \{I, B, F\}$. The sample space for the finalists will be $\Omega = \{IB, BF, IF\}$. If the sample space of the first and second place of the world cup is needed, the sample space increases its cardinality and will become $\Omega = \{IB, BI, BF, FB, IF, FI\}$

Since there is many options to assemble an event A with a given sample space Ω, the it is interesting to define a set that contains all of the possible events.

Definition

A **power set** of Ω is defined as a set that contains as elements all possible subsets of Ω, and it can be written formally as $\mathcal{P}(\Omega)$. Any subset of $\mathcal{P}(\Omega)$ is called a family of sets of Ω.

Example

If $\Omega = \{A, B\}$, then $\mathcal{P}(\Omega) = \{\{\}, \{A\}, \{B\}, \{A, B\}\}$

So basically, power set can be viewed as an operation over a certain set, and it operates in such a way that it creates a set that contains all the possible combinations of the original set.

Definition

A $\sigma - algebra$ of a given set Ω is a collection Σ of subsets of Ω, that is $A_i \subset \Omega$, such that $\Sigma = \bigcup_{i}^{N} A_i$, that is closed under complementation and satisfies that it's members are countable unions. More formally, a subset $\Sigma \subset \mathcal{P}(\Omega)$ is a $\sigma - algebra$ with the following properties:

1. Σ is nonempty, such that $\varnothing$,$\Omega \in \Sigma$
2. If $A \in \Sigma$, then $A^C \in \Sigma$ is also satisfied, where A^C is the complement of A, that can also be viewed as $\Sigma\ A$.
3. The union of countably many sets in Σ is also in Σ, as well as the intersection.

$$\bigcup_{j=1}^{\infty} A_j, \bigcap_{j=1}^{\infty} A_j \in \Sigma$$

Having in mind the general concepts of power set and $\sigma - algebra$, it is possible to return to the primary interest of measuring certain events.

Definition

Given any set Ω of that represents a sample space, and a $\sigma - algebra$ Σ on Ω, then P is a probability measure if:

1. P is non negative
2. $P(\varnothing) = 0$
3. $P(\Omega) = 1$

Now what rests to do is to define for certain events of interest, the probability measure, taking into account the sample space.

Definition

The probability of an event $A \in \Omega$, such that $A \in \Sigma$ is defined as:

$$P(A) = \sum_{\omega \in A} P(\{\omega\}) \tag{1}$$

where P is a measure that needs to be defined a priori taking into account the previous definition and Σ is any of the possible $\sigma - \textit{algebra}$ that can be formed from the original sample space Ω.

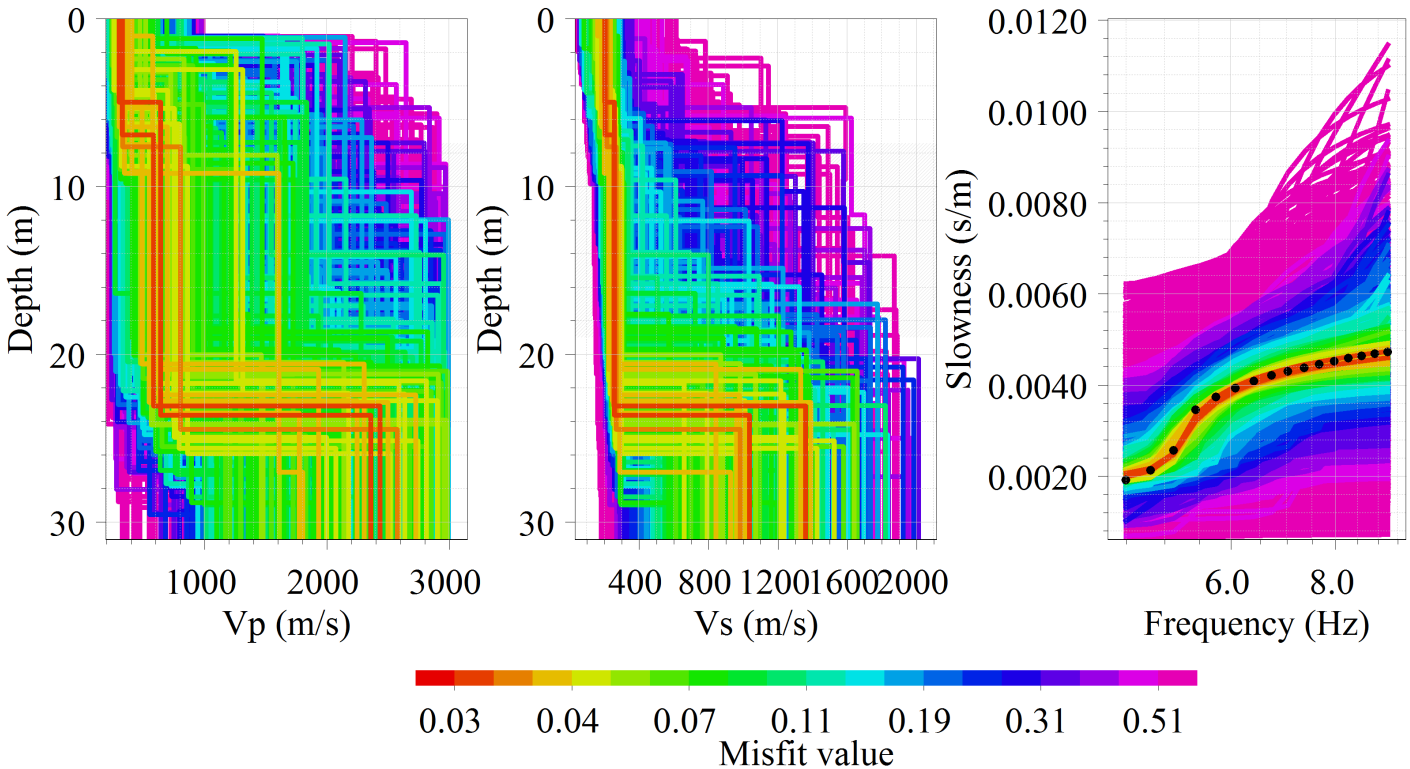
When it comes to quantify the variability of random phenomena a priori of the realization of the experiment, then this type of variability is usually referred to as **aleatory variability**, which has to be treated separately from **epistemic variability**. Epistemic variability is the attempt to quantify the variability of a trial that has already taken place, but the results of the trial are unknown. In other words it can be viewed as variability due to the lack of information or knowledge of the system in study.

Example

A very simply yet enlightening example is to roll die in a cup. Before the die are rolled, it would be the case of aleatory uncertainty, but once the die are rolled, the uncertainty turns into epistemic, which can be reduced if you are a cheat, taking little peeks as to gain more knowledge of your hand.

Example

A real engineering example to understand epistemic uncertainty can be the mechanical properties of a $1D$ soil column. If there is no type of prospection, then nothing can be said about the mechanical properties and as the amount of prospection methods is increased, then it can be said that the uncertainty is reduced.



Continuous random variables and random vectors

It is interesting to construct mappings from the sample space Ω to $\mathbb{R}$, given a probability space (Ω, Σ, P), because probability spaces are not directly observable as opposed to the quantifiable values of an experiment.

Definition

A **random variable** η, is a function that maps a sample space into the real numbers and it can be written as follows:

$$\eta : \Omega \rightarrow \mathbb{R} \tag{2}$$

Since the random variable already has a assigned probability measure P, then it is rather easy to compute the probability of certain simple events, and from there, to define elementary concepts to understand better the behavior of generical random variables.

Definition

The **cumulative distribution function** (CDF) $F_\eta(z)$, is defined as the probability of the event in which the random variable is less or equal than a certain threshold that is in the real numbers, this can be read more formally as:

$$F_\eta(z) = P(\{\omega : \eta(\omega) \leq z\}) = P(\eta \leq z) \tag{3}$$

with $z \in \mathbb{R}$ and it can be easily checked that this function increases monotonically in the interval of $[0, 1]$ as variable z increases.

From the CDF, another function can be defined, which resembles a "weight" function or mass function, that physically tells where is the value of the random variable more likely to be before the trial is performed.

Definition

The **probability density function function** (PDF) is defined as the derivative of the CDF

$$p_\eta(z) = F'_\eta(z) \tag{4}$$

provided that F_η is differentiable at least 1 time.

It can be noted that the PDF is non-negative, and that $\int_{\mathbb{R}} p_\eta(z)dz = 1$, which comes from the previous definition of the P measure.

The **expected value** of a random variable is:

$$E[\eta] = \mu_1 = \int_{\mathbb{R}} z dF_\eta(z) = \int_{\mathbb{R}} z p_\eta(z) dz \tag{5}$$

and a generalization of the **moments** is given by:

$$\mu_p = E[\eta^p], p = 1, 2, \ldots \tag{6}$$

the **first centered moment**, also known as the variance is:

$$V[\eta] = \mu_2' = E[(\eta - E[\eta])^2] \tag{7}$$

and the higher **centered moments** will be:

$$\mu_p' = E[(\eta - E[\eta])^p], p = 1, 2, \ldots \tag{8}$$

finally, the **standard deviation** is defined as follows:

$$\sigma = \sqrt{V[\eta]} \tag{9}$$

with these properties, it is not possible to posses the CDF or PDF of a random variable, but it is possible to have an approximation of the latter functions

Example

A collection of some of the most important distributions is presented.

- Gaussian $\rightarrow \eta \sim N(\mu, \sigma^2)$

$$p_\eta(z) = \varphi(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(z-\mu)^2}{2\sigma^2}} \tag{10}$$

$E[\eta] = \mu$ and $V[\eta] = \sigma^2$

- Uniform $\rightarrow \eta \sim \mathcal{U}(a, b)$

$$p_\eta(z) = \begin{cases} \frac{1}{b-a} & \text{for} \quad a \leq z \leq b \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

- Exponential $\rightarrow \eta \sim Exp(\lambda)$

$$p_\eta(z) = \begin{cases} \lambda e^{-\lambda z} & \text{for} \quad 0 \leq z \leq \infty \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

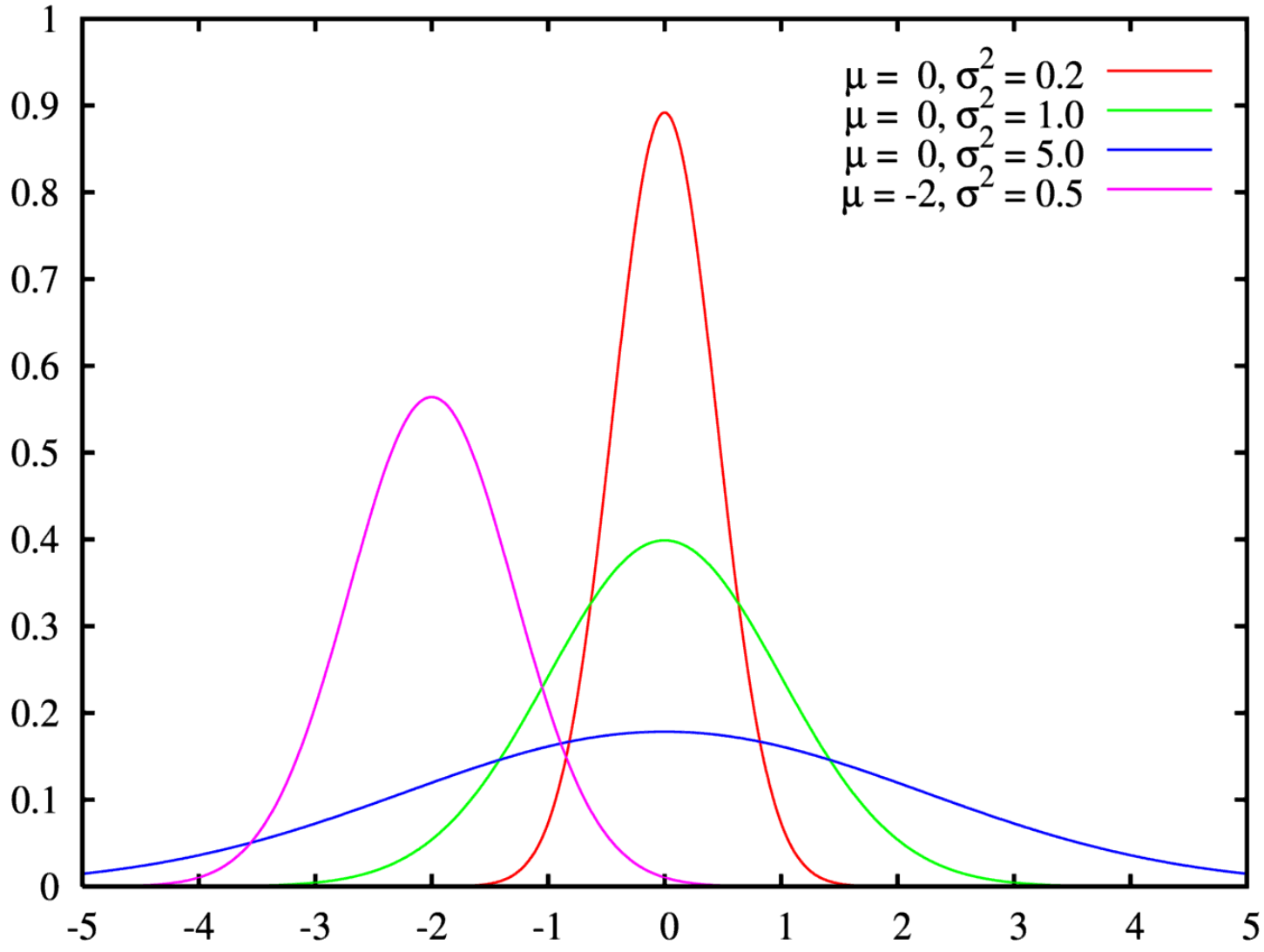


Figure: This figure corresponds to Gaussian distribution.

Functions of random variables

Clearly, a function $g : \mathbb{R} \to \mathbb{R}$ that has as an argument a random variable η such that $\zeta = g(\eta)$, then ζ is also a random variable. The CDF is given by:

$$F_\zeta = P(g(\eta) \leq y) = \int_{\mathbb{R}} dF_\zeta(y) = \int_{\mathbb{R}} \mathbf{I}_{[\eta:\, g(\eta) \leq y]} dF_\eta(z) \tag{13}$$

where $\mathbf{I}_A$ is the **indicator function** of a set $A \subset \mathbb{R}$ such that:

$$\mathbf{I}_A = \begin{cases} 1 & \text{if} \quad \eta \in A \\ 0 & \text{otherwise} \end{cases} \tag{14}$$

from this, the expected value and the variance are respectively:

$$\begin{aligned} E[g(\eta)] &= \int_{\mathbb{R}} g(z) dF_\eta(z) \\ V[g(\eta)] &= \int_{\mathbb{R}} \left(g(z) - E[g(\eta)]\right)^2 dF_\eta(z) \end{aligned} \tag{15}$$

Random vectors

A **random vector** of dimension N, is function $\boldsymbol{\eta}\colon\ [\Omega_1 \times \Omega_2 \ldots \times \Omega_N] \rightarrow \mathbb{R}^N$ is defined as a collection of N real valued random variables, such that:

$$\boldsymbol{\eta}(\boldsymbol{\omega}) = [\eta_1(\omega_1), \ldots, \eta_N(\omega_N)]^T \tag{16}$$

Each component of the random vector has a sample space that can be different from each other and to this generic random vector, it is possible to define the **joint cumulative distribution function** (JCDF) as

$$F_\eta(\mathbf{z}) = P(\eta_1 \leq z_1, \ldots, \eta_N \leq z_N) \tag{17}$$

The **joint probability distribution function** (JPDF) naturally becomes:

$$p_\eta(\mathbf{z}) = \frac{\partial^N F_\eta(\mathbf{z})}{\partial z_1 \cdots \partial z_N} \tag{18}$$

provided that F_η is differentiable. From the previous definitions, it is possible to define the expected value of a random vector.

$$E[\boldsymbol{\eta}] = \int_{\mathbb{R}^N} \boldsymbol{\eta} dF_\eta(\mathbf{z}) \in \mathbb{R}^N \tag{19}$$

A useful concept in multivariate analysis is the **covariance matrix**, which is simply:

$$C[\boldsymbol{\eta}] = E[(\boldsymbol{\eta} - E[\boldsymbol{\eta}])(\boldsymbol{\eta} - E[\boldsymbol{\eta}])^T] \in \mathbb{R}^{N \times N} \tag{20}$$

this matrix can give information related to two random variables that belong to a random vector. A particular case of this matrix is the variance of each random variable belonging to the random vector, which is basically the diagonal of the covariance matrix.

$$V[\boldsymbol{\eta}] = diag(C[\boldsymbol{\eta}]) \in \mathbb{R}^N \tag{21}$$

In multivariate analysis it is very interesting to obtain the CDF or PDF of any component of the random vector, this is referred to as the **marginal distribution function** $F_{\eta_i} : \mathbb{R} \rightarrow \mathbb{R}$, which can be computed simply as:

$$F_{\eta_i} = F_{\eta}(\infty, \ldots, \infty, z_i, \infty, \ldots, \infty) \tag{22}$$

It is quite clear that $p_{\eta_i} = F'_{\eta_i}$ provided that F_{η_i} is differentiable. The **conditional probability density function**, $\boldsymbol{\eta}_j = [\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_N]$ can be computed as:

$$p_{\eta_{\mathbf{j}}|\eta_i}(\mathbf{z_j}|z_i) = \frac{p_{\eta}(\mathbf{z})}{p_{\eta_i}(z_i)} \tag{23}$$

Remark

The random variables $\eta_1, \ldots, \eta_N$ are said to be **independent** if the distribution F_η can be expressed as:

$$F_\eta(z_1, \ldots, z_N) = \prod_{i=1}^{N} F_{\eta_i}(z_i) \tag{24}$$

and also the PDF

$$p_\eta(z_1, \ldots, z_N) = \prod_{i=1}^{N} p_{\eta_i}(z_i) \tag{25}$$

Change of Variables

Any random vector $\boldsymbol{\eta}$, with a distribution function $F_{\eta}(\mathbf{z})$ can be expressed as a deterministic function of N independent random variables ζ_i, $i = 1, 2, \ldots, N$, each with absolutely continuous distribution function $F_{\zeta}(\mathbf{y})$ i.e. there exists $G : \mathbb{R}^N \rightarrow \mathbb{R}^N$ such that $\boldsymbol{\eta} = G(\zeta_1, \ldots, \zeta_N)$ A possible way to construct such a mapping G is by using the **Rosenblatt transformation** [Rosenblatt(1952)]:

$$\begin{aligned} F_{\zeta_1}(y_1) &= F_{\eta_1}(z_1) \\ F_{\zeta_2}(y_2) &= F_{\eta_2}(z_2|\eta_1 = z_1) \\ F_{\zeta_3}(y_3) &= F_{\eta_3}(z_3|\eta_1 = z_1, \eta_2 = z_2) \\ &\vdots \\ F_{\zeta_N}(y_N) &= F_{\eta_N}(z_N|\eta_1 = z_1, \ldots, \eta_{N-1} = z_{N-1}) \end{aligned} \tag{26}$$

From equation (80), if the right side is multiplied, as well as the left side and then equated, yields the following result:

$$\prod_{i=1}^{N} F_{\zeta_i}(y_i) = \prod_{i=1}^{N} F_{\eta_N}(z_N|\eta_1 = z_1, \ldots, \eta_{i-1} = z_{i-1}) \tag{27}$$

Thus it is possible to show that $\boldsymbol{\zeta}$ has independent components.

Diagonalization of the covariance matrix

The covariance matrix $C[\boldsymbol{\eta}]$ is a symmetric and positive semi-definite. Hence, it has real eigenvalues $\lambda_1, \ldots, \lambda_N \geq 0$ and a complete set of orthonormal eigenvectors $[\mathbf{v}_1, \ldots, \mathbf{v}_N]$:

$$C[\boldsymbol{\eta}] = VDV^T = \sum_{i=1}^{N} \lambda_i \mathbf{v}_i (\mathbf{v}_i)^T \tag{28}$$

with D diagonal and V orthogonal. Assuming that all the eigenvalues λ_i are strictly positive, then it is possible to define the random vector $\boldsymbol{\zeta} = [\zeta_1, \ldots, \zeta_N]$

$$\boldsymbol{\zeta} = D^{-\frac{1}{2}} V^T (\boldsymbol{\eta} - E[\boldsymbol{\eta}]) \Longrightarrow \zeta_i = \frac{1}{\sqrt{\lambda_i}} \mathbf{v}_i \cdot (\boldsymbol{\eta} - E[\boldsymbol{\eta}]) \tag{29}$$

It is easy to show that the random variables $\boldsymbol{\zeta}_i$ have zero mean, unit variance and are uncorrelated, but this does not imply that they are independent. A more compact notation of equation (29)

$$\Longrightarrow \boldsymbol{\eta} = E[\boldsymbol{\eta}] + VD\boldsymbol{\zeta} \tag{30}$$

Stochastic processes and random fields

A **Random process** or also known as **Random fields** emerged as a necessety to extend the concept of random variables, because in certain trials, the outcome is not a number, but a function of one or more parameters that posses a certain level of continuity. With this in mind, the definition of a random process is a family of random variables and it can be stated in a more formal way as

Definition

Let $\mathcal{D} \subset \mathbb{R}^d$ be a domain. A **random field** $\kappa(\mathbf{x}, \omega) \colon \mathcal{D} \times \Omega \longrightarrow \mathbb{R}$ is a collection of infinite random variables $\kappa(\mathbf{x}, \omega)$, for each point $\mathbf{x} \in \mathcal{D}$.

Remark

The sample space is considered the same for each random variable assigned to a point in the physical domain, simply for convenience, but in reality, this can clearly change. Another point that can be made out is that the domain is not reserved only to a space domain, it can be any type of domain, such as a time domain, in which the this particular random field is usually denoted as a **stochastic process**, $\kappa(t, \omega)$.

It is clearly quite cumbersome to characterize the infinite collection of random variables that belong to the physical space $\mathcal{D}$, that is why a discrete approach might be more interesting. A random field can be viewed finite dimensional distribution of order n,with n points in the physical domain $\mathcal{D}$ $\mathbf{x}_1, \ldots, \mathbf{x}_n$ or in simple words as a random vector where each component of the vector is associated to a random variable that belongs to the random field. From (17) the JCDF is simply

$$F_n(z_1, \ldots, z_n; \mathbf{x}_1, \ldots, \mathbf{x}_n) = P(\kappa(\mathbf{x}_1, \omega_1) \leq z_1, \ldots, \kappa(\mathbf{x}_n, \omega_n) \leq z_n) \tag{31}$$

The random process is fully characterized by the distribution functions of any order $n = 1, 2, \ldots$ and any set of points $\mathbf{x}_1, \ldots, \mathbf{x}_n$, provided that they satisfy some consistency and symmetry conditions.

The covariance function for two points $\mathbf{x}$ and $\mathbf{y}$ that belong to $\mathcal{D}$ is:

$$C[\kappa(\mathbf{x}), \kappa(\mathbf{y})] = E[(\kappa(\mathbf{x}, \omega) - E[\kappa(\mathbf{x})])(\kappa(\mathbf{y}, \omega) - E[\kappa(\mathbf{y})])] = C_{\kappa\kappa}(\mathbf{x}, \mathbf{y}) \tag{32}$$

from (116), the variance is simply:

$$V[\kappa(\mathbf{x})] = C[\kappa(\mathbf{x}), \kappa(\mathbf{x})] \tag{33}$$

Definition

$\kappa(\mathbf{x}, \omega)$ is said to be a second order random field if

$$V[\kappa(\mathbf{x})] < \infty \tag{34}$$

for all $\mathbf{x} \in \mathcal{D}$

Example

In this example, a realization of a random field is performed, given a correlation field function, and considering that the random field is gaussian.

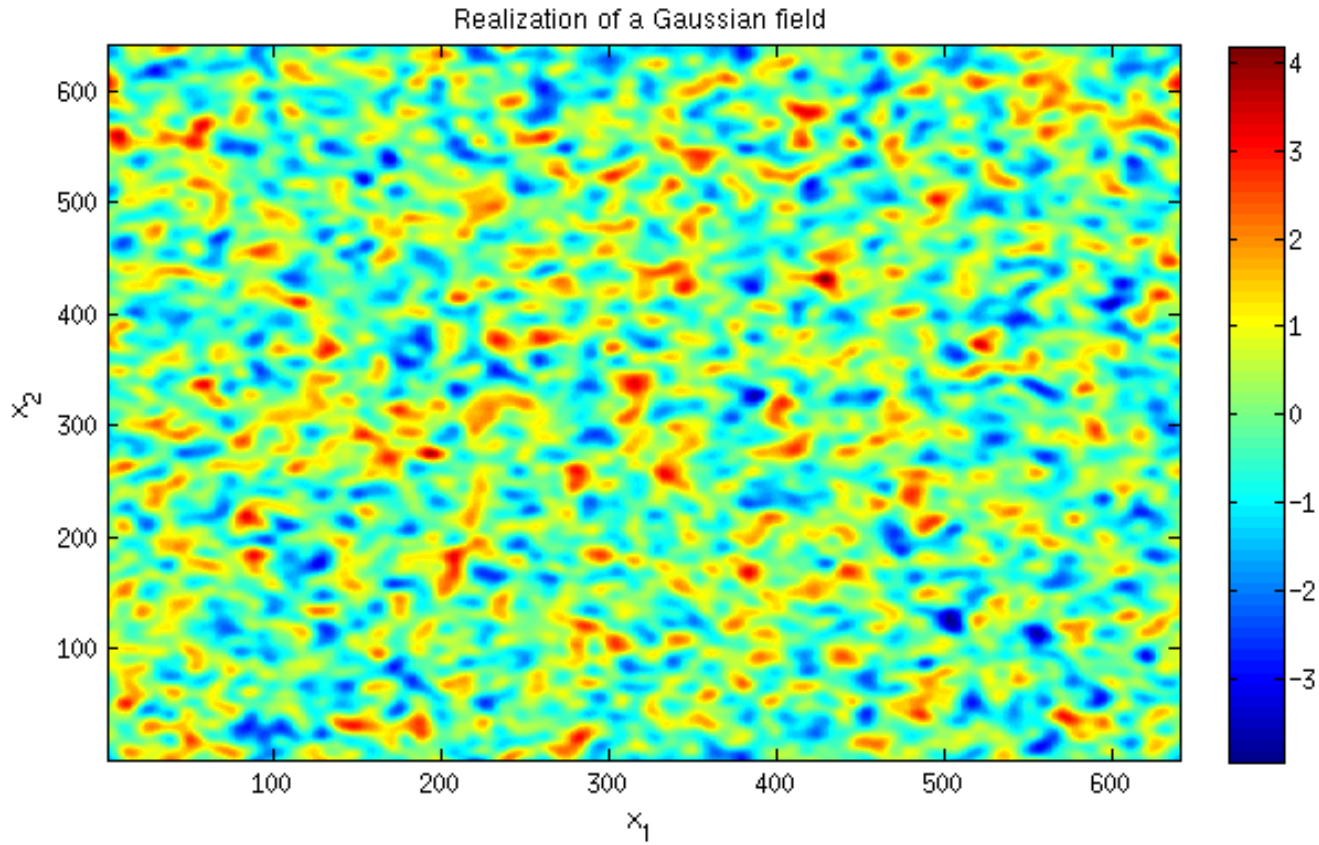


Figure: This figure corresponds to the realization of a gaussian random field that possessed a certain correlation field in a two dimensional field.

Stationary stochastic processes

Stationary stochastic models of the gaussian type was first introduced by [Housner(1947)]. Since then, they have been used by a great number of authors [Bycroft(1960)], [Tajimi(1960)], [Housner & Jennings(1964)] and [Brady(1966)]). Even though these models do not represent accurately what is really happening, they are a good first approximation to estimate the probabilistic response of linear structural systems given that the used frequency content is the predominant of the time history [Tajimi(1960)].

Non stationary separable stochastic processes

Since stationary stochastic processes present limitations, some authors introduced gaussian non stationary stochastic processes, among the ones that can be mentioned are [Bolotin(1960)], [Bogdanoff & Kosin(1961)], [Goldberg et al.(1964)Goldberg, Bogdanoff & Sharpe], [Amin & Ang(1966)], [Shinozuka & Sato(1967)], [Jennings & Housner(1968)] and [Iyengar & Iyengar(1969)]. It must be noted that these models are just a empirical way to capture the behavior of strong ground motion, based on the observation of real accelerograms. These models are made with a stationary stochastic process, that provides the frequency content, modulated by a deterministic function that multiplies the stationary process to give the amplitude evolution in time.

$$f(t) = \psi(t)s(t) \tag{35}$$

where $\psi(t)$ is the modulating function that provides the variations of amplitude in time, and $s(t)$ is the gaussian stochastic process that provides the frequency content.

Some of the basic properties of these processes are as follow:

$$E[s(t)] = \mu_s(t) = 0 \tag{36}$$

$$E[s^2(t)] = \sigma_s^2(t) = 1 \tag{37}$$

The latter yields a non stationary stochastic process $f(t)$, which in this particular case, is seen as the ground acceleration. Using equations (36) and (37), the mean of the process can be computed for any moment in time.

$$E[f(t)] = E[\psi(t)s(t)] = \psi(t)\mu_s(t) = 0 \tag{38}$$

The mean square of the process, that is equal to the variance for each moment is given by:

$$E[f^2(t)] = E[\psi^2(t)s^2(t)] = \psi^2(t)\sigma_s^2(t) = \psi^2(t) \tag{39}$$

This is so because $\sigma_x^2(t) = E[x^2(t)] - \mu_x^2(t)$ for any process $x(t)$ and last but not least, the generalized autocorrelation function can be computed as:

$$R_{ff}(t_1, t_2) = E[f(t_1)f(t_2)] \tag{40}$$

and rearranging the terms yields:

$$R_{ff}(t_1, t_2) = \psi(t_1)\psi(t_2)R_{ss}(t_1, t_2) \tag{41}$$

where $R_{ss}(t_1, t_2)$ is the autocorrelation function of the stationary process $s(t)$.

Taking the generalized autocorrelation function to the frequency domain, by means of the bivariate Fourier transform, the following can be obtained:

$$\Phi_{ff}(\omega_1, \omega_2) = \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \psi(t_1)\psi(t_2) R_{ss}(t_1, t_2) e^{-i(\omega_2 t_2 - \omega_1 t_1)} dt_1 dt_2 \tag{42}$$

and using the Wiener-Kchinchine relationship, the following can be written:

$$\Phi_{ff}(\omega_1, \omega_2) = \int_{-\infty}^{+\infty} \Phi_{ss}(\omega)\Psi(\omega_2 - \omega)\Psi^*(\omega_1 - \omega) d\omega \tag{43}$$

where

$$\Psi(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \psi(t) e^{-i\omega t} dt \tag{44}$$

$\Phi_{ff}(\omega_1, \omega_2)$, determined by equation (43) is denoted as the generalized power spectral density function.

Modulation functions

Many modulation functions have been proposed in the literature, and the way to adjust their parameters is also solved differently, depending on the availability and type of the seismological data. [Bolotin(1984)] and [Shinozuka(1970)], proposed a function that is composed by the difference of two exponential functions.

$$\psi(t) = \sqrt{\beta}(e^{-\alpha t} - e^{-\gamma t}) \tag{45}$$

where $\alpha < \gamma$. This function is very inconvenient due to the difficulty to assign the parameters α, β and γ, given any type of seismological data, specifically free field accelerograms.

A model function that is adjustable to accelerograms is gamma function, proposed by [Saragoni & Hart(1974)]. This method supposes that the propagation medium can be modeled as a series of linear oscillators acting in cascade, where each oscillator properties are normally distributed. This model also supposes that the source can be represented by a white noise excitation. Using all of these assumptions, it is possible to arrive to the following expression for the mean square of the accelerations:

$$E[f^2(t)] = \psi^2(t) = \beta e^{-\alpha t} t^{\gamma} \tag{46}$$

where α, β and γ are constants that can be determined by different ways.

Taking into consideration this model,
[Arias et al.(1976)Arias, Holzapfel & Saragoni], developed a similar modulating function as the latter, but forcing the condition that the function reaches zero once the earthquake duration time is over (in the free field). This is why the beta function was selected.

$$\psi(t) = \begin{cases} \sqrt{\beta}\left(\frac{t}{t_f}\right)^{\frac{\alpha}{2}}\left(1-\frac{t}{t_f}\right)^{\frac{\gamma}{2}} & \text{si} \quad t \leq t_f \\ 0 & \text{si} \quad t > t_f \end{cases} \tag{47}$$

Given an accelerogram, the parameters of the gamma and beta method can easily be obtained using the temporal moments of the record.

$$m_{2k} = \int_0^{t_f} t^k f^2(t) dt = \int_0^{t_f} \psi^2(t) t^k dt \tag{48}$$

for $k = 0, 1, \ldots, n$.

Simulation of separable processes

A separable stochastic process can be simulated numerically with the following equation:

$$f(t) = \psi(t) \sum_{k=1}^{m} \sigma_k [U_k \cos(\omega_k t) + V_k \sin(\omega_k t)] \tag{49}$$

where $\psi(t)$ is the modulating function, σ_k is the variance of the process at time step k, which is and can be approximated to:

$$\sigma_k^2 = \int_{\omega_k - \frac{\Delta\omega_k}{2}}^{\omega_k + \frac{\Delta\omega_k}{2}} G(\omega) d\omega \approx G(\omega_k) \Delta\omega_k \tag{50}$$

U_k and V_k are independent random variables, that distribute normally, i.e.

$$U_k \sim N(0, 1)$$

$$V_k \sim N(0, 1)$$

$G(\omega) = 2\Phi(\omega)$ is the one sided power spectral density for ω in the positive reals and $G(\omega) = 0$ elsewhere

Evolutionary stochastic processes

Different techniques have been used to solve this problem, such is the case of [Liu(1972)], who used the instantaneous power spectral density to incorporate both the frequency and the time domain at the same time in the following way:

$$P = \int_0^{t_f} f^2(t) = \int_0^{t_f} \int_{-\infty}^{+\infty} \varphi(t,\omega) d\omega dt \tag{51}$$

[Saragoni & Hart(1974)] used different spectral functions for different intervals of time, but this did not come out right because the sudden changes in the spectral content affected completely the response of simple SDOF's.

$$f(t) = \psi(t) \sum_{i=1}^{n} (H_t(t_{i-1} - t) - H_t(t_i - t)) s_i(t) \tag{52}$$

where $\psi(t)$ is the modulating function, $H_t(t)$ is the heavy side step function, and $s_i(t)$ are the stationary stochastic processes.

[Hammond(1968)], [Shinozuka(1970)] and [Kameda(1980)] used a evolutionary spectral function in time, to calculate ground response, using the Fourier-Stiljes transform.

$$f(t) = \int_{-\infty}^{+\infty} A(t,\omega) e^{-i\omega t} dZ(\omega) \tag{53}$$

where $Z(\omega)$ corresponds to stationary processes with orthogonal increments amongst themselves and $A(t,\omega)$ is a sigma oscillatory function, that has a spectral density that varies in the time domain. On the other hand, we have the models proposed by [Crempien Laborie & Der Kiureghian(1988)], that characterizes ground acceleration with non stationary amplitude and frequency content, using the theory sigma oscillatory processes, that is the characterization of non stationary processes as a sum of independent stochastic processes.

$$f(t) = \sum_{k=1}^{m} \psi_k(t) s_k(t) \tag{54}$$

where the evolutionary power spectral density in time is given by:

$$\Phi_{ff}(t,\omega) = \sum_{i=1}^{m} \psi_k^2(t)\Phi_{ss}(\omega) \tag{55}$$

Amongst these type of models, [Conte & Peng(1997)] can be pointed out as an improved and natural consequence or development of [Crempien Laborie(1988)]. It must be pointed out that both these models are very complicated, and that there is no study in the literature trying to relate the parameters of these models to the parameters of the physical process of rupture, such as stress drops, area of rupture, etc. These methods enlighten the natural phenomena observed in ground motion, but it is very difficult to reproduce with real conditions.

Approximation and quadrature of functions

Let us consider a deterministic function $g\colon \mathbb{R}^N \to \mathbb{R}$ that holds as argument a random vector $\boldsymbol{\eta}\colon (\Omega, \mathcal{F}, P) \to \mathbb{R}^N$ that has a JCDF $F_\eta(\mathbf{z})$, and a density $p_\eta(\mathbf{z})$. It is clear that g is a random variable with

$$E[g] = \int_{\mathbb{R}^N} g(\mathbf{z}) dF_\eta(\mathbf{z}) \quad \text{and} \quad V[g] = E\left[(g(\boldsymbol{\eta}) - E[g])^2\right] \tag{56}$$

Definition

The space of square integrable functions in $\mathbb{R}^N$ is defined as

$$L^2_{p_\eta}(\mathbb{R}^N) = \left\{ g\colon \mathbb{R}^N \to \mathbb{R} \text{ such that } \int_{\mathbb{R}^N} g(\mathbf{z})^2 dF_\eta(\mathbf{z}) < \infty \right\} \tag{57}$$

and it is a *Hilbert space* with

- Inner product: $\langle f, g \rangle = E[fg] = \int_{\mathbb{R}} f(z) g(z) dF_\eta(z)$
- Norm: $\|g\|_{L^2_{p_\eta}} = \sqrt{E[g^2]}$

$L^2_{p_\eta}(\mathbb{R}^N)$ admits an orthonormal basis ψ_i, $i = 0, 1, \ldots$ such that $E[\psi_i \psi_j] = \delta_{ij}$ for all $i, j \geq 0$. Another feature of the functions that belong to this space is that they can be expanded on this basis in the following manner

$$g(\eta) = \sum_{i=1}^{\infty} g_i \psi_i(\eta) \text{ with } g_i = E[g\psi_i] \tag{58}$$

and this expansion satisfies that

$$\lim_{n \to \infty} \|g - \sum_{i=0}^{n} g_i \psi_i(\eta)\|_{L^2_{p_\eta}} = 0 \tag{59}$$

Approximation of square integrable functions

Orthogonality is defined with respect to an inner product, which in turn involves a measure of integration, dF_η. These measures can be absolutely continuous and it can take the following form

$$dF_\eta(z) = p_\eta(z)dz \quad on \quad \Omega \tag{60}$$

where $p_\eta(z)$ is a positive function in Ω which in turn is referred to as the *support* of dF_η. From now on only support functions that are as well PDF functions will be considered. The moments were previously defined as:

$$\mu_p = E[\eta^p], p = 1, 2, \ldots \tag{61}$$

and the assumption for the future is that all of these moments exist and they are of finite value.

The inner product of two polynomials p and q relative to the measure dF_η is then well defined as:

$$\langle p, q \rangle_{dF_\eta} = \int_{\mathbb{R}} p(z)q(z)dF_\eta = E[p(z)q(z)] \tag{62}$$

There are classical weight functions $dF_\eta(z) = p_\eta(z)dz$ where some representative ones are summarized in the following table:

Table: Classical weight functions.

name	$p_\eta(z)$	support	comment
Jacobi	$(1-z)^\alpha(1+z)^\beta$	[-1,1]	$\alpha > -1$ and $\beta > -1$
Laguerre	$z^\alpha e^{-z}$	$[0, \infty]$	$\alpha > -1$
Hermite	$\lvert z\rvert^{2\alpha} e^{-z^2}$	$[-\infty, \infty]$	$\alpha > -\frac{1}{2}$
Legendre	1	[-1,1]	

Now as it can be noticed, these weight functions are not PDF functions, thus it is necessary to redefine them such that orthonormal polynomials can be obtained from them.

Table: Modified weight functions.

name	$p_\eta(z)$	support	comment
Modified Jacobi	$\frac{z^{\alpha-1}(1-z)^{\beta-1}}{\int_0^1 y^{\alpha-1}(1-y)^{\beta-1}dy}$	[0,1]	$\alpha > 0$ and $\beta > 0$
Modified Laguerre	$\lambda e^{-\lambda z}$	$[0, \infty]$	$\lambda > 0$
Modified Hermite	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$	$[-\infty, \infty]$	$\sigma > 0$
Modified Legendre	1	[0,1]	

The orthonormal polynomials can be computed with the following three term recurrence:

$$H_{k+1}(\eta) = (\eta - \alpha_k)H_k(\eta) - \beta_k H_{k-1}(\eta) \tag{63}$$

, with $H_0(\eta) = 1$ and $H_{-1}(\eta) = 0$ where

$$\begin{aligned} \alpha_k &= \frac{E[\eta H_k^2]}{E[H_k^2]} \\ \beta_k &= \frac{E[H_k^2]}{E[H_{k-1}^2]} \end{aligned} \tag{64}$$

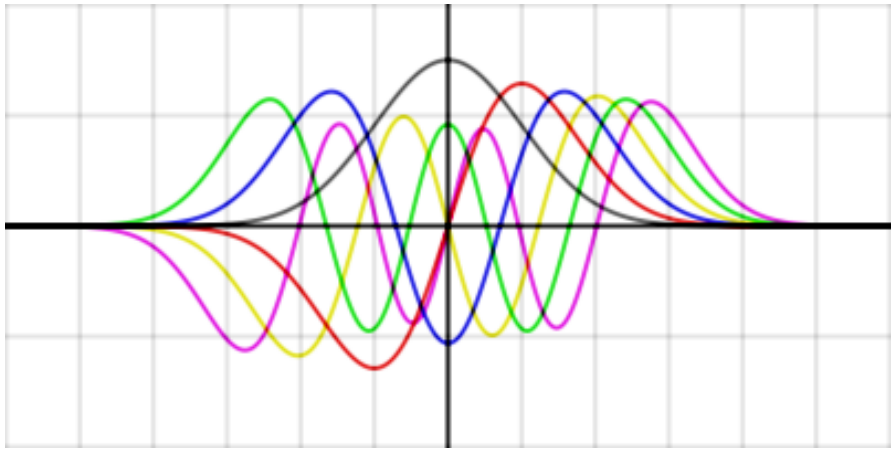


Figure: This figure corresponds to Hermite polynomials.

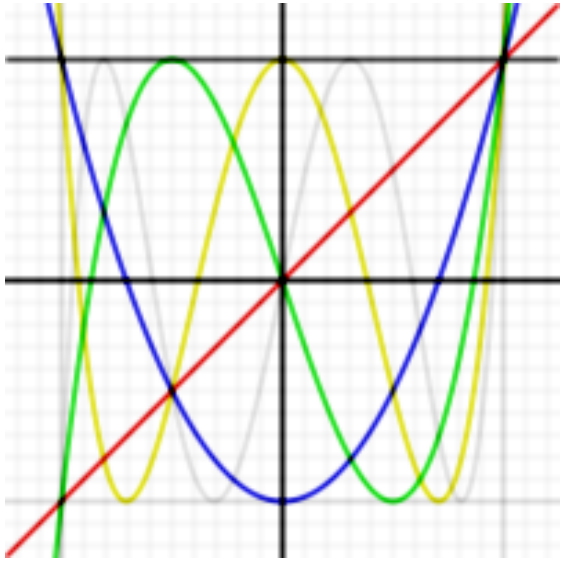


Figure: This figure corresponds Jacobi polynomials.

It can be seen in the three-term recurrence relation expressed in equation (64), the term that defines the series of β_k is not defined for $k = 0$. In the case when the weights are not representing the PDF of a random variable, then the first term of the recurrence relation will be $H_0 = \int_{\mathbb{R}} dp_\eta(z)dz = \beta_0$. it must be noted that in the case of a PDF weight function, $H_0 = 1$ as it was stated before. Placing the coefficients α_k on the diagonal and $\sqrt{\beta_k}$ on the two side diagonals of a matrix produces what is called the *Jacobi matrix* of the measure dF_η,

$$\mathbf{J}(dF_\eta) = \begin{bmatrix} \alpha_0 & \sqrt{\beta_1} & 0 & \dots & 0 \\ \sqrt{\beta_1} & \alpha_1 & \sqrt{\beta_2} & & \\ 0 & \sqrt{\beta_2} & \alpha_2 & \ddots & \\ \vdots & & \ddots & \ddots & \\ 0 & & & & 0 \end{bmatrix} \tag{65}$$

The *Jacobi* matrix is clearly tri-diagonal, real, symmetric of infinite order. With this in mind, then the three term recurrence relation can also be written as:

$$\sqrt{\beta_{k+1}} H_{k+1}(\eta) = (\eta - \alpha_k) H_k(\eta) - \sqrt{\beta_k} H_{k-1}(\eta) \tag{66}$$

, with $H_0(\eta) = \frac{1}{\sqrt{\beta_0}}$ and $H_{-1}(\eta) = 0$ and it can also be written as:

$$\eta \mathbf{H}(\eta) = J_n(dF_\eta) + \sqrt{\beta_n} H_n(\eta) e_n \tag{67}$$

with $\mathbf{H}(\eta) = [H_0(\eta), \ldots, H_{n-1}(\eta)]^T$ and J_n are the eigenvalues of the *Jacobi* which are at the zeros of $\mathbf{H}(\eta)$, and $\mathbf{H}(\overline{\eta})$ are the corresponding eigenvectors where $\overline{\eta}$ are the points of the zeros of $\mathbf{H}(\eta)$.

Example

In the following example, well known orthogonal polynomials will be presented

- Uniform random variables, i.e. $\eta \sim \mathcal{U}(-1, 1)$, which are referred to as Legendre polynomials

$$H_{k+1}(\eta) = \frac{2k+1}{k+1}\eta H_k(\eta) - \frac{k}{k+1}H_{k-1}(\eta)$$
$$E[H_i H_k] = \frac{1}{2(k+\frac{1}{2})}\delta_{ik} \tag{68}$$

Example

- Gaussian random variables, i.e. $\eta \sim \mathcal{N}(0,1)$, referred to as Hermite polynomials.

$$\begin{aligned} H_{k+1}(\eta) &= \eta H_k(\eta) - kH_{k-1}(\eta) \\ E[H_i H_k] &= k!\delta_{ik} \end{aligned} \tag{69}$$

- Exponential random variable, i.e. $\eta \sim Exp(1)$, called Laguerre polynomials

$$\begin{aligned} H_{k+1}(\eta) &= \frac{2k+1-\eta}{k+1}\eta H_k(\eta) - \frac{k}{k+1}H_{k-1}(\eta) \\ E[H_i H_k] &= \delta_{ik} \end{aligned} \tag{70}$$

Lagrange polynomial approximation

Let us consider the recent problem where the n couple values (z_i, y_i), with $i \in [1, \ldots, m]$. We are searching for a polynomial $L_m \in \mathbb{P}_m$, referred to as a interpolatory polynomial, such that

$$L_m(z_i) = a_m z_i^m + \ldots + a_1 z_i + a_0 = y_i, \quad i = 0, \ldots, n \tag{71}$$

If $n \neq m$, then the problem will be either overdetermined or underdetermined. If $n + 1 = m$, then it can be demonstrated that the problem the following theorem.

Theorem

If there is $n + 1$ points with its corresponding values such that (z_i, y_i), then there is a unique polynomial $L_n \in \mathbb{P}_n$ that satisfies $L_n(z_i) = y_i$ for $i = 0, \ldots, n$

There is a specific way to find the polynomial that is given by theorem (17.1), and it is a the sum of polynomials making up a base that can the whole space where the interpolation takes place. This polynomial base is referred to in the literature as the Lagrange interpolation polynomial.

Definition

The Lagrange polynomial can be defined as:

$$L_n(z) = \sum_{i=0}^{n} y_i l_i(z) \tag{72}$$

where $l_i(z) \in \mathbb{P}_n$ is a family of polynomials of order n, such that they form a base in the space of $\mathbb{P}_n$, and it can be written as:

$$l_i(z) = \prod_{\mathbf{j}} \frac{z - z_j}{z_i - z_j} \tag{73}$$

for $i = 0, \ldots, n$ and $\mathbf{j} = 0, 1, \ldots, i-1, i+1, \ldots, n$

It is important to compute the error associated with the use of the interpolating polynomial $L_n f(z)$, a polynomial that is interpolating the values $\{y_i\}$ at the nodes $\{z_i\}$, that are related to the succesive evaluations of the function $f(z)$ in the nodes.

Theorem

Let us assume the nodes $z_0, \ldots, z_n$, that are going to be used to interpolate the function $f(z) \in C^{n+1}(I_z)$, where I_z is the smallest interval length associated with the nodal points $z_1, \ldots, z_{n+1}$, then the error at any point z of the domain in which the function $f(z)$ is defined will be given by the following expression:

$$E_n(z) = f(z) - L_n f(z) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\omega_{n+1}(z) \tag{74}$$

with $\xi \in I_z$ and $\omega_{n+1}(z) = \prod_{i=0}^{n}(z - z_i)$ is the nodal polynomial of degree $n+1$

Remark

It can be shown that the Lagrange polynomials can be written as:

$$L_n(z) = \sum_{i=0}^{n} \frac{\omega_{n+1}(z)}{(z - z_i)\omega'_{n+1}(z_i)} y_i \tag{75}$$

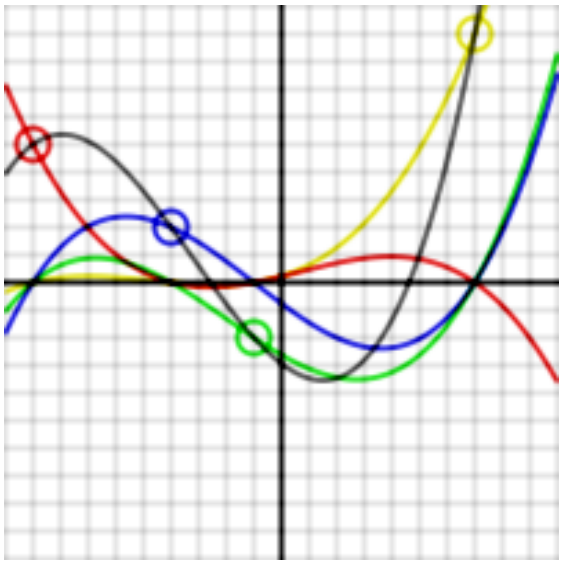


Figure: This figure corresponds to the plots of some Lagrange polynomials.

Gauss formula

Given a positive measure dF_η, the n-point Gaussian quadrature formula associated with the measure dF_η is:

$$\int_{\mathbb{R}} f(z) dF_\eta(z) = \sum_{i=1}^{n} \gamma_i f(\tau_i) + R_n(f) \tag{76}$$

which has a maximum algebraic exactness of $2n-1$

$$R_n(f) = 0 \quad if \quad f \in \mathbb{P}_{2n-1} \tag{77}$$

It is well known that that the nodes τ_i are the zeros of $H(\eta; dF_\eta)$.

Gauss-Radau formula

If there is an interval $[a, \infty]$, $-\infty < a$, containing the support of dF_η, it may be desirable to have an $(n+1)$-point quadrature rule of maximum degree of exactness that has a as a prescribed node,

$$\int_{\mathbb{R}} f(z) dF_\eta(z) = \gamma_0^R f(a) + \sum_{i=1}^{n} \gamma_i^R f(\tau_i^R) + R_n^R(f) \qquad (78)$$

Here, $R_n^R(f) = 0$ for all $f \in \mathbb{P}_{2n}$ and τ_i^R are the zeros of $H(\eta; dF_{\eta_R})$ where $dF_{\eta_R} = (z-a)dF_\eta$. This is called the *Gauss-Radau* formula.

Gauss-Lobatto formula

If the support of dF_η is contained in the finite interval $[a, b]$, it is more attractive to prescribe 2 nodes, the points a and b. Maximazing the degree of exactness subject to these constrains yields the *Gauss-Lobatto* formula,

$$\int_a^b f(z)dF_\eta(z) = \gamma_0^L f(a) + \sum_{i=1}^{n} \gamma_i^L f(\tau_i^L) + \gamma_{n+1}^L f(b) + R_n^{a,b}(f) \quad (79)$$

which is written as an $(n+2)$-point formula; $R_n^L(f) = 0$ for $f \in \mathbb{P}_{2n+1}$. The internal nodes τ_i^L are the zeros of $H(\eta; dF_{\eta_L})$ where $dF_{\eta_L} = (z-a)(b-z)dF_\eta$.

Remark

If there is a domain $\Omega = [a, b]$, then a suitable transformation, $\varphi\colon [a', b'] \to [a, b]$, such that $z = \varphi(\xi) = \frac{b-a}{b'-a'}\xi + \frac{ab'+ba'}{b'-a'}$, which can be used to work in a more simple domain, $\Omega' = [a', b']$

$$\Rightarrow \int_a^b f(z)dz = \frac{b-a}{b'-a'} \int_{a'}^{b'} f(\varphi(\xi))d\xi \quad (80)$$

Clenshaw-Curtis formula

This method proposed by [**?**] consists in a simple change of variable $z = \cos(\theta)$, and withought any loss of generality, if the integration intervale is $[-1, 1]$, then the following integral can also be expressed as

$$\int_{-1}^{1} f(z)dz = \int_{0}^{\pi} f(\cos(\theta)) \sin(\theta)d\theta \tag{81}$$

the latter integral can be resolved very efficiently using cosine series

$$f(\cos(\theta)) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\theta) \tag{82}$$

in which case the integral becomes

$$\int_{0}^{\pi} f(\cos(\theta)) \sin(\theta)d\theta = a_0 + \sum_{k=1}^{\infty} \frac{2a_{2k}}{1-(2k)^2} \tag{83}$$

where the cosine coefficients are

$$a_k = \frac{2}{\pi} \int_0^{\pi} f(\cos(\theta)) \cos(k\theta) d\theta \tag{84}$$

thus, one must perform a numeric integration again, and for periodic functions, Fourier-sereies integrations are very accurate up to the Nyquist frequency, thus the cosine-series integral can be approximated by the discrete cosine transform (DCT)

$$a_k \approx \frac{2}{N} \left[\frac{f(1)}{2} + \frac{f(-1)}{2}(-1)^k + \sum_{n=1}^{N-1} f(\cos[\frac{n\pi}{N}]) \cos(\frac{nk\pi}{N}) \right] \tag{85}$$

for $k = 0, \ldots, N$ to lateer use the explicit quadrature formula. According to [**?**], this method might have a theroretical lower polynomial degree of accuracy, but nevertheless, it has great performance anyway, comparable to Gauss integration schemes, with the advantage that the nodes are much faster to compute.

Approximation of square integrable functions in $\mathbb{R}^N$

Assuming that the random vector $\boldsymbol{\eta}$, that is obtained after approximating by some specific method the random field of interest, has independent components, the JPDF factorises as

$$p_{\eta}(\mathbf{z}) = \prod_{i=1}^{M} p_i(z_i) \tag{86}$$

Let us consider that for each component of the random vector, η_i, and its corresponding density function p_{η_i}, have associated an orthonormal basis $H_1^{(i)}(z_i), \ldots, H_M^{(i)}(z_i)$ of $L^2_{p_{\eta_i}}$. If a multi-index $\mathbf{i} = [i_1, \ldots, i_M] \in \mathbb{N}^M$ is defined, then the previous can be seen in a more compact way

$$H_{\mathbf{i}}(\mathbf{z}) = H_{i_1}^1(z_1) H_{i_2}^2(z_2) \ldots H_{i_M}^M(z_M) \tag{87}$$

and $\{H_{\mathbf{i}}, \forall \mathbf{i} \in \mathbb{N}^M\}$ will be an orthonormal basis in $L^2_{p_{\eta}}(\mathbb{R}^M)$, it is also referred to as a *tensor product basis*.

Approximation using tensor product spaces

Definition

A tensor product space is defined as

$$\Xi_N^T = span\{H_{\mathbf{i}}, \forall \mathbf{j}(\mathbf{i}) \leq N\} \tag{88}$$

So basically a *tensor product space* is the inclusion of approximating terms in each of the components up to the N^{th} term. If in each of the components there is N basis functions, the dimension of the tensor space Ξ_N^T will be N^M, this means that the dimension of the space grows exponentially fast with M, the number of random variables (*the curse of dimensionality*).

- If Ξ_N^T, the space of polynomials of degree at most N in each component of the random vector, then its dimesion will be

$$dim(\Xi_N^T) = (N+1)^M \tag{89}$$

- If the space Ξ_N^T corresponds to a constant piecewise approximation over a cartesian partition of $\mathbb{R}^N$ with N intervals in each component of the random vector, then its dimension is

$$dim(\Xi_n^T) = 2^{NM} \tag{90}$$

Because of the *curse of dimensionality*, tensor product approximations can be used with few random variables.

Approximation using sparse product spaces

Definition

A sparse product space is defined as

$$\Xi_N^S = span\{H_{\mathbf{i}}, \forall |\mathbf{j}(\mathbf{i})| \leq N\} \tag{91}$$

These type of spaces overcome partially the *curse of dimensionality* while keeping almost the same approximability properties as tensor product spaces.

- The space Ξ_N^S that corresponds to polynomials of total degree at most of N will have the following dimension

$$dim(\Xi_N^S) = \sum_{|\mathbf{j}(\mathbf{i})| \leq N} 1 = \frac{(M+N)!}{N!M!} = \binom{M+N}{l} \ll (1+N)^M \tag{92}$$

- In a sparse wavelet space the dimension will be

$$dim(\Xi_N^S) = \sum_{|\mathbf{j}(\mathbf{i})| \leq N} 1 = 2^N \frac{(M+N)!}{N!M!} = 2^N \binom{M+N}{N} \ll 2^{NM} \tag{93}$$

for a large M, these relations behave in an asymptotical way. So a sparse projection using a polynomial basis can be constructed as

$$P_N^S g(\eta) = \sum_{j(k) \leq N} g_{\mathbf{k}} H_{\mathbf{k}}(\eta) \tag{94}$$

where $g_{\mathbf{k}} = E[gH_{\mathbf{k}}]$, and the amount of this coefficients is equal to the dimension of the approximating space Ξ_N^S

Sparse interpolation

As a reminder for the Lagrange interpolation in one dimension is

$$\mathcal{I}_N g(\eta_i) = \sum_{k=0}^{N_i-1} g(\eta_{ik}) L_k(\eta_i) \tag{95}$$

$$\int_{\mathbb{R}^N} p_\eta(\mathbf{z}) f(\mathbf{z}) d\mathbf{z} = \prod_{i=1}^{N} \left(\int_{\mathbb{R}} p_{\eta_i}(z_i) \mathcal{P}_i(L(\eta_i)) dz_i \right) \tag{96}$$

Now, if the difference between 2 levels of interpolation is performed as

$$\Delta_j(g) = \mathcal{I}_j(g) - \mathcal{I}_{j-1}(g) \tag{97}$$

where $\mathcal{I}_{-1}(g) = 0$, then the sparse interpolation reads

$$\mathcal{I}_N^S g(\boldsymbol{\eta}) = \sum_{|\mathbf{j}| \leq N} \bigotimes_{n=1}^{M} \Delta_{j_n} g(\boldsymbol{\eta}) \tag{98}$$

This expression can also be written as

$$\mathcal{I}_N^S g(\boldsymbol{\eta}) = \sum_{N-M+1 \le |\mathbf{j}| \le N} (-1)^{N-|\mathbf{j}|} \binom{M-1}{N-|\mathbf{j}|} \mathcal{I}_N^T g(\boldsymbol{\eta}) \tag{99}$$

where

$$\mathcal{I}_N^T g(\boldsymbol{\eta}) = \sum_{k_1=0}^{N_{j_1}} \cdots \sum_{k_N=0}^{N_{j_N}} g(\eta_{1k_1}, \ldots, \eta_{Nk_N}) L_{k_1}(\eta_1) \cdots L_{k_N}(\eta_N) \tag{100}$$

Hence the sparse interpolation is obtained as a *linear combination* of tensor product interpolations where the tensor grid (set of points where the function is evaluated) is

$$\mathcal{H}_{\mathbf{j}}^T = \{\eta_{\mathbf{k}} = (\eta_{1k_1}, \ldots, \eta_{Nk_N}),\ 0 \le k_i \le N_j - 1\} \tag{101}$$

and

$$\mathcal{H}_{\mathbf{j}}^S = \bigcup_{N-M+1 \le |\mathbf{j}| \le N} \mathcal{H}_{\mathbf{j}}^T \tag{102}$$

is the collection of tensor grids used in sparse interpolation.

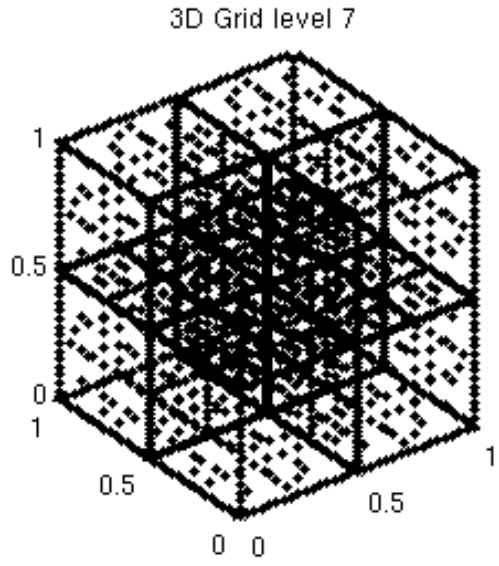


Figure: This figure corresponds to a 3D grid of Clenshaw-Curtis points.

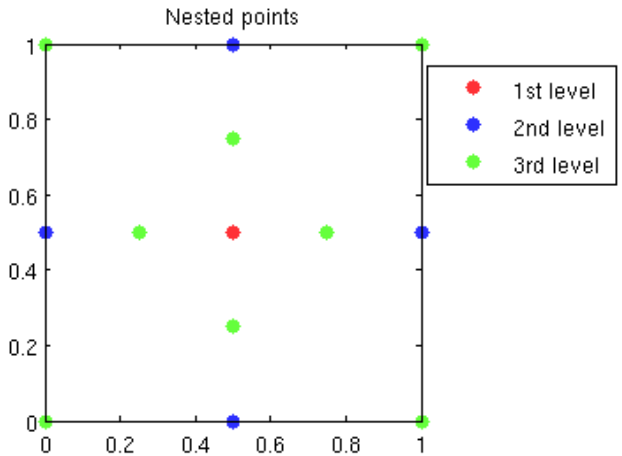


Figure: This figure corresponds to nested Clenshaw-Curtis points.

Quadrature of square integrable functions in $\mathbb{R}^N$

In [Barthelmann et al.(2000)Barthelmann, Novak & Ritter], it is possible to find different methods to compute integrals using *cubature* methods (the same as *quadratures* but in multiple dimesions). If our integral is in a d-dimensional cube, then the integral of any function can be expressed as

$$I_d(f) = \int_{[-1,1]^d} f(\mathbf{z}) d\mathbf{z} \tag{103}$$

A tensor product cubature is

$$I_d(f_1 \otimes \cdots \otimes f_d) = I_1(f_1) \otimes \cdots \otimes I_d(f_d) \tag{104}$$

then the approximation in the i^{th} component is (using any of the previously described $1-D$ approximations methods)

$$U^i(f) = \sum_{j=1}^{m_i} f(z_{ij}) \cdot \gamma_{ij} \tag{105}$$

then using (104), it is possible to compute numerically (103) as

$$(U^{i_1} \otimes \cdots \otimes U^{i_d})(f) = \sum_{j_1=1}^{m_{i_1}} \cdots \sum_{j_d=1}^{m_{i_d}} f(z_{i_1 j_1}, \ldots, z_{i_d j_d}) \cdot (\gamma_{i_1 j_1} \otimes \cdots \otimes \gamma_{i_d j_d}) \tag{106}$$

, which is a tensor cubature. This has the shortcoming that as the dimension of the integral increases, the number of points increases dramatically, thus not making this method unefficient. To overcome this shortcoming, sparse interpolation gives the oportunity to develope sparse numerical integration schemes. Let

$$\Delta^i = U^i - U^{i-1} \tag{107}$$

for $i \in \mathbb{N}$, and $|\mathbf{i}| = i_1 + \cdots + i_d$ for $\mathbf{i} \in \mathbb{N}^d$, then the Smolyak algorith reads

$$A(q,d) = \sum_{|\mathbf{i}| \leq q} (\Delta^{i_1} \otimes \cdots \otimes \Delta^{i_d}) \tag{108}$$

thus simply using for each dimension the desired quadrature method, to later use a method that is proven by [Babuška et al.(2007)Babuška, Nobile & Tempone] and [Nobile et al.(2008)Nobile, Tempone & Webster] to be convergent to the tensor case, which decreases the amount of points needed to compute the integral with respect to the full tensor scheme.

Discretization of random fields

The principal concern is to reduce a infinite amount of information (random variables that belong to a random field) to a finite amount of random variables such that the main information of the random field is well represented. The approximate random field does not only need to be well represented, but also with the minimum amount of random variables to decrease the complexity of calculations. In the literature there are many methods which will be presented. Let $\kappa(\mathbf{x}, \omega) \colon \mathcal{D} \times \Omega \longrightarrow \mathbb{R}$ be a random field with a certain autocovariance function $C_{\kappa\kappa}(\mathbf{x}, \mathbf{y}) = C[\kappa(\mathbf{x}, \omega), \kappa(\mathbf{y}, \omega)]$, then an approximation can be

$$\kappa_N(\mathbf{x}, \boldsymbol{\omega}) = \sum_{i=1}^{N} \eta_i \phi_i(\mathbf{x}) \tag{109}$$

This can be viewed as an expansion of each possible realization of $\kappa(\mathbf{x}, \omega_0)$ over a basis $\{\phi_i\}$, where $\boldsymbol{\eta}_i(\omega)$ are the coordinates of this expansion. This can be done since this is a Hilbert space (Neveu, don have this paper yet), thus it accepts expansions of the sort.

Karhunen-Loève expansion

The Karhunen-Loève expansion of a random field $\kappa(\mathbf{x}, \omega)$ is a spectral decomposition of the autocovariance function, so that there exists a sequence of values $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_k \ldots \geq 0$, with $\lim_{k \to \infty} \lambda_k = 0$ and a corresponding sequence of functions $\varphi_i(\mathbf{x}) \colon \mathcal{D} \to \mathbb{R}$, $i = 1, 2, \ldots$ such that

$$\int_{\mathcal{D}} C_{\kappa\kappa}(\mathbf{x}, \mathbf{y}) \varphi_i(\mathbf{y}) d\mathbf{y} = \lambda_i \varphi_i(\mathbf{x}) \tag{110}$$

which is a Fredholm integral equation, where the kernel is the covariance function, being bounded, symmetric and positive definite. The eigenfunctions satisfy $\int_{\mathcal{D}} \varphi_i(\mathbf{x}) b_j(\mathbf{y}) d\mathbf{x} = \delta_{ij}$ where δ_{ij} is the Kronecker delta, thus being orthogonal. Now, it is possible to define a sequence of random variables $\eta_i(\omega)$, $i = 1, 2, \ldots$ such that

$$\eta_i(\omega) = \frac{1}{\sqrt{\lambda_i}} \int_{\mathcal{D}} (\kappa(\mathbf{x}, \omega) - E[\kappa(\mathbf{x})]) \varphi_i(\mathbf{x}) d\mathbf{x} \tag{111}$$

and if the covariance between the original random field and the Karhunen-Loève expansion of such random field $C[\kappa(\mathbf{x}, \omega), \hat{\kappa}(\mathbf{x}, \omega)]$ is computed, it is possible to conclude that $E[\eta_i \eta_j] = \delta_{ij} = 0$, thus proving that these variables are uncorrelated with zero mean and unit variance and can be viewed as the coordinates of the expansion and of any possible realization.

With all this in mind, the random field $\kappa(\mathbf{x}, \omega)$ can be represented as the infinite series

$$\kappa(\mathbf{x}, \omega) = E[\kappa(\mathbf{x})] + \sum_{i=1}^{\infty} \sqrt{\lambda_i} \varphi_i(\mathbf{x}) \eta_i(\omega) \tag{112}$$

thus creating a separation of the space and the random variables of the field $\kappa(\mathbf{x}, \omega)$. The Karhunen-Loève expansion has very interesting properties, such as:

- Due to non accumulation of the eigenvalues around a zero value, it is possible to order the terms of the expansion in descending way, such that if a truncation of the series defined in (112) gives an approximated Karhunen-Loève expansion

$$\kappa_N(\mathbf{x}, \boldsymbol{\omega}) = E[\kappa(\mathbf{x})] + \sum_{i=1}^{N} \sqrt{\lambda_i} \varphi_i(\mathbf{x}) \eta_i(\omega) \tag{113}$$

- The covariance eigenfunction basis $\{\varphi(\mathbf{x})\}$ is optimal in the sense of the mean square error

$$err = \int_{\mathcal{D}} E[(\kappa(\mathbf{x}, \omega) - \kappa_N(\mathbf{x}, \omega))^2] d\mathbf{x} \tag{114}$$

- The closed form solution for each random variable in (111) is easy to obtain, and if $\kappa(\mathbf{x}, \omega)$ is a *Gaussian field*, then each random variable η_i is gaussian, and for this specific case, $\{\eta_i\}$ forms a set of independent standard normal random variables. It can also be proven [Loève(1977)] that the expansion of Gaussian random fields is almost surely convergent (I guess this is very important for time stochastic processes).
- The variance error, after truncating the series in the N^{th} term is given by

$$\begin{aligned} V[\kappa(\mathbf{x}, \omega) - \kappa_N(\mathbf{x}, \omega)] &= \sigma^2(\mathbf{x}) - \sum_{i=1}^{N} \lambda_i \varphi_i^2(\mathbf{x}) \\ &= V[\kappa(\mathbf{x}, \omega)] - V[\kappa_N(\mathbf{x}, \omega)] \end{aligned} \tag{115}$$

The right hand side of the previous equation is always positive, hence the truncated Karhunen-Loève expansion always under-represents the variance of the field.

Example

If there is a covariance function

$$C(x, y) = \sigma^2 e^{-|x-y|} \tag{116}$$

with $x \in [0, 1]$, then the eigenfunctions will be:

$$\varphi_i(x) = \frac{w_i \cos(\omega_i x) + \sin(\omega_i x)}{\sqrt{\frac{\omega^2}{2} + (1 + \frac{\sin(2\omega)}{2\omega}) + \frac{1}{2}(1 - \frac{\sin(2\omega)}{2\omega}) + \sin^2(\omega)}} \tag{117}$$

and where ω_i is related to the eigenvalues and can be obtained from the following equation:

$$2\omega \cos(\omega) + (1 - \omega^2) \sin(\omega) = 0 \tag{118}$$

Example

It is very easy to obtain these roots with any method, such as the Newton scheme. so then it is possible to obtain the eigenvalues from the following relationship:

$$\omega_i^2 = \frac{2 - \lambda_i}{\lambda_i}\sigma^2 \tag{119}$$

thus making it possible to characterize the random field represented by the covariance in equation (116) using the Karhunen-Loève expansion.

This type of expansion has a limited number of closed form solutions for the set of covariance functions, thus it is necessary to perform other types of numerical expansions to have a spectral representation of the random field given a covariance function.

Galerkin expansion method

A *Galerkin*-type procedure was suggested by [Spanos & Ghanem(1991a)] and it consists in defining a complete basis of the Hilbert space $\{\phi_i(\mathbf{x})\}_{i=1}^{\infty} \in L^2_{p_\eta}(\mathcal{D})$. Each eigenfunction of $C_{\kappa\kappa}(\mathbf{x}, \mathbf{y})$ may be represented by an expansion of the form

$$\varphi_j(\mathbf{x}) = \sum_{i=1}^{\infty} d_{ij}\phi_i(\mathbf{x}) \tag{120}$$

where d_{ij} are the unknown coordinates. The Galerkin procedure aims at obtaining the best approximation of φ_j when truncating the above series after the N^{th} term, which is accomplished by projecting φ_j onto the space $\mathcal{H}_N$ spanned by $\{\phi_i\}_{i=1}^{N}$. The approximation of each eigenfunction is possible because they belong to the space of square integrable functions.

The residual can be written as

$$\epsilon_N(\mathbf{x}) = \sum_{i=1}^{N} d_{ij} \left[\int_{\mathcal{D}} C_{\kappa\kappa}(\mathbf{x}, \mathbf{y}) \phi_i(\mathbf{y}) d\mathbf{y} - \lambda_j \phi_i(\mathbf{x}) \right] \tag{121}$$

which is orthogonal to $\mathcal{H}_N$, thus the following is true

$$\langle \epsilon_N, \phi_j \rangle \equiv \int_{\mathcal{D}} \epsilon_N(\mathbf{x}) \phi_j(\mathbf{x}) d\mathbf{x} \quad j = 1, \ldots, N \tag{122}$$

which leads to a simple linear system

$$\mathbf{CD} = \Lambda \mathbf{BD} \tag{123}$$

where the different matrices are defined as follows

$$\begin{aligned}
\mathbf{B}_{ij} &= \int_{\mathcal{D}} \phi_i(\mathbf{x})\phi_j(\mathbf{x})d\mathbf{x} \\
\mathbf{C}_{ij} &= \int_{\mathcal{D}}\int_{\mathcal{D}} C_{\kappa\kappa}(\mathbf{x},\mathbf{y})\phi_i(\mathbf{x})\phi_j(\mathbf{y})d\mathbf{x}d\mathbf{y} \\
\mathbf{D}_{ij} &= d_{ij} \\
\Lambda_{ij} &= \delta_{ij}\lambda_j
\end{aligned} \tag{124}$$

thus rendering a series of linear systems to be solved.

Orthogonal series expansion method

Many times there are no closed for solution to the problem of obtaining eigen-pairs given a kernel function such as the covariance. Taking this into mind, [Zhang & Ellingwood(1994)] proposed a method to avoid the computation of the eigen-pairs just by selecting prior to any calculation, a complete set of orthogonal functions. A similar idea had been used previously by Lawrence (1987), (don't have this paper). Let $\{\varphi_i(\mathbf{x})\}_{i=1}^{\infty}$ be a family of orthogonal functions forming a basis in L^2, and without any loss of generality, let this family be orthonormal as well

$$\int_{\mathcal{D}} \phi_i(\mathbf{x})\phi_j(\mathbf{x})d\mathbf{x} = \delta_{ij} \tag{125}$$

If $\kappa(\mathbf{x}, \omega)$ is a random fiels, with a given covariance function $C_{\kappa\kappa}(\mathbf{x}, \mathbf{y})$, any realization of the field is a function of L^2 which can be expanded by means of the previously selected set of orthogonal functions, and this expansion can be

$$\kappa(\mathbf{x}, \omega) = \mu(\mathbf{x}) + \sum_{i=1}^{\infty} \chi_i(\omega)\phi_i(\mathbf{x}) \tag{126}$$

where $\chi_i(\omega)$ are zero-mean random variables. Using the orthogonality properties described before, and also with the help of basic algebra, it is possible to show that

$$\chi(\omega) = \int_{\mathcal{D}} \left[\kappa(\mathbf{x}, \omega) - \mu(\mathbf{x})\right] \phi_i(\mathbf{x}) d\mathbf{x} \tag{127}$$

and

$$(C_{\chi\chi})_{kl} \equiv E[\chi_k \chi_l] = \int_{\mathcal{D}} \int_{\mathcal{D}} C_{\kappa\kappa}(\mathbf{x}, \mathbf{y}) h_k(\mathbf{x}) h_l(\mathbf{y}) d\mathbf{x} d\mathbf{y} \tag{128}$$

If $\kappa(\mathbf{x}, \omega)$ is a zero mean Gaussian field, thenit is easy to prove that $\{\chi_i\}_{i=1}^{\infty}$ are zero mena Gaussian random variables, but unfortunately, they might bee correlated, thus it is necessary to construct the covariance matrix of these random variables, and to perform a transformation into uncorrelated random variables using the spectral decomposition method described previously.

Optimal linear estimation method

The method was proposed by [Li & Kiureghian(1993)]. It is also mentioned in the literature as *Kriging method*. It is a special case of the method of regression on linear functionals, and the approximation has the following form

$$\kappa_N(\mathbf{x},\omega) = a(\mathbf{x}) + \sum_{i=1}^{N} b_i(\mathbf{x})\chi_i \tag{129}$$

where N is the total number of nodal points involved in the approximation. The functions $a(\mathbf{x})$ and $b_i(\mathbf{x})$ are obtained using a nonlinear regression, minimizing in each point $\mathbf{x}$ the error of the variance, subjected to having an unbiased estimator of the real random field in the mean, these consitions are expressed as

$$\begin{array}{rl} \forall \mathbf{x} \in \mathcal{D} & \\ \text{Minimize} & V[\kappa(\mathbf{x},\omega) - \kappa_N(\mathbf{x},\omega)] \\ \text{subjected to} & E[\kappa(\mathbf{x},\omega) - \kappa_N(\mathbf{x},\omega)] = 0 \end{array} \tag{130}$$

The condition of problem (130) requires that

$$E[\mathbf{x}] = a(\mathbf{x}) + \mathbf{b}^T(\mathbf{x}) \cdot E[\boldsymbol{\chi}] \tag{131}$$

then, the error of the variance, which is always a positive value, holds true

$$\begin{aligned} &V[\kappa(\mathbf{x},\omega) - \kappa_N(\mathbf{x},\omega)] \\ &= \sigma^2(\mathbf{x}) - 2\sum_{i=1}^{N} b_i(\mathbf{x}) C[\kappa(\mathbf{x},\omega),\chi_i] + \sum_{i=1}^{N}\sum_{j=1}^{N} b_i(\mathbf{x}) b_j(\mathbf{x}) C[\chi_i,\chi_j] \end{aligned} \tag{132}$$

The minimization problem is solved for each $b_i(\mathbf{x})$, thus requiring that the partial derivatives of $b_i(\mathbf{x})$ be equal to zero for each i yields

$$\forall i = 1, \ldots, N \quad -C[\kappa(\mathbf{x}, \omega), \chi_i] + \sum_{j=1}^{N} b_j(\mathbf{x}) C[\chi_i, \chi_j] = 0 \tag{133}$$

which can also be written in matricial form

$$-C_{\kappa\chi}(\mathbf{x}) + C_{\chi\chi} \cdot b(\mathbf{x}) = 0 \tag{134}$$

with this result, the final estimation is written as

$$\kappa_N(\mathbf{x}, \omega) = E[\mathbf{x}] + C^T_{\kappa\chi}(\mathbf{x}) \cdot C^{-1}_{\chi\chi}(\boldsymbol{\chi} - E[\boldsymbol{\chi}]) \tag{135}$$

Separating the deterministic terms from the stochastic ones gives

$$\kappa_N(\mathbf{x}, \omega) = \left[E[\mathbf{x}] - C^T_{\kappa\chi}(\mathbf{x}) \cdot C^{-1}_{\chi\chi} \cdot E[\boldsymbol{\chi}]\right] + \sum_{i=1}^{N} \chi_i (C^{-1}_{\chi\chi} \cdot C_{\kappa\chi}) \tag{136}$$

and since the variance errors are always positive, then this approximation always underestimates the variance.

Expansion optimal linear estimation method

The expansion optimal linear estimation method was proposed by [Li & Kiureghian(1993)]. It is an extension of OLE using a spectral vector of nodal variables χ. Assuming that $\kappa(\mathbf{x}, \omega)$ is Gaussian, the spectral decomposition of the covariance matrix C_χ, where $\chi = \{\kappa(\mathbf{x}_1, \omega)\}, \ldots, \kappa(\mathbf{x}_N, \omega)$ is

$$\chi(\omega) = \boldsymbol{\mu}_\chi + \sum_{i=1}^{N} \sqrt{\lambda_i}\zeta_i(\omega)\varphi_i(\mathbf{x}) \tag{137}$$

where the set of random variable $\{\zeta_i\}_i^N$ are independent and standard normal. The pair (λ_i, φ_i) are the eigenvalues and eigenvectors of the covariance matrix $C_{\boldsymbol{\chi}\boldsymbol{\chi}}$ that satisfies

$$C_{\chi\chi}\varphi_i = \lambda_i\varphi_i \tag{138}$$

hence, if the substitution of (138) into (129) and solving the OLE problem yields

$$\kappa_N(\mathbf{x},\omega) = \mu(\mathbf{x}) + \sum_{i=1}^{N} \frac{\zeta_i(\omega)}{\sqrt{\lambda_i}} \varphi_i^T C_{\kappa\chi} \tag{139}$$

As in the Karhunen-Loève expansion, the series can be truncated after r terms, of course after sorting the eigenvalues λ_i in a descending order. The variance error of this methos will be

$$V[\kappa(\mathbf{x},\omega) - \kappa_N(\mathbf{x},\omega)] = \sigma^2(\mathbf{x}) - \sum_{i=1}^{r} \frac{1}{\lambda_i} (\varphi_i^T C_{\kappa\chi})^2 \tag{140}$$

The problem with this method is the choice of points in the random field, such to assure a good result. This was not aborded by the author of the method.

Operators with uncertain parameters

Let us consider the following problem

$$\mathcal{L}(\kappa)(u) = f \quad in \quad \mathcal{D} \tag{141}$$

where $\mathcal{L}$ is an operator of a certain kind withought loss of generality, and it is defined in a domain D. Both $\kappa = \kappa(\mathbf{x}, \omega)$ and $f = f(\mathbf{x}, \omega)$ can be considered as *random fields* of a certain kind. The latter variables can describe any kind of scalar field, such as the Poisson ratio, Young's modulus or the external forces acting on $\partial\mathcal{D}$. The random field on the right hand side and left hand side of equation (141) are considered to be functions of a finite amount of random variables such that it can be written as:

$$\begin{aligned} \kappa(\mathbf{x}, \omega) &= \kappa(\mathbf{x}; \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_N) \\ f(\mathbf{x}, \omega) &= f(\mathbf{x}; \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_N) \end{aligned} \tag{142}$$

This approximation of a random field using a random vector is possible and it can be done using different methodologies such as the truncation of the Karhunen-Loève expansion, etc. Many assumptions can be made of the chosen random vector that represents the both random fields $\kappa(\mathbf{x}, \omega)$ and $f(\mathbf{x}, \omega)$, such as the correlation between the components of the random vector as $C[\eta_i, \eta_j] = \delta_{ij}$, the type of distribution that each component has, etc.

Sometimes, the uncertain coefficients of the operator $\mathcal{L}$ or the forcing term of equation (141) need to be more specified, an example is illustrated.

Example

If the coefficient $\kappa(\mathbf{x}, \omega)$ is always greater than κ_{min} for it to have sense, then the following transformation guarantees that this will always be certain.

$$log(\kappa - \kappa_{min})(\mathbf{x}, \omega) = b_0(\mathbf{x}) + \sum_{i=1}^{N} \sqrt{\lambda_i} b_i(\mathbf{x}) \eta_i(\omega) \tag{143}$$

The latter is simple the truncation of the Karhunen-Loève expansion of $log(\kappa - \kappa_{min})$. The same can be done for the forcing term $f(\mathbf{x}, \omega)$

Lemma

Let $\boldsymbol{\eta}$ be an N-dimensional random vector defined on $(\Omega^N, \mathcal{F}, P)$ and let ζ be a σ-measurable random variable defined on the same space. Then $\zeta = g(\boldsymbol{\eta})$ for some Borel measurable $g : \mathbb{R}^N \to \mathbb{R}$, this means that ζ will depend only on the components of $\boldsymbol{\eta}$, that is, $\zeta(\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_N)$

This lemma can be extended to problem (141) for the case of finite dimensional noise, which can be viewed in the following way:

$$\mathcal{L}(\kappa_N)(u_N) = f_N \quad in \quad \mathcal{D} \tag{144}$$

using lemma (28.2) it can be concluded that $u_N = u_N(\mathbf{x}; \boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_N)$

Remark

It can be noticed that in the definition of the finite noise problem (142), for both the forcing term and the term associated with the operator have dependency on the same random variables. It is clear that in most of the real cases this is not, so the problem can be viewed in the following way:

$$\begin{aligned} \kappa(\mathbf{x}, \omega) &= \kappa(\mathbf{x}; \boldsymbol{\eta}_a) \\ f(\mathbf{x}, \omega) &= f(\mathbf{x}; \boldsymbol{\eta}_f) \end{aligned} \tag{145}$$

thus $u_N = u_N(\mathbf{x}; \boldsymbol{\eta}_a, \boldsymbol{\eta}_f)$

Stochastic collocation methods

Let $u_h(x, \boldsymbol{\eta})$ be the semi-discrete solution of

$$\int_{\mathcal{D}} \kappa(\mathbf{x}, \omega) \nabla u_h(\mathbf{x}, \omega) \cdot \nabla v_h(\mathbf{x}) d\mathbf{x} = \int_{\mathcal{D}} f(\mathbf{x}) v_h(\mathbf{x}) d\mathbf{x} \quad \forall v_h \in V_h \tag{146}$$

Consider a sparse interpolation formula $\mathcal{I}_N^S \colon L^2_{p_\eta}(\mathbb{R}^N) \to \Xi_N^S$ that uses a sparse grid $\mathcal{H}_N^S$. The stochastic collocation FEM solution is obtained by simply interpolating the semi-discrete solution on the sparse grid

$$u_h^N(\mathbf{x}, \boldsymbol{\eta}) = \mathcal{I}_N^S u_h(\mathbf{x}, \boldsymbol{\eta}) \in V_h \otimes \Xi_N^S \tag{147}$$

This means that if the points in the probability domain are $\boldsymbol{\eta}^{(1)}, \ldots, \boldsymbol{\eta}^{(M)} \in \mathcal{H}_N$ that belong to the sparse grid, then using the pre-defined sparse interpolation, the solution can be achieved simply by computing M deterministic solutions $u_h^{(j)}$, $j = 1, \ldots, M$, using the points that belong to the sparse grid, and projecting the solution of these points into the sparse interpolation formula.

Finite elements for beams using a residual formulation

According to the virtual work principal, the external work is equal to the internal work, thus yielding

$$L_v^{int} = L_v^{ext} \tag{148}$$

where the internal work is $L_v^{int} = \int_{\mathcal{D}} \sigma \delta \varepsilon dV$. On the other hand, the kinematics of the beam can be stated as

$$\varepsilon = \varepsilon_0 - y\chi = u_{,x} - yv_{,xx} \tag{149}$$

which means that plane sections remain plane. Clearly $\delta\varepsilon = \delta u_{,x} - y\delta v_{,xx}$, therefore

$$L_v^{int} = \int_{\mathcal{D}} \sigma(\delta u_{,x} - y\delta v_{,xx})dV = \int_l (N\delta u_{,x} + M\delta v_{,xx})dl \tag{150}$$

since $N = \int_A \sigma dA$ and $M = -\int_A y\sigma dA$

Using a finite element approximation of the following form for the beam as

$$\begin{aligned} u &\simeq \mathbf{N}^u \hat{\mathbf{u}} \quad \mathbf{u}_{,x} \simeq \mathbf{B}^u \hat{\mathbf{u}} \\ v &\simeq \mathbf{N}^v \hat{\mathbf{v}} \quad \mathbf{v}_{,xx} \simeq \mathbf{B}^v \hat{\mathbf{v}} \end{aligned} \tag{151}$$

introducing these approximations in the equation of virtual work will give a set of equations

$$L_v^{int} = \delta \hat{\mathbf{u}}^T \int_l \mathbf{B}^{uT} N dl + \delta \mathbf{v}^T \int_l \mathbf{B}^{vT} M dl \tag{152}$$

which can also be written as

$$L_v^{int} = \delta \hat{\mathbf{u}}^T \mathbf{F}_{axi}^{int} + \delta \mathbf{v}^T \mathbf{F}_{bend}^{int} \tag{153}$$

and recalling that

$$L_v^{ext} = \delta \hat{\mathbf{u}}^T \mathbf{F}_{axi}^{ext} + \delta \mathbf{v}^T \mathbf{F}_{bend}^{ext} \tag{154}$$

The principal of virtual work implies that

$$\mathbf{R}(\hat{\mathbf{b}}, \hat{\mathbf{v}}) = [\mathbf{R}_{axi}(\hat{\mathbf{b}})\ \mathbf{R}_{bend}(\hat{\mathbf{v}})]^T = 0 \tag{155}$$

There are many algorithms available in the literature to solve this kind of nonlinear system of equations, for instance the Newton-Raphson scheme. For the sake of simplcity, the integration rule for each cross section is done using a composite midpoint rule over n_s horizontal stripes of width b and height h/n_s.

$$\Rightarrow N = \int_A \sigma \, dA \simeq \frac{bh}{n_s} \sum_{\text{I}=1}^{n_s} \sigma_i \tag{156}$$

and

$$-\int_A \sigma y dA \simeq -\frac{bh}{n_s}\sum_{i=1}^{n_s}\sigma_i y_i \tag{157}$$

where $y_i = \frac{h}{2}\left(\frac{2i-1}{n_s} - 1\right)$. To compute the internal forces a simple 3 point Simpson quadrature is more than enough, using the first end, the midpoint and the end of the element as P_1, P_2 and P_3, thus the expresion for the internal force reduces to

$$\begin{bmatrix} \mathbf{F}^{int}_{ax} \\ \mathbf{F}^{int}_{bend} \end{bmatrix} = \begin{matrix} \int_l \mathbf{B}^{uT} N dl \\ \int_l \mathbf{B}^{vT} M dl \end{matrix} \backsimeq \frac{l}{6}\begin{bmatrix} \mathbf{B}^{uT} N|P_1 + 4\mathbf{B}^{uT} N|P_2 + \mathbf{B}^{uT} N|P_3 \\ \mathbf{B}^{vT} M|P_1 + 4\mathbf{B}^{vT} M|P_2 + \mathbf{B}^{vT} M|P_3 \end{bmatrix} \tag{158}$$

and $\mathbf{R} = \mathbf{F}^{int} - \mathbf{F}^{ext}$

Beams with uncertain mechanical properties

Let us consider now the same problem as stated previously, but considering uncertain mechanical properties along the length of the beam. For this a collocation method will be used and the solution of the problem will be expanded

$$\mathcal{I}_N^S \mathbf{g}(\boldsymbol{\eta}) = \sum_{N-M+1 \leq |\mathbf{j}| \leq N} (-1)^{N-|\mathbf{j}|} \binom{M-1}{N-|\mathbf{j}|} \mathcal{I}_N^T \mathbf{g}(\boldsymbol{\eta}) \tag{159}$$

where $\mathbf{g} = [\mathbf{u}\ \mathbf{v}\ \boldsymbol{\theta}]^T$ The mechanical properties can be viewed as a RF with the following covariance function

$$C_{\kappa\kappa}(x, y) = e^{-\frac{(x-y)^2}{L^2}} \tag{160}$$

for $x, y \in [0, 1]$, where L represents the correlation length. The KL expansion of this covariance function is given by

$$\kappa(x, \boldsymbol{\eta}) = 1 + \eta_1(\omega) \left(\frac{\sqrt{\pi} L}{2} \right) + \sum_{k=2}^{N} \lambda_k b_k(x) \eta_k(\omega) \tag{161}$$

with

$$\lambda_k = \sqrt{\sqrt{\pi} L e^{\left(-\frac{\lfloor \frac{k}{2} \rfloor^2 \pi^2 L^2}{8}\right)}} \tag{162}$$

$$b_k(x) = \begin{cases} \sin(\lfloor k/2 \rfloor \pi x) & \text{if} \quad k\text{odd} \\ \sin(\lfloor k/2 \rfloor \pi x) & \text{if} \quad k\text{odd} \end{cases} \tag{163}$$

the random coordinates are uniformaly distributed in $[-\sqrt{3}, \sqrt{3}]$ with zero mean and unit variance. and $E(x, \eta) = 1/2 + e^{\kappa(x, \eta)}$ represents the elastic modulus along the length of the beam.

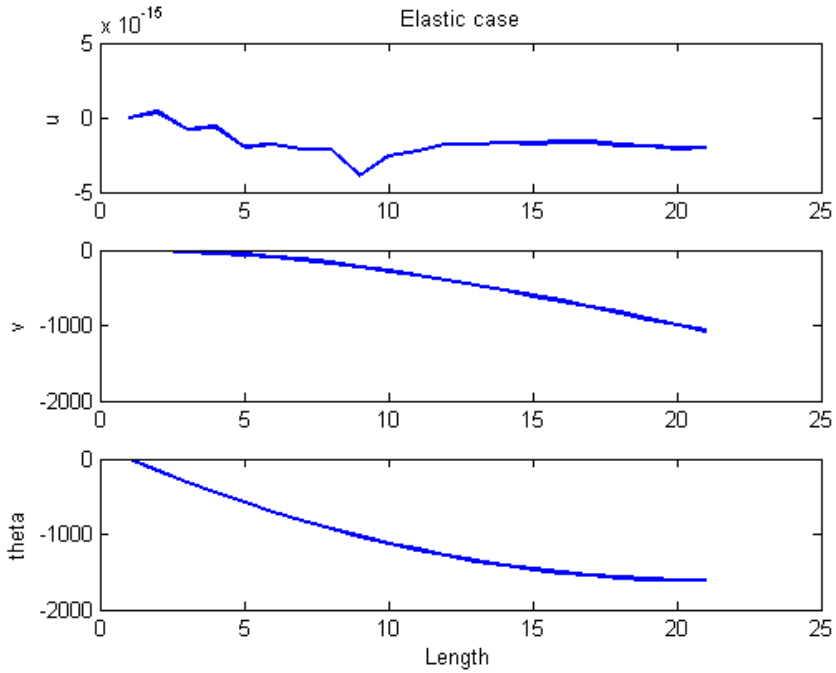


Figure: Linear mean response.

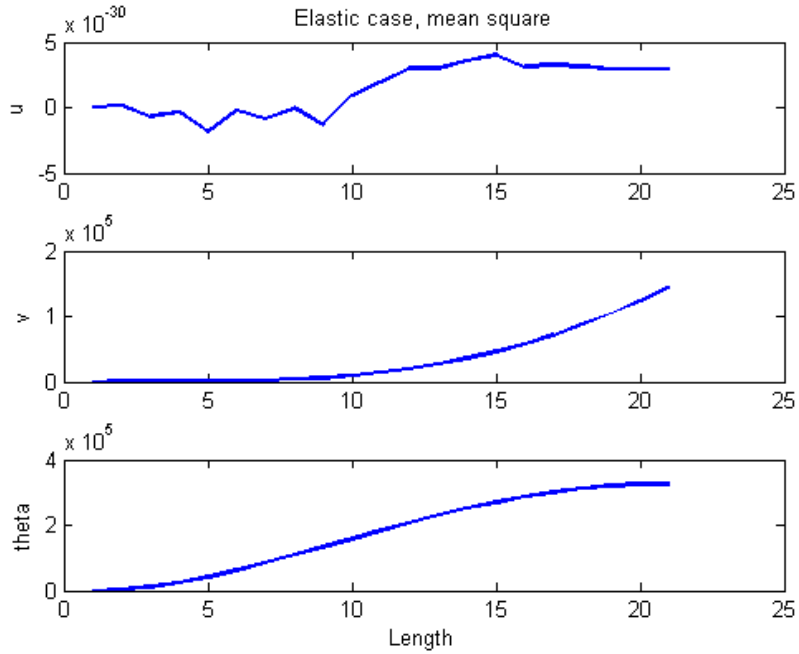


Figure: Linear mean square response.

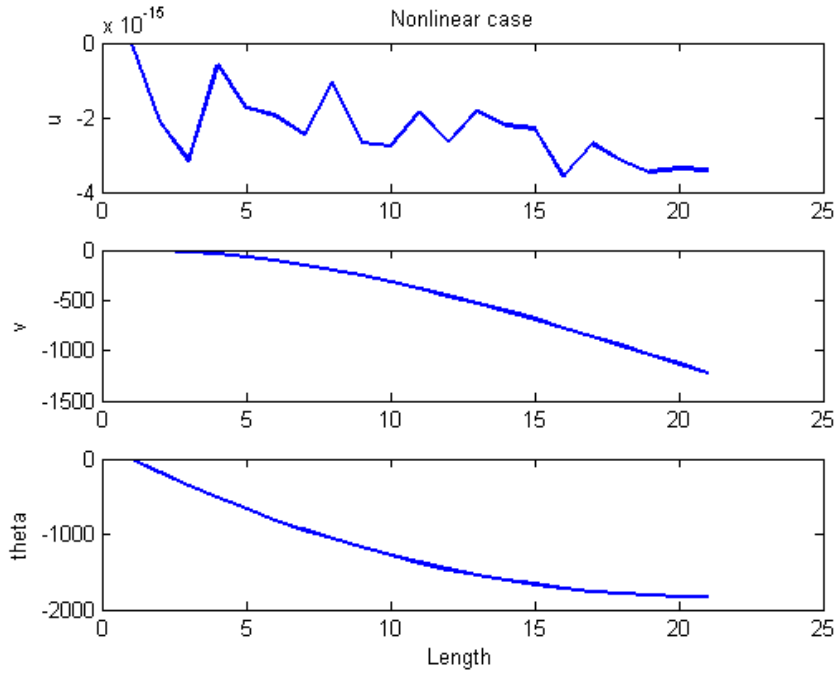


Figure: Nonlinear mean response.

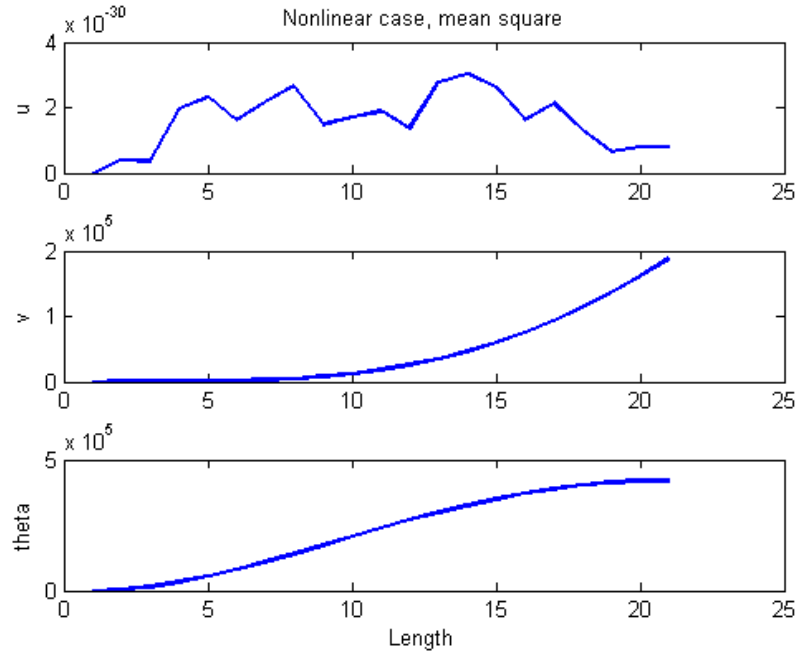
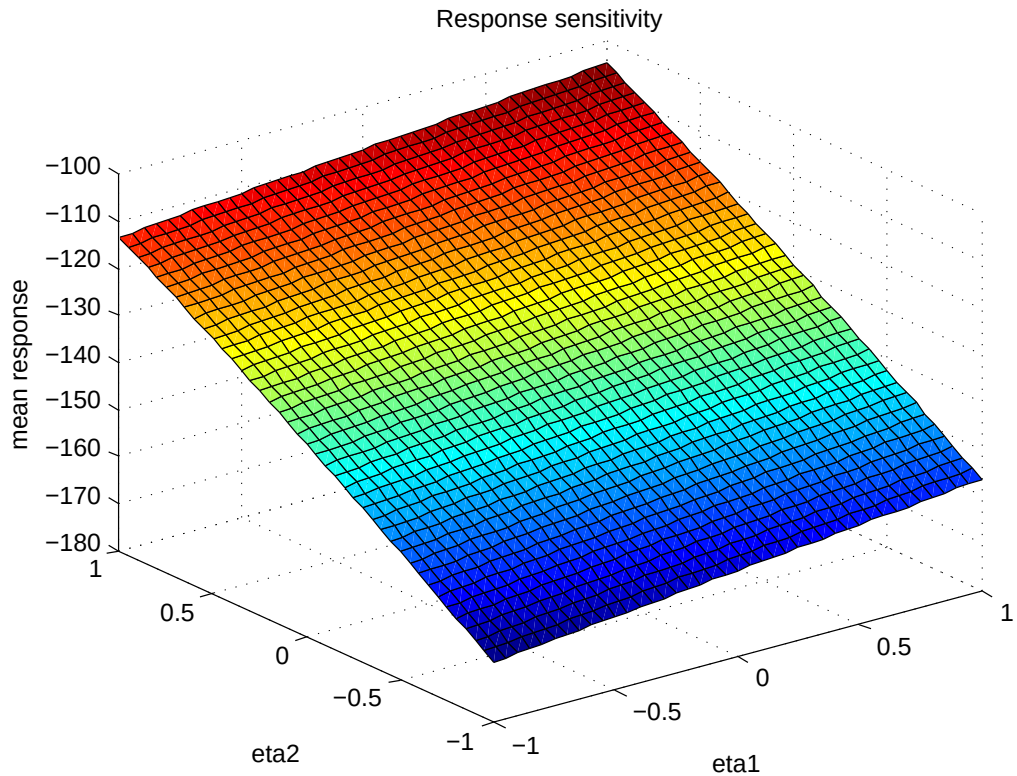


Figure: Nonlinear mean square response.


Response sensitivity
mean response
−100
−110
−120
−130
−140
−150
−160
−170
−180
1
0.5
0
−0.5
−1
eta2
−1
−0.5
0
0.5
1
eta1

Stochastic Galerkin approximation using double orthogonal polynomials

Let us consider the diffusion problem with no reaction for simplicity

$$\begin{cases} \nabla \cdot (\kappa(\mathbf{x}, \omega) \nabla u) & \text{in} \quad \mathcal{D} \subset \mathbb{R}^d \\ u = 0 & \text{on} \quad \partial \mathcal{D} \end{cases} \tag{164}$$

[Babuška et al.(2004)Babuška, Tempone & Zouraris] developed an efficient method to solve stochastic finite elements, using a stochastic Galerkin approach, that incorporates double orthogonal polynomials in the probabilistical domain, to obtain a number of undecoupled systems, each of the size of a deterministic realization of the same problem. Using a $p \times h$ finite element version, and withought any loss of generality, attention is focused in finding a solution $u_h^p \in V^h \otimes \Xi_p^S$ such that

$$E\left[\int_{\mathcal{D}} \kappa(\mathbf{x}, \omega) \nabla u_h^p(\mathbf{x}, \omega) \cdot \nabla v(\mathbf{x}, \omega) d\mathbf{x}\right] = E\left[\int_{\mathcal{D}} f(\mathbf{x}, \omega) v(\mathbf{x}, \omega) d\mathbf{x}\right] \tag{165}$$

Let $\{H(\boldsymbol{\eta})\}$ be a basis for the subspace $\mathbf{z}^p \subset L^2_{p_\eta}(\Omega)$, and $\{\phi(\mathbf{x})\}$ be a basis for the subspace $V_h \subset H^1_0(\mathcal{D})$. Then an approximation of the solution can be

$$u_h^p(\mathbf{x}, \eta) = \sum_{j,i} u_{ij} H_j(\omega)\phi_i(\mathbf{x}) \tag{166}$$

and the test functions can be taken as

$$v(\mathbf{x}, \omega) = H_k(\omega)\phi_l(\mathbf{x}) \tag{167}$$

to find u_{ij} coefficients, then equation (32) will be equivalent to

$$\begin{aligned} \sum_{j,i} E\left[H_k(\mathbf{z})H_j(\mathbf{z})\int_{\mathcal{D}} \kappa(\mathbf{x},\omega)\nabla\phi_i(\mathbf{x})\cdot\nabla\phi_l(\mathbf{x})d\mathbf{x}\right] u_{ij} \\ = E\left[H_k(\mathbf{z})\int_{\mathcal{D}} f(\mathbf{x},\mathbf{z})\phi(\mathbf{x})d\mathbf{x}\right] \quad \forall k, l \end{aligned} \tag{168}$$

Defining

$$G_{il}(\mathbf{z}) = \int_{\mathcal{D}} \kappa(\mathbf{x}, \omega) \nabla \phi_i(\mathbf{x}) \nabla \phi_l(\mathbf{x}) d\mathbf{x}$$
$$\text{and } f_l(\mathbf{z}) = \int_{\mathcal{D}} f(\mathbf{x}, \mathbf{z}) \phi_l(\mathbf{x}) d\mathbf{x} \tag{169}$$

If the random field is expanded into a KL expansion, then

$$G_{il}(\mathbf{z}) = \int_{\mathcal{D}} \left(E[\kappa(\mathbf{x}, \omega)] + \sum_{p=1}^{N} b_p z_p \right) \tag{170}$$

where

$$G_{il}^0 \equiv \int_{\mathcal{D}} E[\kappa(\mathbf{x}, \omega)] \nabla \phi_i(\mathbf{x}) \cdot \nabla \phi_l(\mathbf{x}) d\mathbf{x}$$
$$\text{and } G_{il}^p = \int_{\mathcal{D}} b_p(\mathbf{x}) \nabla \phi_i(\mathbf{x}) \cdot \nabla \phi_l(\mathbf{x}) d\mathbf{x} \tag{171}$$

Since $H_k \in \Xi_p^s$, with a multi-index $p = [p_1, \ldots, p_N]$, it is enough to take it as a product

$$H_k(\mathbf{z}) = \prod_{r=1}^{N} H_{kr}(z_r) \tag{172}$$

where $H_{kr} : \Omega_r \to \mathbb{R}$ is a basis function of the subspace $Z^{p_r} = span\{H_{hr} : h = 1, \ldots, p_r + 1\}$. Keeping the choice of H_k in mind,

$$\begin{aligned}\int_\Omega p_\eta(\mathbf{z}) H_k H_j G_{il}(z) d\mathbf{z} = G_{il}^0 \int_\Omega \prod_{m=1}^{N} p_{\eta_m}(z_m) H_{km}(z_m) H_{jm}(z_m) d\mathbf{z} \\ + \sum_{n=1}^{N} G_{il}^n \int_\Omega z_n \prod_{m=1}^{N} p_{\eta_m}(z_m) H_{km}(z_m) H_{jm}(z_m) d\mathbf{z}\end{aligned} \tag{173}$$

Now, for every set $\Omega_n,\ n = 1, \ldots, N$, the polynomials $H_j(\mathbf{z}) = \prod_{n=1}^{N} H_{jn}(z_n)$ need to be chose, and they can be selected to satisfy biorthogonality, so that they can satisfy

$$\begin{aligned} &\int_{\Omega_n} p_{\eta_n}(z_n) H_{kn}(z_n) H_{jn}(z_n) dz_n = \delta_{kj} \\ \text{and } &\int_{\Omega_n} z_n p_{\eta_n}(z_n) H_{kn}(z_n) H_{jn}(z_n) dz_n = c_{kn}\delta_{kj} \end{aligned} \tag{174}$$

To find these polynomials, N eigen-problems need to be solved, each of them having a size of $(1 + p_n)$. The computational work required by these eigen-problems is negligible compared with the effort to solve for u_{ij} Anyway, there is also the possibility of keeping these polynomials, for a given support Ω_n, and performing a simple change of variables using the property stated in (80) will reduce computational complexity. With this last finding, the right hand side of equation (168) will be

$$\begin{aligned}
&\sum_{j,i} \left(\int_{\Omega} p_\eta(\mathbf{z}) H_k(\mathbf{z}) H_j(\mathbf{z}) G_{il}(\mathbf{z}) d\mathbf{z} \right) \\
&= G^0_{il} \int_{\Omega} p_\eta(\mathbf{z}) H_k(\mathbf{z}) H_j(\mathbf{z}) d\mathbf{z} + \sum_{n=1}^{N} G^n_{il} \int_{\Omega} z_n p_\eta(\mathbf{z}) H_k(\mathbf{z}) H_j(\mathbf{z}) d\mathbf{z} \\
&= \left(G^0_{il} + \sum_{n=1}^{N} c_{kn} G^n_{il} \right) \delta_{kj}
\end{aligned} \tag{175}$$

thus rendering a N decoupled problem

Solution of hyperbolic problems with uncertain parameters

Many authors have worked on this specific problem with different angles of approach. Such is the case of [Lin(1965)], who is one of the first to work on this field. Lin studied the response of linear systems, subjected to aleatory pulse sequences. [Roberts(1965)] used the same approach to solve the problem, and in both cases, a non-stationary noise method, developed by [Parzen(1962)] was used which is quite similar to the one developed by [Rice(1944)]. This same methodology was used by [Wen(1974)] to study the of lightweight equipment joined to structural systems. [Hammond(1968)] studied the response of systems subjected to base excitation by stochastic processes that have a evolutionary power spectral density function with the principal aim of including in the structural response the variation of the frequency content in time.

[Roberts(1965)] used relations in the frequency domain, assuming that the excitation was periodical and non-stationary. He was careful to define a period of time larger than the duration of the duration of the external loading to avoid aliasing in the signal. This method however presents problems for the computation of the response, if the selected period of time is not long enough. [Holman & Hart(1974)] computed the response of linear systems excited by segmented and modulated stationary processes. They considered that the frequency content was invariant in each time segment. [Corotis & Vanmarcke(1975)] proposed a method to compute the response of linear structural systems subjected to sudden white noise.

[Crempien Laborie & Saragoni(1978)] studied the influence of strong ground motion duration in the response of linear structural systems. For this they assumed a time segment of the excitation to be stationary. This proved the necessity to study the response of non-stationary response of structural systems.
[Spanos & Lutes(1980)] also used an evolutionary power spectral density function to compute the response assuming that the response can be modelled as a Markov process. The approach that will be developed now considering that the excitation is of Gaussian nature, both non-stationary in time and in frequency and the structural system is linear.

Response of a simple oscillator

The response of a simple oscillator can be deduced from its ordinary differential equation that is stated as.

$$m\ddot{u}(t) + c\dot{u}(t) + ku(t) = -mF(t) \tag{176}$$

where m is the mass of the oscillator, c is the viscous damping and k is the elastic rigidity of the oscillator. u is the displacement of the oscillator. u is the displacement of the oscillator, considering the location were the oscillator is at rest as the reference state position. The external excitation is considered as a basal acceleration $F(t)$ which later on will be considered as a stochastic process. The problem stated in (176) has an equivalent formulation when normalizing my the mass m

$$\ddot{u}(t) + 2\xi\omega_n\dot{u}(t) + \omega_n^2 u(t) = f(t) \tag{177}$$

In this last equation, $2\xi\omega_n = \frac{c}{m}$ and $\omega_n^2 = \frac{k}{m}$ where ω_n is the circular natural frequency of the system and ξ is the critical damping ratio of the system. With these 2 constants, the dynamical behavior of the system can be characterized.

The response of the system can be written as a Duhammel integral solution, which is simply a convolution integral given by

$$u(t) = \int_{-\infty}^{+\infty} h(t-\tau) f(\tau) d\tau \tag{178}$$

where $h(t)$ is the response of the system to an unitary impulse excitation, that can be considered as a delta Dirac function, thus yielding

$$h(t) = \begin{cases} \frac{1}{\omega_a} e^{-\xi\omega_n t} \sin(\omega_a t) & \text{si} \quad t \geq 0 \\ 0 & \text{si} \quad t < 0 \end{cases} \tag{179}$$

where $\omega_a = \omega_n \sqrt{1-\xi^2}$ If the expected value of the response is taken, then

$$E[u(t)] = \int_{-\infty}^{+\infty} h(t-\tau) E[f(\tau)] d\tau \tag{180}$$

and since the expected value of $E[f(t)] = 0$ is equal to zero, then equation (180) will become

$$E[u(t)] = 0 \tag{181}$$

To compute the autocorrelation of the response, the following can be done

$$R_{uu}(t_1, t_2) = E\left[\int_{-\infty}^{+\infty} h(t_1 - \tau_1) f(\tau_1) d\tau_1 \int_{-\infty}^{+\infty} h(t_2 - \tau_2) f(\tau_2) d\tau_2\right] \tag{182}$$

using the properties of commutativity , then

$$R_{uu}(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(t_1 - \tau_1) h(t_2 - \tau_2) E[f(\tau_1) f(\tau_2)] d\tau_1 d\tau_2 \tag{183}$$

$$R_{uu}(t_1, t_2) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} h(t_1 - \tau_1) h(t_2 - \tau_2) R_{ff}(\tau_1, \tau_2) d\tau_1 d\tau_2 \tag{184}$$

And in this last equation, the autocorrelation of the response has been computed, in function of the autocorrelation of the process.

Response to a stationary process

A stationary stochastic process which has a autocorrelation function that depends only on the difference between two instances of time t_1 and t_2

$$R_{ff}(t_1, t_2) = R_{ff}(t_2 - t_1) \tag{185}$$

which can be related to the power spectral density function (PSD) $S_{ff}(\omega)$ through the Wiener-Kchintchine theorem

$$R_{ff}(t_2 - t_1) = \int_{-\infty}^{+\infty} S_{ff}(\omega) e^{i\omega(t_2 - t_1)} d\omega \tag{186}$$

replacing this last equation in (184) and alternating the order in the integral yields

$$R_{uu}(t_1, t_2) = \int_{-\infty}^{+\infty} S_{ff}(\omega) \int_{-\infty}^{+\infty} h(t_1 - \tau_1) e^{-i\omega\tau_1} d\tau_1 \int_{-\infty}^{+\infty} h(t_2 - \tau_2) e^{i\omega\tau_2} d\tau_2 d\omega \tag{187}$$

and with a proper variable change as $\xi_1 = t_1 - \tau_1$, $\xi_2 = t_2 - \tau_2$, $\tau = t_2 - t_1$, it is possible to obtain

$$R_{uu}(\tau) = \int_{-\infty}^{+\infty} S_{ff}(\omega) e^{-i\omega\tau} \int_{-\infty}^{+\infty} h(\xi_1) e^{-i\omega\xi_1} d\xi_1 \int_{-\infty}^{+\infty} h(\xi_2) e^{i\omega\xi_2} d\xi_2 d\omega \tag{188}$$

and the integral becomes

$$\int_{-\infty}^{+\infty} h(\tau) e^{-i\omega\xi_i} d\xi_i = 2\pi \hat{h}(\omega) \tag{189}$$

where $\hat{h}(\omega)$ is the Fourier transform of $h(t)$, given by

$$\hat{h}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} h(\tau) e^{-i\omega\xi_i} d\xi_i = \frac{1}{2\pi((\omega_n^2 - \omega^2) - 2i\xi\omega_n\omega)} \tag{190}$$

therefore, the autocorrelation function can be written as

$$R_{uu}(\tau) = \int_{-\infty}^{+\infty} S_{ff}(\omega) e^{-i\omega\tau} \hat{h}(\omega) \hat{h}^*(\omega) d\omega \tag{191}$$

If $t_1 = t_2$, then $\tau = 0$. This corresponds to the mean square response

$$R_{uu}(0) = E[u^2(t)] = \int_{-\infty}^{+\infty} \|\hat{h}(\omega)\|^2 S_{ff}(\omega) d\omega \tag{192}$$

If S_{ff} varies slowly with respect to $\|\hat{h}(\omega)\|$, then the system will behave like a narrow band filter, this means that the highest contribution of the excitation of the process will be around the natural frequency ω_n [Caughey & Stumpf(1961)], thus equation (192) will take the following form

$$E[u^2(t)] = S_{ff}(\omega_n) \int_{-\infty}^{+\infty} \|\hat{h}(\omega)\|^2 d\omega \tag{193}$$

[Caughey & Stumpf(1961)], among many, use the residual theorem to solve equation (193), leaving the following equation

$$E[u^2(t)] = \frac{\pi S_{ff}(\omega_n)}{2\xi\omega_n^3} \tag{194}$$

If the mean value is considered to be zero, then equation (194 completes the probabilistical structure of the response in the case of an Gaussian excitation.

Response to a non-stationary separable process

if f is considered to be a non-stationary separable process, then using the property stated in equation (183), the autocorrelation function can be found, which is given by

$$R_{ff}(t_1, t_2) = \psi(t_1)\psi(t_2)R_{ss}(t_1, t_2) \tag{195}$$

now equation (184) can be used to find the autocorrelation of the response

$$R_{uu}(t_1, t_2) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} h(t_1 - \tau_1)h(t_2 - \tau_2)\psi(\tau_1)\psi(\tau_2) R_{ss}(\tau_1, \tau_2)d\tau_1 d\tau_2 \tag{196}$$

Using the Wiener-Kchintchine relationship, the autocorrelation function can be expressed in terms of the autocorrelation of the stationary process, an the latter, can be expressed in terms of the power spectral density function of the stationary process as follows

$$R_{ss}(\tau_1, \tau_2) = \int_{-\infty}^{+\infty} S(\omega) e^{i\omega(\tau_2 - \tau_1)} d\omega \tag{197}$$

making some algebraic arrangements the next expression can be obtained

$$\begin{gathered} R_{uu}(t_1, t_2) = \int_{-\infty}^{+\infty} S(\omega) \int_{-\infty}^{+\infty} h(t_1 - \tau_1)\psi(\tau_1) e^{-i\omega\tau_1} d\tau_1 \\ \int_{-\infty}^{+\infty} h(t_2 - \tau_2)\psi(\tau_2) e^{i\omega\tau_2} d\tau_2 d\omega \end{gathered} \tag{198}$$

Calling non-stationary transfer function to

$$\Upsilon(t,\omega) = \int_{-\infty}^{+\infty} h(t-\tau)\psi(\tau)e^{-i\omega\tau}d\tau \tag{199}$$

and replacing this term in equation (198), it is possible to obtain

$$R_{uu}(t_1,t_2) = \int_{-\infty}^{+\infty} \Upsilon(t_1,\omega)\Upsilon^*(t_2,\omega)S(\omega)d\omega \tag{200}$$

where $\Upsilon^*(t,\omega)$ is the complex conjugate of $\Upsilon(t,\omega)$. In the case that $t_1 = t_2 = t$, then the mean square response will be

$$E[u^2(t)] = \int_{-\infty}^{+\infty} \|\Upsilon(t,\omega)\|^2 S(\omega)d\omega \tag{201}$$

figures

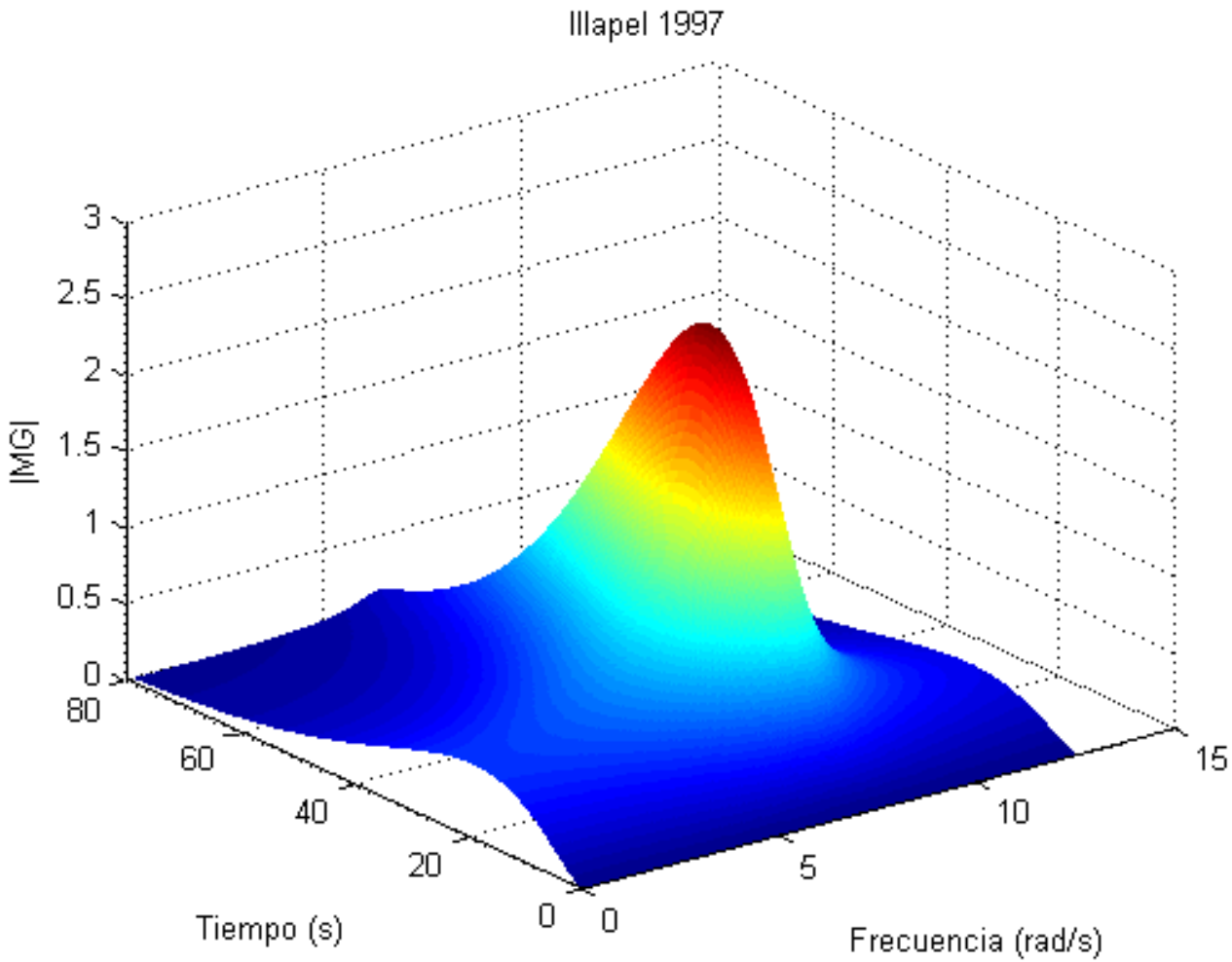


Figure: Nonstationary transfer function.

To obtain the probabilistical structure of the response, it is convenient to have the correlation functions of the displacement-velocity, $E[\dot{u}(t)u(t)]$ and the mean square response of the velocity $E[\dot{u}^2(t)]$. To obtain these functions, the methodology used by [Crempien Laborie & Crempien de la Carrera(2005)] can be used, this is achieved derivating with respect to time equation (201), and developing these expressions, the correlation between u y $\dot{u}$ can be obtained.

$$\frac{\partial}{\partial t}(E[u^2(t)]) = \frac{\partial}{\partial t}\int_{-\infty}^{+\infty} \Upsilon(t,\omega)\Upsilon^*(t,\omega)S(\omega)d\omega \qquad (202)$$

The partial derivative can be taken into the integral

$$\frac{\partial}{\partial t}(E[u^2(t)]) = \int_{-\infty}^{+\infty} \left[\dot{\Upsilon}(t,\omega)\Upsilon^*(t,\omega) + \dot{\Upsilon}^*(t,\omega)\Upsilon(t,\omega)\right] S(\omega)d\omega \qquad (203)$$

The same with the expected operator

$$E[2\dot{u}(t)u(t)] = \int_{-\infty}^{+\infty} \left[\dot{\Upsilon}(t,\omega)\Upsilon^*(t,\omega) + \left(\dot{\Upsilon}(t,\omega)\Upsilon^*(t,\omega)\right)^*\right] S(\omega)d\omega \qquad (204)$$

If the latter expression is further manipulated, the correlation can be reached

$$2E[\dot{u}(t)u(t)] = 2\int_{-\infty}^{+\infty} \mathrm{Real}\left[\dot{\Upsilon}(t,\omega)\Upsilon^*(t,\omega)\right] S(\omega)d\omega \quad (205)$$

$$E[\dot{u}(t)u(t)] = \int_{-\infty}^{+\infty} \mathrm{Real}\left[\dot{\Upsilon}(t,\omega)\Upsilon^*(t,\omega)\right] S(\omega)d\omega \quad (206)$$

If equation (200) is partially derived with respect to any two different instances of time, then

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_{uu}(t_1,t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} \int_{-\infty}^{+\infty} \Upsilon(t_1,\omega)\Upsilon^*(t_2,\omega)S(\omega)d\omega \quad (207)$$

The partial derivatives can enter the integral in the following way

$$\frac{\partial^2}{\partial t_1 \partial t_2} R_{uu}(t_1,t_2) = \int_{-\infty}^{+\infty} \dot{\Upsilon}(t_1,\omega)\dot{\Upsilon}^*(t_2,\omega)S(\omega)d\omega \quad (208)$$

If $t_1 = t_2 = t$, then the mean square of the velocity will be obtained

$$E[\dot{u}^2(t)] = \int_{-\infty}^{+\infty} \|\dot{\Upsilon}(t,\omega)\|^2 S(\omega)d\omega \quad (209)$$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