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A THREE-DIMENSIONAL FINITE-DEFORMATION SMALL-STRAIN BEAM MODEL: FROM A CONSISTENT PRINCIPLE OF VIRTUAL WORK TO THE DEVELOPMENT OF A COMPUTATIONAL FINITE ELEMENT APPROACH

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ANNO ACCADEMICO 2005-2006

Contents

Notation and some preliminaries

In this brief section we give some notation and introduce quantities and properties that will be often used in the thesis.

Some algebraic structures.

- Group: it is called *group* a set G provided with an operation, $\cdot$, which satisfies the following properties
  - $\forall\, \mathbf{g}_1, \mathbf{g}_2 \in G, \quad \mathbf{g}_1 \cdot \mathbf{g}_2 = \mathbf{g}_3$ with $\mathbf{g}_3 \in G$, i.e. $\cdot$ is an internal operation;
  - $\forall\, \mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3 \in G, \quad (\mathbf{g}_1 \cdot \mathbf{g}_2) \cdot \mathbf{g}_3 = \mathbf{g}_1 \cdot (\mathbf{g}_2 \cdot \mathbf{g}_3)$, i.e. $\cdot$ is associative;
  - $\exists\, \mathbf{u} \in G \;\mid\; \mathbf{u} \cdot \mathbf{g} = \mathbf{g} \cdot \mathbf{u} = \mathbf{g}, \;\forall\, \mathbf{g} \in G$, i.e. there exists a neuter element $\mathbf{u}$;
  - $\forall\, \mathbf{g} \in G, \;\exists\, \mathbf{g}^{-1} \in G \;\mid\; \mathbf{g} \cdot \mathbf{g}^{-1} = \mathbf{g}^{-1} \cdot \mathbf{g} = \mathbf{u}$, i.e. there exists an inverse element $\mathbf{g}^{-1}$ for every $\mathbf{g}$.
- Linear space: given a scalar field K, it is called *linear space on K* a set V provided with the following properties
  - there exists in V an internal operation $+$ such that
    - $\forall\, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in V, \quad (\mathbf{v}_1 + \mathbf{v}_2) + \mathbf{v}_3 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_3)$;
    - $\exists\, \mathbf{0} \in V \;\mid\; \mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}, \;\forall\, \mathbf{v} \in V$;
    - $\forall\, \mathbf{v} \in V, \;\exists\, -\mathbf{v} \in V \;\mid\; \mathbf{v} + (-\mathbf{v}) = -\mathbf{v} + \mathbf{v} = \mathbf{0}$;
    - $\forall\, \mathbf{v}_1, \mathbf{v}_2 \in V, \;\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$.
  - there exists an external operation called *scalar multiplication* which associates to every couple $(\lambda, \mathbf{v})$, with $\lambda \in K$ and $\mathbf{v} \in V$, an element of V, indicated with $\lambda\mathbf{v}$ and called *multiple* of $\mathbf{v}$ through λ, and such that the following properties hold $\forall\, \mathbf{v}, \mathbf{w} \in V, \quad \forall\, \lambda, \mu \in K$
    - $\lambda(\mathbf{v} + \mathbf{w}) = \lambda\mathbf{v} + \lambda\mathbf{w}$;
    - $(\lambda + \mu)\mathbf{v} = \lambda\mathbf{v} + \mu\mathbf{v}$; $(\lambda\mu)\mathbf{v} = \lambda(\mu\mathbf{v})$.
  - indicated by 1 the multiplicative unit of K, it holds $1\mathbf{v} = \mathbf{v}, \;\forall\, \mathbf{v} \in V$

Mathematical sets often used into the thesis.

- $\mathcal{R}^3$: linear space of three-dimensional vectors
- $\mathcal{G}^{orth+}$: multiplicative group of rotation tensors (nonlinear manifold)
- $so(3)$: linear space of $[3 \times 3]$ skew-symmetric tensors. A tensor $\mathbf{\Theta}$ is skew-symmetric if

  $$\mathbf{\Theta}^T = -\mathbf{\Theta}.$$

  We recall that

- $so(3)$ is isomorphic with $\mathcal{R}^3$, i.e. there exists a one-one correspondence between elements of $so(3)$ and elements of $\mathcal{R}^3$: given a tensor $\mathbf{\Theta} \in so(3)$ there exists a unique vector $\boldsymbol{\theta} \in \mathcal{R}^3$ and viceversa such that

$$\mathbf{\Theta a} = \boldsymbol{\theta} \times \mathbf{a} \qquad \forall\, \mathbf{a} \in \mathcal{R}^3 \tag{0.0.1}$$

  with

$$\mathbf{\Theta} = \begin{bmatrix} 0 & -\theta_3 & \theta_2 \\ \theta_3 & 0 & -\theta_1 \\ -\theta_2 & \theta_1 & 0 \end{bmatrix}, \boldsymbol{\theta} = \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix}.$$

  $\boldsymbol{\theta}$ is called *axial vector* of $\mathbf{\Theta}$ and we will refer to the latter relation as *axial vector relation*. Often we will use the notation $[\boldsymbol{\theta}\times]$ to indicate the skew tensor $\mathbf{\Theta}$ in order to emphasize its axial vector.

- For a skew-tensor $\mathbf{\Theta}$ and its axial vector $\boldsymbol{\theta}$ the following identity holds

$$\mathbf{\Theta}^2\mathbf{b} = \mathbf{\Theta}(\mathbf{\Theta b}) = [\boldsymbol{\theta} \otimes \boldsymbol{\theta} - \theta^2\mathbf{I}]\mathbf{b} \qquad \forall\, \mathbf{b} \in \mathcal{R}^3, \tag{0.0.2}$$

  where $\theta = \parallel \boldsymbol{\theta} \parallel$.

- Given two skew tensors $\mathbf{\Theta}$ and $\mathbf{W}$ and their axial vectors respectively $\boldsymbol{\theta}$ and $\mathbf{w}$ the following identity holds

$$[\mathbf{\Theta W} - \mathbf{W\Theta}]\mathbf{a} = (\boldsymbol{\theta} \times \mathbf{w}) \times \mathbf{a} \qquad \forall\, \mathbf{a} \in \mathcal{R}^3. \tag{0.0.3}$$

  The term in brackets is called *Lie brackets*.

- $T_{\mathbf{\Lambda}}\mathcal{G}^{orth+}$: tangent space to the group of rotation tensors at the point $\mathbf{\Lambda}$. If $\mathbf{\Lambda} = \mathbf{I}$, where $\mathbf{I}$ is the identity tensor, then the notation become $T_{\mathbf{I}}\mathcal{G}^{orth+}$. It can be proved that the linear space of skew-symmetric tensors is the tangent space to the rotation manifold at any point, i.e.

$$T_{\mathbf{\Lambda}}\mathcal{G}^{orth+} \equiv so(3) \qquad \forall\, \mathbf{\Lambda} \in \mathcal{G}^{orth+}.$$

Index notation. The following indices will be used

- Latin indices, i, j, k..., which range from 1 to 3
- Greek indices, α, β..., which can take values 1 and 2

Some properties of vector operations. The following properties of cross and scalar product will be often used. Given $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathcal{R}^3$

- cross product between parallel vectors $\mathbf{a} \times (\alpha\mathbf{a}) = \mathbf{0}$;
- cross product anticommutativity: $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$;
- mixed product identities $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})$;
- double cross product identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.

Given also a rotation tensor $\mathbf{\Lambda} \in \mathcal{G}^{orth+}$:

- invariance of scalar product under rotation

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{\Lambda a}\cdot\mathbf{\Lambda b}; \tag{0.0.4}$$

- distributivity of cross product with respect to product with a rotation

$$\mathbf{\Lambda}(\mathbf{a}\times\mathbf{b}) = \mathbf{\Lambda a}\times\mathbf{\Lambda b}. \tag{0.0.5}$$

Orthonormality conditions for 3 vectors: given a triad of vectors $\mathbf{t}_i \in \mathcal{R}^3$, they form an orthonormal triad with right-hand-rule if

$$\mathbf{t}_i\cdot\mathbf{t}_j = \delta_{ij}, \quad \mathbf{t}_3 = \mathbf{t}_1\times\mathbf{t}_2,$$

where δ_{ij} is the Kroneker delta.

Derivative of an exponential mapping. Given an exponential mapping, the expression of its derivative is obtained studying the solution of an ordinary differential equation (for more details see [9] page 227). Consider a tensor $\mathbf{A}$ and the initial-value problem

$$\begin{aligned}\dot{\mathbf{X}}(t) &= \mathbf{AX}(t) \quad t>0,\\ \mathbf{X}(0) &= \mathbf{I},\end{aligned}$$

for a tensor function $\mathbf{X}(t), 0 \leqslant t < \infty$. The solution of the problem is

$$\mathbf{X}(t) = \exp[t\mathbf{A}] \quad \longrightarrow \quad \dot{\overline{\exp[t\mathbf{A}]}} = \mathbf{A}\exp[t\mathbf{A}].$$

If $\mathbf{A}$ is a skew tensor, then $\exp[t\mathbf{A}]$ is a rotation $\forall\ t>0$

Notation and some rules in tensor analysis. Given a smooth *material* tensor field $\mathbf{H}(\mathbf{X},t)$, its material gradient $\nabla_{\mathbf{X}}\mathbf{H}$ and material divergence $\mathrm{div}_{\mathbf{X}}\mathbf{H}$ are defined as

$$\nabla_{\mathbf{X}}\mathbf{H} = \frac{\partial\mathbf{H}(\mathbf{X},t)}{\partial\mathbf{X}}, \qquad \mathrm{div}_{\mathbf{X}}\mathbf{H} = \mathrm{tr}\big[\nabla_{\mathbf{X}}\mathbf{H}\big].$$

Given a smooth *spatial* tensor field $\mathbf{G}(\boldsymbol{x},t)$, its spatial gradient $\nabla\mathbf{G}$ and spatial divergence $\mathrm{div}\mathbf{G}$ are defined as

$$\nabla\mathbf{G} = \frac{\partial\mathbf{G}(\boldsymbol{x},t)}{\partial\boldsymbol{x}}, \qquad \mathrm{div}\mathbf{G} = \mathrm{tr}\big[\nabla\mathbf{G}\big].$$

Given three arbitrary tensor, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$, a *symmetric* tensor $\mathbf{A}$ and a vector field $\mathbf{g}$, we will use the following properties

$$\mathbf{D}:(\mathbf{BC}) = (\mathbf{B}^T\mathbf{D}):\mathbf{C} = (\mathbf{DC}^T):\mathbf{B}; \tag{0.0.6}$$

$$\mathrm{div}(\mathbf{D}^T\mathbf{g}) = \mathbf{D}:\nabla\mathbf{g} + \mathbf{g}\cdot\mathrm{div}\mathbf{D}; \tag{0.0.7}$$

$$\mathbf{A}:\mathbf{B} = \mathbf{A}:\mathrm{sym}[\mathbf{B}]. \tag{0.0.8}$$

Introduction

In this work we present a three-dimensional elastic beam theory capable of describing the deformation and the static equilibrium of a beam with no restrictions on either displacements or rotations and taking into account shear strain. A small-strain hypothesis is introduced. The model was first proposed by Reissner in [18, 19]. Subsequently it was revised by Simo in [23, 24, 25] and Simo first introduced the still used terminology *geometrically exact beam* to indicate the model. These works are considered the pioneering ones on the subject. After them a wide literature have been produced, especially addressed to the investigation of computational aspects.

Our work moves from the Simo's ones and proposes new developments for the beam model following two main goals. The first one is the recovering of the one-dimensional weak form of equilibrium equations proposed by Simo in [23] exploiting the *three-dimensional continuum principle of virtual work*. The second one is the illustration of two different forms of equilibrium equations related with two different linearization procedures for *rotation tensors*, the tensors used to describe the finite rotations of beam cross-sections in the three-dimensional space.

In order to achieve the goals we first investigate the three-dimensional principle of virtual work (chapter one) and then we propose "an excursion into finite rotation", to cite Argyris [1] (chapter two). Both topics are quite complex: the first one demand to manage with three-dimensional nonlinear stress and strain measures, the second one with concepts of differential geometry. Chapter three is devoted to develop explicitly the proposed goals and hence the various forms of finite-deformation small-strain model equations are obtained. In presentation of the equations attention is posed on the introduction of a special polar decomposition for the beam deformation gradient and on the explanation of the small-strain hypothesis role into the model.

A third and last goal of the work is the presentation of the *Finite Element formulation* developed to solve the model and subsequently implemented. Some tests are presented in order to state the reliability of the finite element.

Have a good reading.

Chapter 1

Three-dimensional equilibrium equations in non-linear continuum mechanics

The chapter opens with a brief section which introduces the definition of reference and current configurations and their associated three-dimensional stress measures. Subsequently we provide strong form of equilibrium equations for a three-dimensional deformable continuum body using the various stress measures just introduced. From the strong form we finally obtain the principle of virtual work, both in reference and current configuration.

1.1 Stress measures

Consider a continuum body $\mathcal{B}$ occupying currently, at time t, an arbitrary region Ω with boundary surface $\partial\Omega$ and, at time $t = 0$, the region Ω_0 with boundary surface $\partial\Omega_0$. We refer to Ω as current configuration and to Ω_0 as reference configuration. The position of body material points with respect to a fixed reference system are denoted by the vector field $\boldsymbol{x}$ in the current configuration Ω and by the vector field $\mathbf{X}$ in the reference configuration Ω_0. Consider the actual infinitesimal force, $\mathrm{d}\mathbf{f}$, acting in the current configuration Ω on an infinitesimal surface plane element, $\mathrm{d}a$, internal to the body with normal unit vector $\mathbf{n}$ and located at point $\boldsymbol{x}$, see figure (1.1). $\mathrm{d}a$, $\mathbf{n}$ and $\boldsymbol{x}$ are denoted respectively by $\mathrm{d}a_0$, $\mathbf{n}_0$ and $\mathbf{X}$ when considered in the reference configuration Ω_0. The force $\mathrm{d}\mathbf{f}$ is given by

$$\boxed{\mathrm{d}\mathbf{f} = \mathbf{t}_{\mathbf{n}}\mathrm{d}a = \mathbf{t}_{\mathbf{n}_0}\mathrm{d}a_0}, \tag{1.1.1}$$

where, for the Cauchy's stress theorem,

$$\mathbf{t}_{\mathbf{n}}(\boldsymbol{x}, \mathbf{n}, t) = \boldsymbol{\sigma}(\boldsymbol{x}, t)\mathbf{n} \quad \text{and} \quad \mathbf{t}_{\mathbf{n}_0}(\mathbf{X}, \mathbf{n}_0, t) = \mathbf{P}(\mathbf{X}, t)\mathbf{n}_0. \tag{1.1.2}$$

$\boldsymbol{\sigma}$ is the *Cauchy stress tensor* and $\mathbf{P}$ is the *first Piola-Kirchhoff stress tensor*. The former linearly maps the current unit area vector $\mathbf{n}\mathrm{d}a$ into the current infinitesimal force $\mathrm{d}\mathbf{f}$

while the latter linearly maps the reference unit area vector $\mathbf{n}_0 \mathrm{d}a_0$ again into the current infinitesimal force $\mathrm{d}\mathbf{f}$. $\boldsymbol{\sigma}$ is defined in the current configuration and it is also called *true stress* tensor since it is the physical stress of the true-current configuration. $\mathbf{P}$ instead is a *two-point tensor* since maps a vector defined in the reference configuration into a vector defined in the current configuration. Substituting the Cauchy's theorem into equation (1.1.1) we get

$$\boxed{\mathrm{d}\mathbf{f} = \boldsymbol{\sigma}\mathbf{n}\mathrm{d}a = \mathbf{P}\mathbf{n}_0\mathrm{d}a_0}. \tag{1.1.3}$$

Consider the relation which maps the reference area vector $\mathbf{n}_0 \mathrm{d}a_0$ into the current one $\mathbf{n}\mathrm{d}a$ (Nanson's formula, see [10] page 75)

$$\mathbf{n}\mathrm{d}a = J\mathbf{F}^{-T}\mathbf{n}_0\mathrm{d}a_0$$

where $\mathbf{F}$ is the deformation gradient and $J = \det\mathbf{F}$. Substituting the above equation into (1.1.3) we obtain the relation between Cauchy and First Piola-Kirchhoff stress tensors

$$\mathbf{P} = J\boldsymbol{\sigma}\mathbf{F}^{-T} \tag{1.1.4}$$

and

$$\boldsymbol{\sigma} = J^{-1}\mathbf{P}\mathbf{F}^{T}. \tag{1.1.5}$$

Moreover, balance of angular momentum implies Cauchy tensor to be symmetric

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \tag{1.1.6}$$

and hence it follows from (1.1.5) that

$$J^{-1}\mathbf{P}\mathbf{F}^T = J^{-1}\left(\mathbf{P}\mathbf{F}^T\right)^T = J^{-1}\mathbf{F}\mathbf{P}^T \quad \Rightarrow \quad \mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T. \tag{1.1.7}$$

Consequently, $\mathbf{P}$ is, in general, not symmetric.

Consider the infinitesimal force $\mathrm{d}\mathbf{f}_0 = \mathbf{F}^{-1}\mathrm{d}\mathbf{f}$, which is the current infinitesimal force $\mathrm{d}\mathbf{f}$ mapped back to the reference configuration. From this quantity the **second Piola-Kirchhoff stress tensor S** is defined as

$$\boxed{\mathrm{d}\mathbf{f}_0 = \mathbf{S}\mathbf{n}_0\mathrm{d}a_0}. \tag{1.1.8}$$

Equation (1.1.8) states that $\mathbf{S}$ maps the reference unit area vector $\mathbf{n}_0\mathrm{d}a_0$ into the reference infinitesimal force $\mathrm{d}\mathbf{f}_0$. It can be shown that $\mathbf{S}$ is a symmetric tensor (see [10] page 127)

$$\mathbf{S} = \mathbf{S}^T. \tag{1.1.9}$$

The relation between $\mathbf{P}$ and $\mathbf{S}$ is obtained substituting the definition of $\mathrm{d}\mathbf{f}_0$ into equation (1.1.8) and multiplying with $\mathbf{F}$

$$\mathbf{F}\mathbf{F}^{-1}\mathrm{d}\mathbf{f} = \mathbf{F}\mathbf{S}\mathbf{n}_0\mathrm{d}a_0 \quad \Rightarrow \quad \mathrm{d}\mathbf{f} = \mathbf{F}\mathbf{S}\mathbf{n}_0\mathrm{d}a_0.$$

Comparing the above relation with $(1.1.3)_2$ we get

$$\mathbf{P} = \mathbf{F}\mathbf{S}. \tag{1.1.10}$$

Alternative forms of stress tensor are not considered in this thesis. For further information see [10] and [22].

Box1: Definition of stress tensors

- Cauchy's stress tensor: $\boldsymbol{\sigma}\mathbf{n}da = \mathrm{d}\mathbf{f}$
- First Piola-Kirchhoff stress tensor: $\mathbf{P}\mathbf{n}_0 da_0 = \mathrm{d}\mathbf{f}$
- Second Piola-Kirchhoff stress tensor: $\mathbf{S}\mathbf{n}_0 da_0 = \mathrm{d}\mathbf{f}_0$ where $\mathrm{d}\mathbf{f}_0 = \mathbf{F}^{-1}\mathrm{d}\mathbf{f}$

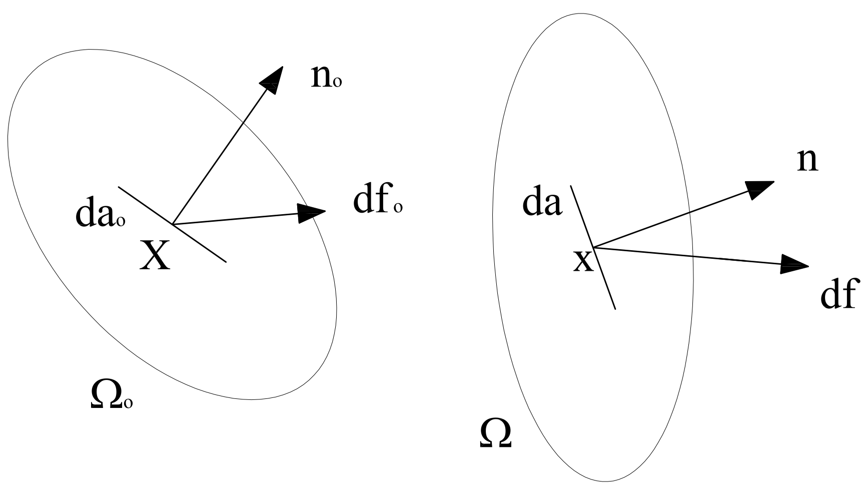


Figure 1.1: Reference configuration (left). Current configuration (right).

1.2 Strong (differential) equilibrium equations

In this section we present translational and rotational differential equilibrium equations for the three-dimensional continuum body $\mathcal{B}$ in the static regime. These equations are clearly valued on the real current configuration Ω and involve d$\mathbf{f}$, but since the differential infinitesimal current force d$\mathbf{f}$ can be expressed in term of both the *true* Cauchy's tensor $\boldsymbol{\sigma}$ and the *two-point* first Piola-Kirchhoff tensor $\mathbf{P}$, they can be formulated in two ways, one for each stress measure adopted.

Strong equilibrium with the *true* Cauchy's stress $\boldsymbol{\sigma}$

Consider a generical part $\mathcal{P}$ of the current region Ω, with boundary region $\partial\mathcal{P}$. The force resultant, $\mathbf{r}$, and the moment resultant, $\mathbf{m}$, relative to $\mathcal{P} \subset \Omega$ can be defined as

$$\mathbf{f} = \int_{\mathcal{P}} \mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{t}_{\mathbf{n}}\,\mathrm{d}a, \tag{1.2.1}$$

$$\mathbf{m} = \int_{\mathcal{P}} \mathbf{r}\times\mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{r}\times\mathbf{t}_{\mathbf{n}}\,\mathrm{d}a. \tag{1.2.2}$$

where

- $\mathbf{b} = \mathbf{b}(\boldsymbol{x})$ is the vector field of body force per unit *current* volume;
- $\mathbf{r} = \boldsymbol{x} - \boldsymbol{x}_o$ is the position vector computed with respect to a generic momentum pole o in $\boldsymbol{x}_o$;
- $\mathbf{t}_{\mathbf{n}}$ is the traction vector introduced in the previous section;
- $dv \subset \mathcal{P}$ is the current infinitesimal volume;
- $da \subset \partial\mathcal{P}$ is the current infinitesimal area.

The static equilibrium axiom postulates that a deformable body is in equilibrium if and only if the force resultant and the momentum resultant are zero on each portion of the body, i.e. a body $\mathcal{B}$ in a configuration Ω is in equilibrium if and only if

$$\int_{\mathcal{P}} \mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{t}_{\mathbf{n}}\,\mathrm{d}a = \mathbf{0} \qquad \forall\,\mathcal{P}\subset\Omega, \tag{1.2.3}$$

$$\int_{\mathcal{P}} \mathbf{r}\times\mathbf{b}\,\mathrm{d}v + \int_{\partial\mathcal{P}} \mathbf{r}\times\mathbf{t}_{\mathbf{n}}\,\mathrm{d}a = \mathbf{0} \qquad \forall\,\mathcal{P}\subset\Omega. \tag{1.2.4}$$

The equations are respectively the specialization of linear and angular momentum balance laws to the static regime and are known respectively as *translational equilibrium* and *rotational equilibrium* equations. Using the Cauchy theorem $(1.1.2)_1$ and the divergence theorem, the surface integral in the translational equilibrium (1.2.3) can be given as

$$\int_{\partial\mathcal{P}} \mathbf{t}_{\mathbf{n}}\,\mathrm{d}a = \int_{\partial\mathcal{P}} \boldsymbol{\sigma}\mathbf{n}\,\mathrm{d}a = \int_{\mathcal{P}} \mathrm{div}\boldsymbol{\sigma}\,\mathrm{d}v$$

where the operator "div(·)" is the spatial divergence operator (see section). By substituting this result into the translational equilibrium (1.2.3) we obtain

$$\int_{\mathcal{P}} \mathrm{div}\boldsymbol{\sigma} + \mathbf{b}\,\mathrm{d}v = \mathbf{0} \qquad \forall\, \mathcal{P} \subset \Omega$$

which, holding for any part $\mathcal{P}$ of the region Ω, yields the *Cauchy's equation of equilibrium*

$$\boxed{\mathrm{div}\boldsymbol{\sigma} + \mathbf{b} = \mathbf{0}} \tag{1.2.5}$$

Note that both $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{x})$ and $\mathbf{b} = \mathbf{b}(\boldsymbol{x})$, as well as the "div(·)" operator, are all defined with respect to the current configuration.

From the rotational equilibrium (1.2.4), manipulated using again the Cauchy's theorem and the divergence theorem, it can be shown[1] that

$$\boxed{\boldsymbol{\sigma} = \boldsymbol{\sigma}^T} \tag{1.2.6}$$

Strong equilibrium with the *two-point* first Piola-Kirchhoff stress P

The translational and rotational equilibriums can be expressed as integral over the region $\mathcal{P}_0$, subset of the reference region Ω_0, with boundary region $\partial\mathcal{P}_0$. With this task, we write the force resultant $\mathbf{f}$ and the moment resultant $\mathbf{m}$ with respect to the reference configuration Ω_0. From the relation between the current and reference volume, $\mathrm{d}v = J\mathrm{d}v_0$, the integral of body forces can be given as

$$\int_{\mathcal{P}} \mathbf{b}(\boldsymbol{x})\mathrm{d}v = \int_{\mathcal{P}_0} \mathbf{b}_0(\mathbf{X})\mathrm{d}v_0, \qquad \text{with} \quad \mathbf{b}_0 = J\mathbf{b},$$

where $\mathbf{b}_0$ are called reference body forces and $J = \det\mathbf{F}$. From the equivalence of infinitesimal forces $\mathrm{d}\mathbf{f}$, (1.1.1), the surface integral of the traction forces can be given as

$$\int_{\partial\mathcal{P}} \mathbf{t}_{\mathbf{n}}(\boldsymbol{x})\,\mathrm{d}a = \int_{\partial\mathcal{P}_0} \mathbf{t}_{\mathbf{n}_0}(\mathbf{X})\,\mathrm{d}a_0.$$

With these equations in hand, $\mathbf{f}$ and $\mathbf{m}$ take the form

$$\mathbf{f} = \int_{\mathcal{P}_0} \mathbf{b}_0\,\mathrm{d}v_0 + \int_{\partial\mathcal{P}_0} \mathbf{t}_{\mathbf{n}_0}\,\mathrm{d}a_0, \tag{1.2.7}$$

$$\mathbf{m} = \int_{\mathcal{P}_0} \mathbf{r} \times \mathbf{b}_0\,\mathrm{d}v_0 + \int_{\partial\mathcal{P}_0} \mathbf{r} \times \mathbf{t}_{\mathbf{n}_0}\,\mathrm{d}a_0. \tag{1.2.8}$$

It must be emphasized that force and moment resultants even thought described with respect to the reference configuration are still valued in the current configuration. Note that

[1]for this demonstration see [10] page 147

the position vector $\mathbf{r}$ has not been affected by any transformation. The static equilibrium conditions with respect to the reference configuration takes hence the form

$$\int_{\mathcal{P}_0} \mathbf{b}_0 \,\mathrm{d}v_0 + \int_{\partial\mathcal{P}_0} \mathbf{t}_{\mathbf{n}_0} \,\mathrm{d}a_0 = \mathbf{0} \qquad \forall\, \mathcal{P}_0 \subset \Omega_0, \tag{1.2.9}$$

$$\int_{\mathcal{P}_0} \mathbf{r} \times \mathbf{b}_0 \,\mathrm{d}v_0 + \int_{\partial\mathcal{P}_0} \mathbf{r} \times \mathbf{t}_{\mathbf{n}_0} \,\mathrm{d}a_0 = \mathbf{0} \qquad \forall\, \mathcal{P}_0 \subset \Omega_0. \tag{1.2.10}$$

Using the Cauchy theorem (1.1.2) and the divergence theorem, the surface integral in (1.2.9) can be given as

$$\int_{\partial\mathcal{P}_0} \mathbf{t}_{\mathbf{n}_0} \,\mathrm{d}a_0 = \int_{\partial\mathcal{P}_0} \mathbf{P}\mathbf{n}_0 \,\mathrm{d}a_0 = \int_{\mathcal{P}_0} \mathrm{Div}\mathbf{P} \,\mathrm{d}v_0$$

where the operator “Div” is the material divergence operator (see section ()). By substituting this result into the translational equilibrium (1.2.9), we obtain

$$\int_{\mathcal{P}_0} \mathrm{Div}\mathbf{P} + \mathbf{b}_0 \,\mathrm{d}v_0 = \mathbf{0} \qquad \forall\, \mathcal{P}_0 \subset \Omega_0$$

which, holding for any part $\mathcal{P}_0$ of the region Ω_0, yields the *reference form of Cauchy's equation of equilibrium*

$$\boxed{\mathrm{Div}\mathbf{P} + \mathbf{b}_0 = \mathbf{0}.} \tag{1.2.11}$$

From the rotational equilibrium (1.2.10), manipulated using again the Cauchy's theorem and the divergence theorem, it can be shown that

$$\boxed{\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T.} \tag{1.2.12}$$

1.3 Principle of virtual work

In order to develop the principle of virtual work, we introduced the displacement field $\mathbf{u}$ defined as the difference between the position vector field $\boldsymbol{x}$ individuating each point of the current region Ω and the position vector field $\mathbf{X}$ individuating each point of the reference region Ω_0

$$\mathbf{u} = \boldsymbol{x} - \mathbf{X}. \tag{1.3.1}$$

We define the virtual variation of the displacement field $\delta\mathbf{u}$ as *infinitesimal arbitrary and virtual* change of the displacement field $\mathbf{u}$ at fixed time t

$$\delta\mathbf{u} = \mathbf{u}_\varepsilon - \mathbf{u} \tag{1.3.2}$$

where $\mathbf{u}_\varepsilon$ stands for an infinitesimal perturbed displacement field. The virtual variation is defined such that

$$\delta\mathbf{u}(\boldsymbol{x}) \in \delta\mathcal{U} \qquad \text{where} \qquad \delta\mathcal{U} = \{\delta\mathbf{u}(\boldsymbol{x}) : \delta\mathbf{u}(\boldsymbol{x}) \in C^0, \quad \delta\mathbf{u} = \mathbf{0} \quad \text{on} \quad \partial\Omega_{\bar{\mathbf{u}}}\}, \tag{1.3.3}$$

where $\partial\Omega_{\bar{\mathbf{u}}} \subset \partial\Omega$ is the boundary region where displacements are assigned. With this definition in hand, we can enter into the study of the principle of virtual work. Also this kind of equilibrium equations can be given in two forms, depending on which is the domain of integration: the current configuration Ω or the reference configuration Ω_0. In the next paragraphs we derive both of them. We recall that we deal with static equilibrium equations, i.e. we do not consider acceleration term $\ddot{\mathbf{u}}$, as done for the differential case.

1.3.1 Principle of virtual work on current configuration Ω

In this paragraph we consider the weak (or integral) form of the Cauchy's equilibrium equation which leads to the formulation of *principle of virtual work* written with respect to the *current configuration* Ω.

Consider the virtual displacement $\delta\mathbf{u}(\boldsymbol{x})$ as an arbitrary weighting vector function. Multiplying the differential Cauchy's equilibrium equation (1.2.5) by the weighting function $\delta\mathbf{u}$ and integrating over the current domain we obtain

$$\int_\Omega \Big[(\mathrm{div}\boldsymbol{\sigma} + \mathbf{b}) \cdot \delta\mathbf{u}\Big] dv = 0 \qquad \forall\, \delta\mathbf{u} \in \delta\mathfrak{U}. \tag{1.3.4}$$

which is the *integral scalar-valued weak equilibrium* equation written on the current configuration. The *fundamental lemma of calculus of variations* guaranties this weak equation to be equal to the strong one (for further details see [30]). Splitting the integral, the previous equation is written as

$$\int_\Omega \mathrm{div}\boldsymbol{\sigma} \cdot \delta\mathbf{u}\, dv + \int_\Omega \mathbf{b} \cdot \delta\mathbf{u}\, dv = 0 \qquad \forall\, \delta\mathbf{u} \in \delta\mathfrak{U}. \tag{1.3.5}$$

Consider the first integral $\int_\Omega \mathrm{div}\boldsymbol{\sigma} \cdot \delta\mathbf{u}\, dv$. The scalar product between the divergence of a tensor and a vector can be expressed in term of the vector gradient by the rule (0.0.7). Hence in our case we can write the equality

$$\mathrm{div}\boldsymbol{\sigma} \cdot \delta\mathbf{u} = \mathrm{div}(\boldsymbol{\sigma}^T \delta\mathbf{u}) - \boldsymbol{\sigma} : \nabla\delta\mathbf{u}.$$

Since $\boldsymbol{\sigma}$ is a symmetric tensor ($\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$), trivially we have

$$\mathrm{div}\boldsymbol{\sigma} \cdot \delta\mathbf{u} = \mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{u}) - \boldsymbol{\sigma} : \nabla\delta\mathbf{u}$$

and therefore the considered integral becomes

$$\int_\Omega \mathrm{div}\boldsymbol{\sigma} \cdot \delta\mathbf{u}\, dv = \int_\Omega \mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{u})\, dv - \int_\Omega \boldsymbol{\sigma} : \nabla\delta\mathbf{u}\, dv. \tag{1.3.6}$$

In order to rearrange this expression, we examine first the term $\int_\Omega \boldsymbol{\sigma} : \nabla\delta\mathbf{u}\, dv$. Since $\boldsymbol{\sigma}$ is symmetric, we can use the rule of double contraction between a tensor and a symmetric tensor (0.0.8) to rewrite

$$\int_\Omega \boldsymbol{\sigma} : \nabla\delta\mathbf{u}\, dv = \int_\Omega \boldsymbol{\sigma} : \mathrm{sym}\big[\nabla\delta\mathbf{u}\big]\, dv = \int_\Omega \boldsymbol{\sigma} : \delta\mathbf{e}\, dv, \tag{1.3.7}$$

where we recognized in $\mathrm{sym}\big[\nabla\delta\mathbf{u}\big]$ the *Euler-Almansi* strain tensor's virtual variation $\delta\mathbf{e}$[2]. Consider now the other term $\int_\Omega \mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{u})\, dv$. Applying first the divergence theorem and then the symmetry of $\boldsymbol{\sigma}$, this integral can be given as

$$\int_\Omega \mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{u}) dv = \int_{\partial\Omega} \mathbf{n}\cdot\boldsymbol{\sigma}\delta\mathbf{u}\, da = \int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\cdot\delta\mathbf{u}\, da.$$

Since $\delta\mathbf{u}$ vanishes on the boundary region $\partial\Omega_{\bar{\mathbf{u}}}$ where displacements are assigned, the integral over the whole boundary region reduces to an integral over $\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}$, i.e. the region where traction are assigned,

$$\int_{\partial\Omega} \boldsymbol{\sigma}\mathbf{n}\cdot\delta\mathbf{u}\, da = \int_{\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}} \bar{\mathbf{t}}_\mathbf{n}\cdot\delta\mathbf{u}\, da.$$

Therefore by substitution into previous equation we get

$$\int_\Omega \mathrm{div}(\boldsymbol{\sigma}\delta\mathbf{u}) dv = \int_{\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}} \bar{\mathbf{t}}_\mathbf{n}\cdot\delta\mathbf{u}\, da. \tag{1.3.8}$$

Recollecting results of equations (1.3.7) and (1.3.8) and substituting them into (1.3.6) the expression of the term $\int_\Omega \mathrm{div}\boldsymbol{\sigma}\cdot\delta\mathbf{u}\, dv$ takes the form

$$\int_\Omega \mathrm{div}\boldsymbol{\sigma}\cdot\delta\mathbf{u}\, dv = \int_{\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}} \bar{\mathbf{t}}_\mathbf{n}\cdot\delta\mathbf{u}\, da - \int_\Omega \boldsymbol{\sigma} : \delta\mathbf{e}\, dv. \tag{1.3.9}$$

Substituting this expression into equation (1.3.5) we obtain

$$\int_{\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}} \bar{\mathbf{t}}_\mathbf{n}\cdot\delta\mathbf{u}\, da - \int_\Omega \boldsymbol{\sigma} : \delta\mathbf{e}\, dv + \int_\Omega \mathbf{b}\cdot\delta\mathbf{u}\, dv = 0 \qquad \forall\, \delta\mathbf{u}\in\delta\mathfrak{U},$$

which changing sign and reordering terms finally becomes the *principle of virtual work* written on *current configuration*

$$\boxed{\int_\Omega \boldsymbol{\sigma} : \delta\mathbf{e}\, dv - \int_\Omega \mathbf{b}\cdot\delta\mathbf{u}\, dv - \int_{\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}} \bar{\mathbf{t}}_\mathbf{n}\cdot\delta\mathbf{u}\, da = 0 \qquad \forall\, \delta\mathbf{u}\in\delta\mathfrak{U},} \tag{1.3.10}$$

with boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}} \qquad \text{on } \partial\Omega_{\bar{\mathbf{u}}} \subset \partial\Omega.$$

The principle of virtual work states that at the equilibrium configuration the virtual work $\boldsymbol{\sigma} : \delta\mathbf{e}$ done by fixed $\boldsymbol{\sigma}$ with the virtual variation $\delta\mathbf{e}$ on the whole volume equals the sum of work done with virtual displacement $\delta\mathbf{u}$ by body forces $\mathbf{b}$ on the whole volume and surface tractions $\bar{\mathbf{t}}_\mathbf{n}$ on the boundary area $\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}$.

Usually the terms of the principle are indicated by the notation

$$\int_\Omega \boldsymbol{\sigma} : \delta\mathbf{e}\, dv = \delta L_{int}; \tag{1.3.11}$$

$$\int_\Omega \mathbf{b}\cdot\delta\mathbf{u}\, dv + \int_{\partial\Omega} \bar{\mathbf{t}}_\mathbf{n}\cdot\delta\mathbf{u}\, da = \delta L_{ext}. \tag{1.3.12}$$

[2]for demonstration that $\mathrm{sym}\big[\nabla\delta\mathbf{u}\big] = \delta\mathbf{e}$ see [10] page 376

where the first integral, δL_{int}, is called *internal virtual work* and second one, δL_{ext}, *external virtual work*. Note that stress, body forces, traction vectors are all defined on *current* region Ω, which clearly is also the integral domain.

Because the traction boundary conditions, $\mathbf{t_n} = \bar{\mathbf{t}}_\mathbf{n}$ on $\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}$, arise directly from the weak equilibrium equation they are often called *natural* boundary conditions. Instead the displacements boundary conditions, $\mathbf{u} = \bar{\mathbf{u}}$ on $\partial\Omega_{\bar{\mathbf{u}}}$, are called *essential* since they have to be imposed out of the weak form.

1.3.2 Principle of virtual work on reference configuration Ω_0

We consider here the principle of virtual work wherein the integral domain is the reference region Ω_0. In this case the internal virtual work δL_{int} can be expressed in two equivalent forms, using the first Piola-Kirchhoff stress tensor $\mathbf{P}$ or the second Piola-Kirchhoff stress tensor $\mathbf{S}$. For this reason we study the internal and external work separately. The computation is developed starting from the principle of virtual work written in current configuration [3](see equation (1.3.10)).

Internal work

Internal virtual work in term of first Piola-Kirchhoff stress tensor P. Consider the internal virtual work in current configuration δL_{int} (1.3.11) and rewrite it using $\boldsymbol{\sigma} : \delta\mathbf{e} = \boldsymbol{\sigma} : \nabla\delta\mathbf{u}$

$$\delta L_{int} = \int_\Omega \boldsymbol{\sigma} : \nabla\delta\mathbf{u}\, dv.$$

Recalling the relations between current and reference differential volume, respectively dv and dv_0, and between spatial and material gradient of $\delta\mathbf{u}$, respectively $\nabla[\delta\mathbf{u}(\boldsymbol{x})]$ and $\nabla_\mathbf{X}[\delta\mathbf{u}(\mathbf{X})]$

$$dv = J dv_0 \qquad \text{and} \qquad \nabla\delta\mathbf{u} = \nabla_\mathbf{X}(\delta\mathbf{u})\mathbf{F}^{-1},$$

where $J = \det\mathbf{F}$, the internal work δL_{int} takes the form

$$\delta L_{int} = \int_{\Omega_0} \left[\boldsymbol{\sigma} : \nabla_\mathbf{X}(\delta\mathbf{u})\mathbf{F}^{-1}J\right] dv_0. \tag{1.3.13}$$

The equation states that the internal virtual work integrated on the current configuration domain can be given as an integral on the reference domain of the function in brackets. Using first the property of double contraction between a tensor and a product of tensors, $(0.0.6)_3$, and then the relations between $\boldsymbol{\sigma}$ and $\mathbf{P}$ ($J\boldsymbol{\sigma}\mathbf{F}^{-T} = \mathbf{P}$) and between $\delta\mathbf{u}$ and $\delta\mathbf{F}$ ($\nabla_\mathbf{X}(\delta\mathbf{u}) = \delta\mathbf{F}$) [4] the function in brackets can be rearranged in the form

$$\begin{aligned} \boldsymbol{\sigma} : \nabla_\mathbf{X}[\delta\mathbf{u}]\mathbf{F}^{-1}J &= J\boldsymbol{\sigma}\mathbf{F}^{-T} : \nabla_\mathbf{X}(\delta\mathbf{u}) \\ &= \mathbf{P} : \delta\mathbf{F}. \end{aligned}$$

[3]The virtual work on reference configuration which uses the first Piola-Kirchhoff tensor $\mathbf{P}$ can be obtained also by weighting with virtual displacements and integrating on the reference domain the strong equilibrium form written with $\mathbf{P}$, see [10] page 384

[4]see [10] page 374 for this proof

By substitution of this expression into (1.3.13), we finally obtain the *internal virtual work* written on *reference configuration* in term of *first Piola-Kirchhoff stress tensor* $\mathbf{P}$,

$$\delta L_{int} = \int_{\Omega_0} \Big[\mathbf{P} : \delta\mathbf{F}\Big] dv_0. \tag{1.3.14}$$

The stress tensor $\mathbf{P}$ turns out to be work conjugate with the virtual variation of deformation gradient $\delta\mathbf{F}$.

Internal virtual work in term of second Piola-Kirchhoff stress tensor, S. Consider the internal work on reference configuration in term of $\mathbf{P}$ (equation (1.3.14)). Using that $\mathbf{P} = \mathbf{FS}$ and the double contraction rule $(0.0.6)_2$ it follows that

$$\mathbf{P} : \delta\mathbf{F} = \mathbf{FS} : \delta\mathbf{F} = \mathbf{S} : \mathbf{F}^T\delta\mathbf{F}.$$

Since $\mathbf{S}$ is symmetric, we can use the rule of double contraction between a tensor and a symmetric tensor (0.0.8) to rewrite

$$\mathbf{S} : \mathbf{F}^T\delta\mathbf{F} = \mathbf{S} : \mathrm{sym}[\mathbf{F}^T\delta\mathbf{F}] = \mathbf{S} : \delta\mathbf{E},$$

where we recognize the Green-Lagrange strain tensor's virtual variation $\delta\mathbf{E} = \mathrm{sym}[\mathbf{F}^T\delta\mathbf{F}]$[5]. Hence by substitution we get the equality

$$\mathbf{P} : \delta\mathbf{F} = \mathbf{S} : \delta\mathbf{E}, \tag{1.3.15}$$

which introduced into equation (1.3.14) finally leads to the *internal virtual work* written on *reference configuration* in term of *second Piola-Kirchhoff stress tensor* $\mathbf{S}$

$$\delta L_{int} = \int_{\Omega_0} \Big[\mathbf{S} : \delta\mathbf{E}\Big] dv_0. \tag{1.3.16}$$

External virtual work

Consider the external virtual work written in current configuration, δL_{ext} (equation (1.3.11)). The term associated with the body force, $\int_\Omega \Big[\mathbf{b}\cdot\delta\mathbf{u}\Big] dv$, can be expressed in term of an integral over the reference domain Ω_0 using again the relation ($dv = J dv_0$) between current and reference differential volume

$$\int_\Omega \mathbf{b}\cdot\delta\mathbf{u}\, dv = \int_{\Omega_0} \mathbf{b}_0\cdot\delta\mathbf{u}\, dv_0 \qquad \text{with} \quad \mathbf{b}_0 = J\mathbf{b}. \tag{1.3.17}$$

To express the boundary work $\int_{\partial\Omega_{\bar{\mathbf{t}}_\mathbf{n}}} \bar{\mathbf{t}}_\mathbf{n}\cdot\delta\mathbf{u}\, da$ as an integral over the reference domain, we recall that by the definition of differential force $\mathrm{d}\mathbf{f}$ (1.1.1) and the Cauchy's theorem we have

$$\mathrm{d}\mathbf{f} = \mathbf{t}_\mathbf{n}\mathrm{d}a = \mathbf{t}_{\mathbf{n}_0}\mathrm{d}a_0, \qquad \text{with } \mathbf{t}_n = \boldsymbol{\sigma}\mathbf{n},\ \mathbf{t}_{n_0} = \mathbf{P}\mathbf{n}_0.$$

[5] see [10] page 375 for this demonstration

Specializing these relations for the boundary region where tractions are assigned, we get

$$\bar{\mathbf{t}}_{\mathbf{n}}\mathrm{d}a = \bar{\mathbf{t}}_{\mathbf{n}_0}\mathrm{d}a_0.$$

and consequentially

$$\int_{\partial\Omega_{\bar{\mathbf{t}}_{\mathbf{n}}}} \bar{\mathbf{t}}_{\mathbf{n}} \cdot \delta\mathbf{u}\, da = \int_{\partial\Omega_{\bar{\mathbf{t}}_{\mathbf{n}_0}}} \bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\, da_0. \tag{1.3.18}$$

Substituting the results of (1.3.17) and (1.3.18) into the expression of the external work (1.3.12), we finally get the *external virtual work* written on *reference configuration*

$$\delta L_{ext} = \int_{\Omega_0} \mathbf{b}_0 \cdot \delta\mathbf{u}\, dv_0 + \int_{\partial\Omega_{\bar{\mathbf{t}}_{\mathbf{n}_0}}} \bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\, da_0. \tag{1.3.19}$$

Observing that displacement boundary conditions must be assigned on the reference configuration in order to be consistent with the virtual work, the two complete forms of the *principle of virtual work* on Ω_0 can finally be given

$$\boxed{\int_{\Omega_0} \mathbf{P} : \delta\mathbf{F}\, dv_0 - \int_{\Omega_0} \mathbf{b}_0 \cdot \delta\mathbf{u}\, dv_0 - \int_{\partial\Omega_{\bar{\mathbf{t}}_{\mathbf{n}0}}} \bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\, da_0 = 0 \qquad \forall\, \delta\mathbf{u} \in \delta\mathcal{U},} \tag{1.3.20}$$

$$\boxed{\int_{\Omega_0} \mathbf{S} : \delta\mathbf{E}\, dv_0 - \int_{\Omega_0} \mathbf{b}_0 \cdot \delta\mathbf{u}\, dv_0 - \int_{\partial\Omega_{\bar{\mathbf{t}}_{\mathbf{n}0}}} \bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\, da_0 = 0 \qquad \forall\, \delta\mathbf{u} \in \delta\mathcal{U},} \tag{1.3.21}$$

with boundary conditions

$$\mathbf{u}_0 = \bar{\mathbf{u}}_0 \qquad \text{on } \partial\Omega_{\bar{\mathbf{u}}_0} \subset \partial\Omega_0.$$

1.3.3 Box resume of principle of virtual work

- Principle of virtual work on current configuration $\mathbf{\Omega}$

$$\int_{\Omega} \boldsymbol{\sigma} : \delta \mathbf{e} \, dv - \int_{\Omega} \mathbf{b} \cdot \delta \mathbf{u} \, dv - \int_{\partial \Omega_{\bar{\mathbf{t}}_{\mathbf{n}}}} \bar{\mathbf{t}}_{\mathbf{n}} \cdot \delta \mathbf{u} \, da = 0 \qquad \forall \, \delta \mathbf{u} \in \delta \mathcal{U};$$

  with boundary conditions

$$\mathbf{u} = \bar{\mathbf{u}} \qquad \text{on } \partial \Omega_{\bar{\mathbf{u}}} \subset \partial \Omega.$$

- Principle of virtual work on reference configuration $\mathbf{\Omega}_0$
  - Internal work using first Piola-Kirchhoff stress tensor $\mathbf{P}$ and virtual variation of deformation gradient $\delta \mathbf{F}$

$$\int_{\Omega_0} \mathbf{P} : \delta \mathbf{F} \, dv_0 - \int_{\Omega_0} \mathbf{b}_0 \cdot \delta \mathbf{u} \, dv_0 - \int_{\partial \Omega_{\bar{\mathbf{t}}_{\mathbf{n}0}}} \bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta \mathbf{u} \, da_0 = 0 \qquad \forall \, \delta \mathbf{u} \in \delta \mathcal{U};$$

  - Internal work using second Piola-Kirchhoff stress tensor $\mathbf{S}$ and virtual variation of Green-Lagrange strain tensor $\delta \mathbf{E}$

$$\int_{\Omega_0} \mathbf{S} : \delta \mathbf{E} \, dv_0 - \int_{\Omega_0} \mathbf{b}_0 \cdot \delta \mathbf{u} \, dv_0 - \int_{\partial \Omega_{\bar{\mathbf{t}}_{\mathbf{n}0}}} \bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta \mathbf{u} \, da_0 = 0 \qquad \forall \, \delta \mathbf{u} \in \delta \mathcal{U};$$

  with boundary conditions

$$\mathbf{u}_0 = \bar{\mathbf{u}}_0 \qquad \text{on } \partial \Omega_{\bar{\mathbf{u}}_0} \subset \partial \Omega_0.$$

Table 1.3.1: Principles of virtual work: resume of different forms

Chapter 2

Rotation tensors and their virtual variations

In the considered beam model a crucial role in the description of the beam motion is played by a three-dimensional rotation tensor, $\mathbf{\Lambda}$, which defines the orientation of the cross-section in space. As described in detail below, rotation tensors do not belong to a linear space but to a multiplicative group, $\mathcal{G}^{orth+}$, which is the reason why they are not easily amenable to direct discretization and their linearization is not straightforward. The continuum beam theory, the discrete theory and the finite elements approach are strictly related with rotations and their linearization. Therefore the chapter is dedicated to investigate rotation parameterizations and linearization procedures. A brief recall of properties of rotation tensors and their derivative is first given.

2.1 Rotations: definition, properties and derivative

2.1.1 Definition and properties

A rotation is an **orthogonal** tensor with **unit determinant**. As any orthogonal tensor, a rotation $\mathbf{\Lambda}$ is defined as a linear transformation which satisfies the condition

$$\mathbf{\Lambda u} \cdot \mathbf{\Lambda v} = \mathbf{u} \cdot \mathbf{v} \tag{2.1.1}$$

for all vectors $\mathbf{u} \in \mathcal{R}^n$ and $\mathbf{v} \in \mathcal{R}^n$. The above relation states that a rotation tensor preserves the inner product, i.e. both the angle between $\mathbf{u}$ and $\mathbf{v}$ and their lengths, $\parallel \mathbf{u} \parallel$, $\parallel \mathbf{v} \parallel$, are preserved. Here and in the following we refer to three-dimensional tensors and not to n-dimensional since it is sufficient for our purposes. From (2.1.1) it is straightforward to verify that

$$\mathbf{\Lambda}^T\mathbf{\Lambda} = \mathbf{\Lambda}\mathbf{\Lambda}^T = \mathbf{I} \qquad \text{or} \qquad \mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^T, \tag{2.1.2}$$

which are known as *orthogonality conditions*. Equation $(2.1.2)_1$ is given in index notation as

$$\Lambda_{ij}\Lambda_{ik} = \Lambda_{ji}\Lambda_{ki} = \delta_{jk} \tag{2.1.3}$$

where δ_{jk} is the Kronecher delta. It shows that in matrix term the condition $(2.1.2)_1$ is an orthonormality condition on the columns of matrix $\mathbf{\Lambda}$.
For a general orthogonal tensor $\mathbf{Q}$ it holds that $\det\mathbf{Q} = \pm 1$. In fact from equation (2.1.2) it follows that

$$\det(\mathbf{Q}^T\mathbf{Q}) = \det\mathbf{I} = 1,$$

and from determinant properties

$$\det(\mathbf{Q}^T\mathbf{Q}) = \det(\mathbf{Q}^T)\det(\mathbf{Q}) = \det(\mathbf{Q})^2$$

which implies that

$$\det(\mathbf{Q}) = \pm 1.$$

In particular a rotation is by definition an orthogonal tensor with $\det = 1$.

The set of all rotations, which we refer to as $\mathcal{G}^{orth+}$, has group structure under multiplication, hence it satisfies the following properties

- $\forall\ \mathbf{\Lambda}_1, \mathbf{\Lambda}_2 \in \mathcal{G}^{orth+}\ \ \mathbf{\Lambda} = \mathbf{\Lambda}_1\mathbf{\Lambda}_2$ with $\mathbf{\Lambda} \in \mathcal{G}^{orth+}$, i.e. the product of two rotations is a rotation;
- $\forall\ \mathbf{\Lambda}_1, \mathbf{\Lambda}_2, \mathbf{\Lambda}_3 \in \mathcal{G}^{orth+}\ \ (\mathbf{\Lambda}_1\mathbf{\Lambda}_2)\mathbf{\Lambda}_3 = \mathbf{\Lambda}_1(\mathbf{\Lambda}_2\mathbf{\Lambda}_3)$, i.e. the product of three rotations is associative;
- $\mathbf{I}\ \mathbf{\Lambda}\mathbf{I} = \mathbf{I}\mathbf{\Lambda} = \mathbf{\Lambda}\ \ \forall\ \mathbf{\Lambda} \in \mathcal{G}^{orth+}$, i.e. there exists a neuter element, the identity matrix;
- $\mathbf{\Lambda}\mathbf{\Lambda}^{-1} = \mathbf{\Lambda}^{-1}\mathbf{\Lambda} = \mathbf{I}\ \ \forall\ \mathbf{\Lambda} \in \mathcal{G}^{orth+}$, i.e. there exists an inverse element $\mathbf{\Lambda}^{-1}$ for every $\mathbf{\Lambda}$.

We point out that in general the rotation *product is not commutative*, i.e.

$$\mathbf{\Lambda}_1\mathbf{\Lambda}_2 \neq \mathbf{\Lambda}_2\mathbf{\Lambda}_1 \qquad \text{with} \quad \mathbf{\Lambda}_1, \mathbf{\Lambda}_2\ \in \mathcal{G}^{orth+}.$$

Physically it is easy to see that two inverted sequences of rotations around not parallel fixed axes in space map the same vector into two different positions.
Finally since the set of rotations is not a linear space, rotations *are not additive*, i.e. given two rotations, $\mathbf{\Lambda}_1 \in \mathcal{G}^{orth+}$ and $\mathbf{\Lambda}_2 \in \mathcal{G}^{orth+}$, then $\mathbf{\Lambda}_1 + \mathbf{\Lambda}_2 = \mathbf{H} \notin \mathcal{G}^{orth+}$.

2.1.2 Derivative: spin tensors

In the beam theory the rotation will be defined as a one parameter rotation, i.e. it will be function only of a one independent variable, the reference length of beam. Since we will clearly deal with the derivative of a rotation with respect to this parameter, we introduce here such topic which is not trivial at all.

Consider an arbitrary parameter, t, and a one-parameter rotation $\mathbf{\Lambda} = \mathbf{\Lambda}(t)$. We indicate by $\dot{\mathbf{\Lambda}}$ the derivative of $\mathbf{\Lambda}$ with respect to t. Taking derivative of the orthogonality condition $\mathbf{\Lambda}\mathbf{\Lambda}^T = \mathbf{I}$ (2.1.2), we get

$$\dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T + \mathbf{\Lambda}\dot{\mathbf{\Lambda}}^T = \mathbf{0}$$

which given in the form

$$\dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T = -\mathbf{\Lambda}\dot{\mathbf{\Lambda}}^T \quad \longrightarrow \quad \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T = -(\dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T)^T$$

shows that $\dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T$ is a skew tensor. With this result in hand we define the *spin tensor* $\mathbf{\Omega}$ and consequently its axial vector $\boldsymbol{\omega}$, *spin vector*, as

$$\boxed{\mathbf{\Omega} = [\boldsymbol{\omega}\times] = \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T \qquad \text{where} \quad \mathbf{\Omega} \in so(3), \quad \boldsymbol{\omega} \in \mathcal{R}^3.} \tag{2.1.4}$$

The terminology *spin vector* is taken from rigid body dynamics where the parameter t as the specific meaning of time. In beam theory we preserve this name in order to preserve the analogy.
Reversing (2.1.4) we get the derivative $\dot{\mathbf{\Lambda}}$ as

$$\boxed{\dot{\mathbf{\Lambda}} = \mathbf{\Omega}\mathbf{\Lambda}.} \tag{2.1.5}$$

With a similar procedure, derivative of the other orthogonality condition $\mathbf{\Lambda}^T\mathbf{\Lambda} = \mathbf{I}$ (2.1.2) respect to t yields to the definition of the skew tensor $\mathbf{\Omega}_r$ and its axial vector $\boldsymbol{\omega}_r$ as

$$\boxed{\mathbf{\Omega}_r = [\boldsymbol{\omega}_r\times] = \mathbf{\Lambda}^T\dot{\mathbf{\Lambda}} \qquad \text{where } \mathbf{\Omega}_r \in so(3), \quad \boldsymbol{\omega}_r \in \mathcal{R}^3,} \tag{2.1.6}$$

and to another form of derivative $\dot{\mathbf{\Lambda}}$

$$\boxed{\dot{\mathbf{\Lambda}} = \mathbf{\Lambda}\mathbf{\Omega}_r.} \tag{2.1.7}$$

Note that $\dot{\mathbf{\Lambda}} \notin \mathcal{G}^{orth+}$, i.e. the derivative of a rotation is not a rotation. At contrary, recalling from section () that the skew-tensor space $so(3)$ is the tangent space to the rotation group, equations (2.1.5) and (2.1.7) show that the derivative is a composition of the current rotation with an element of the rotation tangent space. Moreover in the case (2.1.5) the spin tensor $\mathbf{\Omega}$ follows the rotation in the composition product, i.e. it lays in a space that has been already rotated by $\mathbf{\Lambda}$. Accordingly it belongs to the rotation tangent space in $\mathbf{\Lambda}$, $T_{\mathbf{\Lambda}}\mathcal{G}^{orth+}$. In the case (2.1.7) the spin tensor $\mathbf{\Omega}_r$ precedes the rotation in the composition product, i.e. it lays in a space that has not been affected by any rotation. Accordingly it belongs to the rotation tangent space in the identity $\mathbf{I}$, $T_{\mathbf{I}}\mathcal{G}^{orth+}$.

Comparing equations (2.1.5) and (2.1.7) it is easy to obtain

$$\mathbf{\Omega} = \mathbf{\Lambda}\mathbf{\Omega}_r\mathbf{\Lambda}^T \qquad \text{and} \qquad \mathbf{\Omega}_r = \mathbf{\Lambda}^T\mathbf{\Omega}\mathbf{\Lambda} \tag{2.1.8}$$

which shows that $\mathbf{\Omega}$ is the rotated-forward expression of $\mathbf{\Omega}_r$ and $\mathbf{\Omega}_r$ is the rotated-back expression of $\mathbf{\Omega}$.

An equivalent relation can be obtained for axial vector $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_r$. Comparing again (2.1.5) and (2.1.7) it follows that

$$\boldsymbol{\Omega}\boldsymbol{\Lambda}\mathbf{a} = \boldsymbol{\Lambda}\boldsymbol{\Omega}_r\mathbf{a} \qquad \forall\, \mathbf{a} \in \mathcal{R}^3,$$

which using the axial vector relation (0.0.1) can be given in term of $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_r$ as

$$\boldsymbol{\omega} \times \boldsymbol{\Lambda}\mathbf{a} = \boldsymbol{\Lambda}(\boldsymbol{\omega}_r \times \mathbf{a}) \qquad \forall\, \mathbf{a} \in \mathcal{R}^3.$$

Using for the right-hand-side the distributivity property of cross product under rotation (0.0.5), we obtain

$$\boldsymbol{\omega} \times \boldsymbol{\Lambda}\mathbf{a} = \boldsymbol{\Lambda}\boldsymbol{\omega}_r \times \boldsymbol{\Lambda}\mathbf{a} \qquad \forall\, \mathbf{a} \in \mathcal{R}^3,$$

which entails

$$\boldsymbol{\omega} = \boldsymbol{\Lambda}\boldsymbol{\omega}_r \tag{2.1.9}$$

and consequentially

$$\boldsymbol{\omega}_r = \boldsymbol{\Lambda}^T\boldsymbol{\omega}. \tag{2.1.10}$$

Such as for their skew tensors, $\boldsymbol{\omega}$ is the rotated-forward expression of $\boldsymbol{\omega}_r$ and $\boldsymbol{\omega}_r$ is the rotated-back expression of $\boldsymbol{\omega}$.
Spin tensor $\boldsymbol{\Omega}$ and spin vector $\boldsymbol{\omega}$, together with their rotated-back form respectively $\boldsymbol{\Omega}_r$ and $\boldsymbol{\omega}_r$, will play a crucially role in the development of the beam model.

2.2 Parameterizations of rotations

A three-dimensional rotation tensor can be represented by a 3×3 matrix. By the way the minimum number of degree of freedom needed to describe a finite rotation is three. The fact is evident since the nine components of the rotation matrix are related each other by the six orthonormality conditions which define a rotation tensor (see equations (2.1.2) and (2.1.3)). A parametrization of the three-dimensional rotation is a representation of the rotation through three (or in some cases four) parameters instead of the nine parameters employed in the matrix representation. These parametrizations are useful to reduce the costly representation of matrix rotation in a computational context. Some of them is also very useful to develop linearization of rotation, as shown below.

Several rotation parameterizations have been proposed in literature (e.g. see [1], [3], [5]). Here we study in depth the so called *rotation vector* parametrization, which we use in the development of beam theory and in successive finite element approach. Moreover we mention other parametrizations we have found in literature to be related with the beam model.

The *rotation vector* parametrization is extensively treated in [1, 4, 11]. The name is not univocal in literature, so much so that Argyris in [1] refers to it as rotation pseudovector parametrization. The rotation vector is a vector with direction along the physical axis of rotation, versus defined by the right-hand-rule in dependence of the clockwise or counter-clockwise sense of rotation and norm equal to the angle of rotation. This parametrization is demonstrated to coincide with the *exponential map* for rotation which is a chart of fundamental importance in definition of both linearizations of rotations and rotation updating computational techniques. The illustration of rotation vector parametrization for the rotations is followed by the explanation of the same parametrization for the spin tensors.

We consider finally pseudovector parametrizations which follow from different definition of the norm of rotation tensor and quaternion parametrization.

2.2.1 Rotation vector parametrization

Exponential mapping and Rodrigues formula

From a physical point of view, in three-dimensional space a rotation can be described by an axis, defined by a unit vector $\mathbf{e}$, and an angle θ which lies in a plane orthogonal to the axis. All the space rotates around $\mathbf{e}$, see Fig.2.1. We introduce the *rotation vector*, $\boldsymbol{\theta}$,

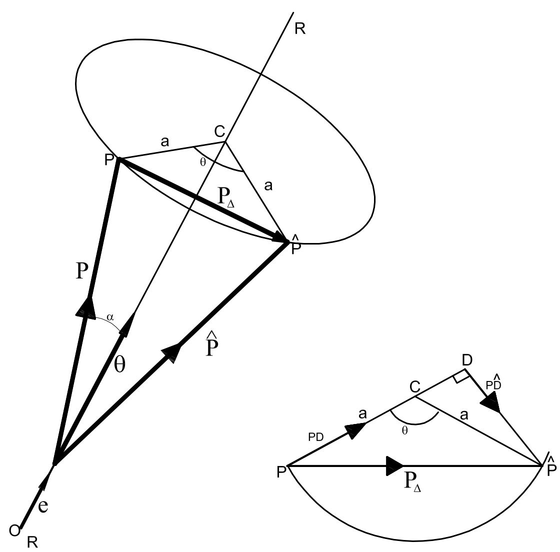


Figure 2.1: Construction of rotation matrix $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ with rotation vector $\boldsymbol{\theta}$

defined as

$$\boldsymbol{\theta} = \theta \mathbf{e}. \tag{2.2.1}$$

Accordingly, by definition, $\boldsymbol{\theta}$ has norm equal to θ and is directed as the unit vector $\mathbf{e}$. Indicating with ϕ, χ, ψ the components of $\boldsymbol{\theta}$ in a cartesian reference system $\{OX_1X_2X_3\}$, $\boldsymbol{\theta}$ can be given as

$$\boldsymbol{\theta} = \left\{ \begin{array}{c} \phi \\ \chi \\ \psi \end{array} \right\} \tag{2.2.2}$$

and its norm $\theta = \parallel \boldsymbol{\theta} \parallel$ as

$$\theta = \sqrt{\phi^2 + \chi^2 + \psi^2}. \tag{2.2.3}$$

Consider a vector $\mathbf{p}$ which is mapped into $\hat{\mathbf{p}}$ by a rotation of an angle θ around the axis in direction of $\mathbf{e}$, Fig.2.1. This mapping has the form

$$\hat{\mathbf{p}} = \mathbf{\Lambda}\mathbf{p} \tag{2.2.4}$$

where $\mathbf{\Lambda}$ is a rotation tensor. Our task is to find an explicit expression of $\mathbf{\Lambda}$ as a function of $\boldsymbol{\theta}$, i.e. to establish the transformation

$$\hat{\mathbf{p}} = \mathbf{\Lambda}(\boldsymbol{\theta})\mathbf{p}. \tag{2.2.5}$$

A very clear derivation can be performed by use of vector calculus [1]. Looking at Fig.2.1 the relation between vector $\mathbf{p}$ and its rotated vector $\hat{\mathbf{p}}$ can be given as

$$\hat{\mathbf{p}} = \mathbf{p} + \mathbf{p}_\Delta. \tag{2.2.6}$$

Our task is to transform the right-hand-side of (2.2.6) into the matrix product of (2.2.5). To this end from the Fig.2.1 we first deduce the following relation

$$\mathbf{p}_\Delta = \overline{PD} + \overline{D\hat{P}}, \tag{2.2.7}$$

where $\overline{D\hat{P}}$ is drawn normal to $\overline{PC}$. We note that the vector $\overline{D\hat{P}}$ stands perpendicular to the plane OPC. Therefore it points in the direction $(\mathbf{e} \times \mathbf{p})$. To find its magnitude we observe that

$$D\hat{P} = a \sin\theta. \tag{2.2.8}$$

On the other hand, we observe that the magnitude of $(\mathbf{e} \times \mathbf{p})$ is

$$\parallel \mathbf{e} \times \mathbf{p} \parallel = 1 \cdot p \sin\alpha = p\frac{a}{p} = a. \tag{2.2.9}$$

It follows in conjunction with (2.2.8) and (2.2.9) that

$$\overline{D\hat{P}} = (\mathbf{e} \times \mathbf{p}) \sin\theta = \frac{\sin\theta}{\theta}(\boldsymbol{\theta} \times \mathbf{p}). \tag{2.2.10}$$

Next we proceed to the determination of the vector $\overline{PD}$. Fig.(2.1) shows immediately that it is not only perpendicular to $(\mathbf{e} \times \mathbf{p})$ but also to $\mathbf{e}$ since it lies in the plane $PC\hat{P}$ normal to $\mathbf{e}$. Hence it may be assigned the direction of $\mathbf{e} \times (\mathbf{e} \times \mathbf{p})$. Now the absolute value of the last vector is clearly again a since $\mathbf{e}$ is a unit vector and it is normal to $(\mathbf{e} \times \mathbf{p})$, i.e.

$$\parallel \mathbf{e} \times (\mathbf{e} \times \mathbf{p}) \parallel = \parallel \mathbf{e} \times \mathbf{p} \parallel = a.$$

At the same time Fig.2.1 yields to

$$PD = a - a\cos\theta = (1 - \cos\theta)a = 2\sin^2\frac{\theta}{2}a. \tag{2.2.11}$$

Hence we deduce, using (2.2.11) and the direction of $\overline{PD}$,

$$\overline{PD} = 2\sin^2\frac{\theta}{2}(\mathbf{e} \times (\mathbf{e} \times \mathbf{p})) = \frac{1}{2}\frac{\sin^2(\theta/2)}{(\theta/2)^2}(\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \mathbf{p})). \tag{2.2.12}$$

Applying (2.2.12) and (2.2.10) in (2.2.7) and (2.2.6), the vector $\hat{\mathbf{p}}$ takes the form

$$\hat{\mathbf{p}} = \mathbf{p} + \frac{\sin\theta}{\theta}(\boldsymbol{\theta}\times\mathbf{p}) + \frac{1}{2}\frac{\sin^2(\theta/2)}{(\theta/2)^2}(\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\mathbf{p})). \tag{2.2.13}$$

Equation (2.2.13) can be written in matrix form

$$\hat{\mathbf{p}} = \mathbf{p} + \frac{\sin\theta}{\theta}\boldsymbol{\Theta}\mathbf{p} + \frac{1}{2}\frac{\sin^2(\theta/2)}{(\theta/2)^2}\boldsymbol{\Theta}^2\mathbf{p} \tag{2.2.14}$$

where

$$\boldsymbol{\Theta} = \begin{bmatrix} 0 & -\psi & \chi \\ \psi & 0 & -\phi \\ -\chi & \phi & 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Theta}^2 = \boldsymbol{\Theta}\boldsymbol{\Theta} = \begin{bmatrix} -(\chi^2+\psi^2) & \chi\phi & \phi\psi \\ \chi\phi & -(\psi^2+\phi^2) & \psi\chi \\ \phi\psi & \psi\chi & -(\phi^2+\chi^2) \end{bmatrix} \tag{2.2.15}$$

$\boldsymbol{\Theta}$ is the skew-symmetric tensor having $\boldsymbol{\theta}$ as axial vector. The equality between equations (2.2.13) and (2.2.14) is easily shown recalling that by definition of axial vector (see equation (0.0.1)), the cross products $\boldsymbol{\theta}\times\mathbf{p}$ and $\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\mathbf{p})$ can be given in the matrix form

$$\boldsymbol{\theta}\times\mathbf{p} = \boldsymbol{\Theta}\mathbf{p} \quad \text{and} \quad \boldsymbol{\theta}\times(\boldsymbol{\theta}\times\mathbf{p}) = \boldsymbol{\Theta}^2\mathbf{p}. \tag{2.2.16}$$

Equation (2.2.14) is the transformation we were looking for, i.e.

$$\hat{\mathbf{p}} = \boldsymbol{\Lambda}(\boldsymbol{\theta})\mathbf{p} \quad \text{with} \quad \boxed{\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \mathbf{I} + \frac{\sin\theta}{\theta}\boldsymbol{\Theta} + \frac{1}{2}\frac{\sin^2(\theta/2)}{(\theta/2)^2}\boldsymbol{\Theta}^2}. \tag{2.2.17}$$

Substituting the trigonometric identity $\sin^2 x = \frac{1-\cos(2x)}{2}$ into the argument of $\boldsymbol{\Theta}^2$ we obtain an equivalent form of the transformation $\boldsymbol{\Lambda}(\boldsymbol{\theta})$

$$\boxed{\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \mathbf{I} + \frac{\sin\theta}{\theta}\boldsymbol{\Theta} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\Theta}^2.} \tag{2.2.18}$$

These equivalent formulae represent the *rotation vector parametrization* of rotation tensor $\boldsymbol{\Lambda}$.

We make now some calculation in order to show the orthogonality of $\boldsymbol{\Lambda}(\boldsymbol{\theta})$. The transpose $\boldsymbol{\Lambda}^T(\boldsymbol{\theta})$ can be easy computed recalling that $\boldsymbol{\Theta}$ is a skew tensor, ($\boldsymbol{\Theta}^T = -\boldsymbol{\Theta}$),

$$\boldsymbol{\Lambda}^T(\boldsymbol{\theta}) = \mathbf{I} - \frac{\sin\theta}{\theta}\boldsymbol{\Theta} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\Theta}^2. \tag{2.2.19}$$

Evaluating the rotation $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ for $\boldsymbol{\theta} = -\boldsymbol{\theta}$ we obtain the right-hand-side of previous expression, hence we can state that

$$\boldsymbol{\Lambda}^T(\boldsymbol{\theta}) = \boldsymbol{\Lambda}(-\boldsymbol{\theta}).$$

$\boldsymbol{\Lambda}(-\boldsymbol{\theta})$ clearly maps the rotated vector $\hat{\mathbf{p}}$ back into $\mathbf{p}$, because $\boldsymbol{\Lambda}(-\boldsymbol{\theta})$ is the rotation around the inverted rotation vector. Therefore $\boldsymbol{\Lambda}(-\boldsymbol{\theta}) = \boldsymbol{\Lambda}^{-1}$. Accordingly we have

$$\boldsymbol{\Lambda}^{-1} = \boldsymbol{\Lambda}(-\boldsymbol{\theta}) = \boldsymbol{\Lambda}^T(\boldsymbol{\theta}).$$

which confirms that $\mathbf{\Lambda}(\boldsymbol{\theta})$ is orthogonal.

Consider the series expansion of $\mathbf{\Theta}$

$$\exp[\mathbf{\Theta}] = \mathbf{I} + \mathbf{\Theta} + \frac{1}{2!}\mathbf{\Theta}^2 + \frac{1}{3!}\mathbf{\Theta}^3 + ... + \frac{1}{n!}\mathbf{\Theta}^n + ... \tag{2.2.20}$$

which is by definition the *exponential function* of the skew-tensor $\mathbf{\Theta}$. It can be proved (see [9] page 228) that the exponential function of a generic skew tensor is a rotation tensor, i.e. the exponential is the chart which maps a skew-symmetric tensor in a rotation tensor. Here we show in fact that the expression of $\mathbf{\Lambda}$ (2.2.17) (or (2.2.18)) yields to the exponential map of $\mathbf{\Theta}$.

Proof

We start expanding in series with respect to $\boldsymbol{\theta}$ the trigonometric functions in (2.2.17), which yields to

$$\mathbf{\Lambda}(\mathbf{\Theta}) = \mathbf{I}+\Big[1-\frac{\theta^2}{3!}+\frac{\theta^4}{5!}+...+(-1)\frac{\theta^{2n}}{(2n+1)!}...\Big]\mathbf{\Theta}+\Big[\frac{1}{2!}-\frac{\theta^2}{4!}+\frac{\theta^4}{6!}+...+(-1)\frac{\theta^{2n}}{(2n+2)!}...\Big]\mathbf{\Theta}^2. \tag{2.2.21}$$

Considering now the skew-symmetric tensor $\mathbf{\Theta}$ with axial vector $\boldsymbol{\theta}$ and $\theta = \parallel \boldsymbol{\theta} \parallel$, by explicit computation it can be demonstrated that

$$-\theta^2\mathbf{\Theta} = \mathbf{\Theta}^3, \quad \theta^4\mathbf{\Theta} = \mathbf{\Theta}^5$$

$$-\theta^2\mathbf{\Theta}^2 = \mathbf{\Theta}^4, \quad \theta^4\mathbf{\Theta}^2 = \mathbf{\Theta}^6$$

which leads to the recurrence formulae

$$(-1)^{n-1}\theta^{2(n-1)}\mathbf{\Theta} = \mathbf{\Theta}^{2n-1}, \qquad (-1)^{n-1}\theta^{2(n-1)}\mathbf{\Theta}^2 = \mathbf{\Theta}^{2n}. \tag{2.2.22}$$

Developing the multiplications in equation (2.2.21) and then substituting into it the right-hand-side of (2.2.22), we obtain $\mathbf{\Lambda}$ as a series expansion of $\mathbf{\Theta}$

$$\mathbf{\Lambda}(\mathbf{\Theta}) = \mathbf{I} + \mathbf{\Theta} + \frac{1}{2!}\mathbf{\Theta}^2 + \frac{1}{3!}\mathbf{\Theta}^3 + ... + \frac{1}{n!}\mathbf{\Theta}^n + ...$$

which proves that

$$\mathbf{\Lambda} = \exp[\mathbf{\Theta}] \qquad \blacksquare$$

Finally the exponential mapping defined in (2.2.17) or (2.2.18) can be recast also in alternative but equivalent forms bringing out the unit vector $\mathbf{e} = \boldsymbol{\theta}\backslash\theta$. They are know in literature as *Rodrigues formulae*. Writing (2.2.18) using the vector notation for skew tensor $\mathbf{\Theta} = [\boldsymbol{\theta}\times]$, we obtain

$$\mathbf{\Lambda}(\boldsymbol{\theta}) = \mathbf{I} + \frac{\sin\theta}{\theta}[\boldsymbol{\theta}\times] + \frac{1-\cos\theta}{\theta^2}[\boldsymbol{\theta}\times[\boldsymbol{\theta}\times]],$$

which becomes, pointing out $\mathbf{e}$, the Rodrigues formula

$$\boxed{\mathbf{\Lambda}(\boldsymbol{\theta}) = \mathbf{I} + \sin\theta[\mathbf{e}\times] + (1-\cos\theta)[\mathbf{e}\times[\mathbf{e}\times]].} \tag{2.2.23}$$

Alternatively, introducing in (2.2.18) the identity (0.0.2)

$$\boldsymbol{\Theta}^2\mathbf{b} = \boldsymbol{\Theta}(\boldsymbol{\Theta}\mathbf{b}) = [\boldsymbol{\theta}\otimes\boldsymbol{\theta} - \theta^2\mathbf{I}]\mathbf{b} \qquad \forall\, \mathbf{b}\in\mathcal{R}^3,$$

we obtain

$$\begin{aligned}\boldsymbol{\Lambda}(\boldsymbol{\theta}) &= \mathbf{I} + \frac{\sin\theta}{\theta}\boldsymbol{\Theta} + \frac{1-\cos\theta}{\theta^2}(\boldsymbol{\theta}\otimes\boldsymbol{\theta} - \theta^2\mathbf{I}) = \\ &= \cos\theta\mathbf{I} + \frac{\sin\theta}{\theta}\boldsymbol{\Theta} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\theta}\otimes\boldsymbol{\theta}.\end{aligned} \tag{2.2.24}$$

Recognizing again $\mathbf{e}$, by substitution into the previous equation we get another form of Rodrigues formula

$$\boxed{\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \cos\theta\mathbf{I} + \sin\theta[\mathbf{e}\times] + (1-\cos\theta)(\mathbf{e}\otimes\mathbf{e}).} \tag{2.2.25}$$

An extensive investigation on the rotational vector parametrization can be found in [1]. By the way, in this article can be found an explanation of the important property for the exponential map

$$\exp[\mathbf{S}_1 + \mathbf{S}_2] \neq \exp[\mathbf{S}_1]\exp[\mathbf{S}_2],$$

where $\mathbf{S}_1$ and $\mathbf{S}_2$ are two general skew tensors. In [9] a small section is devoted just to the exponential map. There it can be found an alternative demonstration that the exponential of a skew tensor is a rotation tensor.

Spin tensors by rotation vector

We have already said that the spin tensor $\boldsymbol{\Omega} = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T$ and its axial vector $\boldsymbol{\omega}$ as well as their rotated-back form $\boldsymbol{\Omega}_r = \boldsymbol{\Lambda}^T\boldsymbol{\Omega}\boldsymbol{\Lambda}$ and $\boldsymbol{\omega}_r = \boldsymbol{\Lambda}^T\boldsymbol{\omega}$, introduced in section (2.1.2), play a crucial role in the kinematics and equilibrium of beam model. We are interested here in study their relations with the rotation vector $\boldsymbol{\theta}$. The relations can be found developing the definition of spin tensors, $\boldsymbol{\Omega} = \dot{\boldsymbol{\Lambda}}\boldsymbol{\Lambda}^T$ and $\boldsymbol{\Omega}_r = \boldsymbol{\Lambda}^T\dot{\boldsymbol{\Lambda}}$, substituting $\boldsymbol{\Lambda}$ with the exponential mapping or the equivalent Rodrigues formulae. The computation shows that both the spin vectors $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_r$ are related *linearly* with the derivative of the rotation vector $\dot{\boldsymbol{\theta}}$ through a *non linear* function of $\boldsymbol{\theta}$; it results in fact that

$$\boldsymbol{\omega} = \mathbf{T}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{2.2.26}$$

$$\boldsymbol{\omega}_r = \mathbf{T}^T(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \tag{2.2.27}$$

where

$$\mathbf{T}(\boldsymbol{\theta}) = \mathbf{I} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\Theta} + \frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\Theta}^2, \tag{2.2.28}$$

$$\mathbf{T}^T(\boldsymbol{\theta}) = \mathbf{I} - \frac{1-\cos\theta}{\theta^2}\boldsymbol{\Theta} + \frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\Theta}^2. \tag{2.2.29}$$

$\mathbf{T}^T(\boldsymbol{\theta})$ is computed from $\mathbf{T}$ changing the sign to the coefficient of $\boldsymbol{\Theta}$, since $\boldsymbol{\Theta}$ is skew ($\boldsymbol{\Theta}^T = -\boldsymbol{\Theta}$). Using identity (0.0.2), the quadratic skew tensor $\boldsymbol{\Theta}^2$ can be given in term of vector $\boldsymbol{\theta}$, hence tensor $\mathbf{T}$ can be rearranged easily as

$$\mathbf{T}(\boldsymbol{\theta}) = \frac{\sin\theta}{\theta}\mathbf{I} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\Theta} + \frac{\theta - \sin\theta}{\theta^3}\boldsymbol{\theta}\otimes\boldsymbol{\theta}.$$

Substituting the expression of $\mathbf{T}$ (2.2.28) into the relation between $\boldsymbol{\omega}$ and $\dot{\boldsymbol{\theta}}$ (2.2.26) and using the axial vector notation for $\boldsymbol{\Theta}$, we obtain

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} + \frac{1-\cos\theta}{\theta^2}\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}} + \frac{\theta-\sin\theta}{\theta^3}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\dot{\boldsymbol{\theta}}),$$

which shows that if $\boldsymbol{\theta}$ and $\dot{\boldsymbol{\theta}}$ are parallel, as in the case of a *plane problem*, $\boldsymbol{\omega} = \dot{\boldsymbol{\theta}}$.

We know that the relation between $\boldsymbol{\Lambda}$ and its rotation vector $\boldsymbol{\theta}$ can be always expressed as the exponential map of the skew tensor $\boldsymbol{\Theta} = [\boldsymbol{\theta}\times]$

$$\boldsymbol{\Lambda} = \exp[\boldsymbol{\Theta}] = \sum_{k=0}^{\infty}\frac{\boldsymbol{\Theta}^k}{k!}.$$

It is of interest that also $\mathbf{T}(\boldsymbol{\theta})$ can be expressed by a series expansion of $\boldsymbol{\Theta}$ which takes the compact form

$$\mathbf{T}(\boldsymbol{\Theta}) = \sum_{k=0}^{\infty}\frac{\boldsymbol{\Theta}^k}{k+1!} \qquad (2.2.30)$$

as clearly shown in [21]. Moreover it is provedin [4] that

$$\boldsymbol{\Lambda}(\boldsymbol{\theta}) = \mathbf{I} + \boldsymbol{\Theta}\mathbf{T}(\boldsymbol{\theta}). \qquad (2.2.31)$$

Finally it is important to recognize that $\mathbf{T}(\boldsymbol{\Theta})$ is *singular* for certain value of θ. Calculating the determinant from expression (2.2.28) we obtain

$$\det(\mathbf{T}) = \frac{2(1-\cos\theta)}{\theta^2},$$

which is null for $\theta = 2n\pi \;\; n = 1, 2, 3, ...$ In order to avoid this problem during computations, some references as [12] and [13] introduce a special incremental updating procedure when solving the nonlinear finite element equations, known as *updated Lagrangian*, other references as [4, 6, 15], introduce a rescaling process on $\boldsymbol{\theta}$.

For proof of equations (2.2.26), (2.2.27) and (2.2.28) see reference [2] and [4].

A family of trigonometric functions for the rotation vector parametrization

Observing $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ and $\mathbf{T}(\boldsymbol{\theta})$, we suppose that their first and second linearization with respect to $\boldsymbol{\theta}$ would be quite complex. In order to carry out these operations in a systematic and easy way, the following family of trigonometric functions is introduced

$$\boxed{a_0(\theta) = \cos\theta, \quad a_1(\theta) = \frac{\sin\theta}{\theta}, \quad a_2(\theta) = \frac{1-\cos\theta}{\theta^2}, \quad a_3(\theta) = \frac{\theta-\sin\theta}{\theta^3},} \qquad (2.2.32)$$

where we recall that $\theta = \| \boldsymbol{\theta} \|$. With this notation in hand, the exponential maps (2.2.18) and (2.2.24) can be rewritten as

$$\begin{aligned} \boldsymbol{\Lambda}(\boldsymbol{\theta}) &= \mathbf{I} + a_1(\theta)\boldsymbol{\Theta} + a_2(\theta)\boldsymbol{\Theta}^2, & (2.2.33) \\ \boldsymbol{\Lambda}(\boldsymbol{\theta}) &= a_0(\theta)\mathbf{I} + a_1(\theta)\boldsymbol{\Theta} + a_2(\theta)\boldsymbol{\theta} \otimes \boldsymbol{\theta}, & (2.2.34) \end{aligned}$$

and the tensor $\mathbf{T}(\boldsymbol{\theta})$ as

$$\begin{aligned} \mathbf{T}(\boldsymbol{\theta}) &= \mathbf{I} + a_2(\theta)\boldsymbol{\Theta} + a_3(\theta)\boldsymbol{\Theta}^2, & (2.2.35) \\ \mathbf{T}(\boldsymbol{\theta}) &= a_1(\theta)\mathbf{I} + a_2(\theta)\boldsymbol{\Theta} + a_3(\theta)\boldsymbol{\theta} \otimes \boldsymbol{\theta}. & (2.2.36) \end{aligned}$$

Note the similarity between equations (2.2.33) and (2.2.35) or between (2.2.34) and (2.2.36) which display a difference of one unit in the indices of the respective $a_i(\theta)$ functions.

In order to compute linearizations of $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ and $\mathbf{T}(\boldsymbol{\theta})$, it results very useful to give the linearizations of trigonometric functions $a_i(\theta)$ with respect to $\delta\theta$, which come from the operation

$$\delta(a_i(\theta)) = \frac{\mathrm{d}a_i(\theta)}{\mathrm{d}\theta}\delta\theta.$$

since $\boldsymbol{\theta}$ belongs to a linear space. Observing that $\theta^2 = \boldsymbol{\theta} \cdot \boldsymbol{\theta}$, linearizing both right-hand and left-hand sides we get

$$2\theta\ \delta\theta = \boldsymbol{\theta} \cdot \delta\boldsymbol{\theta} + \delta\boldsymbol{\theta} \cdot \boldsymbol{\theta} \rightarrow \delta\theta = \frac{\boldsymbol{\theta} \cdot \delta\boldsymbol{\theta}}{\theta}.$$

Substituting the variation $\delta\theta$ into the linearization of a_i we can write

$$\delta(a_i(\theta)) = b_i(\theta)(\boldsymbol{\theta} \cdot \delta\boldsymbol{\theta}) \qquad \text{where } b_i(\theta) = \frac{1}{\theta}\frac{\mathrm{d}a_i(\theta)}{\mathrm{d}\theta}.$$

Each function $b_i(\theta)$ is given explicitly by

$$\begin{aligned} b_0(\theta) &= -\frac{\sin\theta}{\theta}, \qquad b_1(\theta) = \frac{\theta\cos\theta - \sin\theta}{\theta^3}, \\ b_2(\theta) &= \frac{\theta\sin\theta - 2 + 2\cos\theta}{\theta^4}, \qquad b_3(\theta) = \frac{3\sin\theta - 2\theta - \theta\cos\theta}{\theta^5}. \end{aligned} \tag{2.2.37}$$

In an analogous way, we can define functions $c_i(\theta)$ as

$$\delta(b_i(\theta)) = c_i(\theta)(\boldsymbol{\theta} \cdot \delta\boldsymbol{\theta}) \qquad \text{where } c_i(\theta) = \frac{1}{\theta}\frac{\mathrm{d}b_i(\theta)}{\mathrm{d}\theta},$$

which are

$$\begin{aligned} c_0(\theta) &= \frac{\sin\theta - \theta\cos\theta}{\theta^3}, \qquad c_1(\theta) = \frac{3\sin\theta - \theta^2\sin\theta - 3\theta\cos\theta}{\theta^5}, \\ c_2(\theta) &= \frac{8 - 8\cos\theta - 5\theta\sin\theta + \theta^2\cos\theta}{\theta^6}, \qquad c_3(\theta) = \frac{8\theta + 7\theta\cos\theta + \theta^2\sin\theta - 15\sin\theta}{\theta^7} \\ . \end{aligned} \tag{2.2.38}$$

An extensive treatment of this subject can be found in [21].

2.2.2 Pseudovector parameterizations and quaternions

This brief section is devoted to give the expressions of various parametrizations we have found studying the literature about three-dimensional rotations. Consider the vector $\boldsymbol{\rho}$ in direction of the unit vector $\mathbf{e} = \boldsymbol{\theta}\backslash\theta$ defined as

$$\boldsymbol{\rho} = \rho\mathbf{e} = \tan\frac{\theta}{2}\mathbf{e}. \tag{2.2.39}$$

with

$$\rho = \tan\frac{\theta}{2} = \parallel \boldsymbol{\rho} \parallel .$$

Rearranging equation (2.2.13) by trigonometric identities and then substituting (2.2.39), after some manipulation it can be shown that we obtain the following relations

$$\hat{\mathbf{p}} = \boldsymbol{\Lambda}(\boldsymbol{\rho})\mathbf{p} \tag{2.2.40}$$

with

$$\boldsymbol{\Lambda}(\boldsymbol{\rho}) = \mathbf{I} + \frac{2}{1+\boldsymbol{\rho}\cdot\boldsymbol{\rho}}[\mathbf{R} + \mathbf{R}^2], \tag{2.2.41}$$

where $\mathbf{R} = [\boldsymbol{\rho}\times]$ is the skew-symmetric matrix with axial vector $\boldsymbol{\rho}$. Expression (2.2.41) is a pseudovector parametrization of $\boldsymbol{\Lambda}$. This expression is simpler than the exponential mapping expression, but it is not associated to any series expansion. For a detailed proof of (2.2.40) see [1].

Another parametrization of the rotation $\boldsymbol{\Lambda}$, given explicitly in term of $\boldsymbol{\theta}$ components, $\{\varphi, \chi, \psi\}$, and $\boldsymbol{\theta}$ norm, θ, is

$$\boldsymbol{\Lambda} = \begin{bmatrix} 1 - \left[1 - \left(c\frac{\varphi}{\theta}\right)^2\right](1-\cos\theta) & \frac{\varphi\chi}{\theta^2}(1-\cos\theta) - \frac{\psi}{\theta}\sin\theta & \frac{\varphi\psi}{\theta^2}(1-\cos\theta) + \frac{\chi}{\theta}\sin\theta \\ \frac{\varphi\chi}{\theta^2}(1-\cos\theta) + \frac{\psi}{\theta}\sin\theta & 1 - \left[1 - \left(\frac{\chi}{\theta}\right)^2\right](1-\cos\theta) & \frac{\psi\chi}{\theta^2}(1-\cos\theta) - \frac{\varphi}{\theta}\sin\theta \\ \frac{\varphi\psi}{\theta^2}(1-\cos\theta) - \frac{\chi}{\theta}\sin\theta & \frac{\psi\chi}{\theta^2}(1-\cos\theta) + \frac{\varphi}{\theta}\sin\theta & 1 - \left[1 - \left(\frac{\psi}{\theta}\right)^2\right](1-\cos\theta) \end{bmatrix}. \tag{2.2.42}$$

To conclude, consider the element composed by a scalar part q_0 and a vector part $\mathbf{q}$ related to the rotation vector $\boldsymbol{\theta}$ by

$$\{q_0, \mathbf{q}\} = \left\{\cos\frac{\theta}{2}, \boldsymbol{\theta}\sin\frac{\theta}{2}\right\}. \tag{2.2.43}$$

with q_1, q_2, q_3 components of $\mathbf{q}$. This element $\{q_0, \mathbf{q}\}$ is know as *quaternion*. The four parameters which define the quaternion, q_0, q_1, q_2, q_3, are not independent since by definition $\sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2} = 1$, i.e the quaternion norm is unit. The rotation tensor $\boldsymbol{\Lambda}$ can be represented by the quaternion $\{q_0, \mathbf{q}\}$ through the following relation

$$\boldsymbol{\Lambda} = (2q_0^2 - 1)\mathbf{I} + 2q_0\mathbf{Q} + 2\mathbf{q}\otimes\mathbf{q}, \tag{2.2.44}$$

where $\mathbf{Q} = [\mathbf{q}\times]$ is the skew-tensor with axial vector $\mathbf{q}$ and $\mathbf{I}$ is the identity. The quaternions are very useful to represent a rotation compound. Consider two rotations $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ represented respectively by two sets of quaternions, $\{q_0, \mathbf{q}\}$ and $\{p_0, \mathbf{p}\}$. The quaternion $\{r_0, \mathbf{r}\}$ representing the product rotation $\mathbf{\Lambda} = \mathbf{\Lambda}_1\mathbf{\Lambda}_2$ is given as function of $\{q_0, \mathbf{q}\}$ and $\{p_0, \mathbf{p}\}$ by

$$\{r_0, \mathbf{r}\} = \{(p_0q_0 - \mathbf{p}\cdot\mathbf{q}), (p_0\mathbf{q} + q_0\mathbf{p} + \mathbf{p}\times\mathbf{q})\}$$

Hence, $\mathbf{\Lambda} = \mathbf{\Lambda}_1\mathbf{\Lambda}_2$ can be obtained directly from $\{q_0, \mathbf{q}\}$ and $\{p_0, \mathbf{p}\}$ replacing $\{r_0, \mathbf{r}\}$ into (2.2.44). We note in passing that representation (2.2.44) is not a bijection, since from a single rotation tensor one can extract two quaternions. Therefore, a special procedure (e.g. see [29] or [24]) is devised to reduce (although never fully eliminate) the sensitivity of the extraction of a quaternion from a rotation tensor. For further information on this parametrization see [13, 24, 5]

2.3 Virtual variations of rotations and spin tensors

In this section we deal with the *linearization* procedures for rotation and spin tensors, in order to provide quantities afterwards needed to develop the beam *principle of virtual work*. Before proceeding we briefly recall the definition of manifold and tangent spaces, useful to explain the linearization concept in our context. Then we specialize the treatment to the linearization of the rotation manifold $\mathcal{G}^{orth+}$.

Following the definition of B.Doolin [8], a differentiable manifold $\mathcal{M}$ is a topological space that in the neighborhood of each point looks like an open subset of $\mathcal{R}^k$. It can be imagined as a generalization of a surface in the three-dimensional space. A curve on the manifold is a map of an interval of $\mathcal{R}^1$ into a line of the manifold. A tangent vector to a curve of the manifold is the velocity vector of an object which moves along the curve, where the velocity vector as the usual meaning of derivative with respect to the curve parameter. Given a point p of the manifold, the set of all tangent vectors attached to each curve passing through p is the tangent space of the manifold at p, $T_p\mathcal{M}$. The collection of tangent spaces of all points of the manifold is called the tangent space of the manifold: $T\mathcal{M}$.

Specializing the treatment to the rotation group $\mathcal{G}^{orth+}$, it can be demonstrated that $\mathcal{G}^{orth+}$ is a manifold (see [8] page 21). The space of skew-tensors $so(3)$ is the tangent space of the rotation manifold, i.e.

$$T_{\mathbf{\Lambda}}\mathcal{G}^{orth+} = so(3) \quad \forall\, \mathbf{\Lambda}.$$

The last statement can be proved considering the rotation $\mathbf{\Lambda}$ as a one parameter function $\mathbf{\Lambda}(t)$ and then taking derivative with respect to t of the orthogonality condition (2.1.2), as done in previous sections.

With this notation in hand, we can introduce the linearization concept. Consider a manifold $\mathcal{M}$ with a generic element $\mathbf{M}$, its tangent space $T\mathcal{M}$ with a generic element $\mathbf{W}$ and a function $\mathbf{H} = \mathbf{H}(\mathbf{M}) \mid \mathbf{H} : \mathcal{M} \to \mathcal{C}$ where $\mathcal{C}$ is a generic set. We indicate the *tangent operator* or *linearization* of $\mathbf{H}$ by the notation $\delta\mathbf{H}(\mathbf{M})[\mathbf{W}]$ and we call $\mathbf{M}$ the point of linearization and $\mathbf{W}$ the direction of linearization. We state that this operator must be linear in the direction of linearization and we define it as

$$\begin{aligned}\delta\mathbf{H}(\mathbf{M})[\mathbf{W}] &= \lim_{\varepsilon\to 0}\frac{\mathbf{H}(\mathbf{M}_\varepsilon)-\mathbf{H}(\mathbf{M})}{\varepsilon} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{H}\big(\mathbf{M}_\varepsilon\big)\right]_{\varepsilon=0}. \end{aligned} \tag{2.3.1}$$

$\mathbf{M}_\varepsilon$ represent a configuration infinitesimally near to $\mathbf{M}$, obtained perturbing $\mathbf{M}$ in the direction of the tangent element $\mathbf{W}$. The perturbed configuration $\mathbf{M}_\varepsilon$ must satisfy the conditions

$$\mathbf{M}_\varepsilon \in \mathcal{M} \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\mathbf{M}_\varepsilon = \mathbf{M}. \tag{2.3.2}$$

The structure of the tangent operator and also its linear dependence on the direction of linearization result to be connected with the structure of $\mathbf{M}_\varepsilon$.

Consider now the rotation manifold $\mathcal{G}^{orth+}$ with a generic element $\mathbf{\Lambda}$ and the perturbed infinitesimal rotation indicated as $\mathbf{\Lambda}_\varepsilon$. The *tangent operator* of a generic function $\mathbf{H} = \mathbf{H}(\mathbf{\Lambda})$ in a point $\mathbf{\Lambda}$, $\delta\mathbf{H}(\mathbf{\Lambda})$, can be constructed in two ways. They depend on the structure of $\mathbf{\Lambda}_\varepsilon$ which is strictly related with the choice to perform the linearization directly on the manifold or indirectly into the linear space of rotation vectors. Hence these two ways or forms of linearization are called

- **direct linearization form**, where $\mathbf{\Lambda}_\varepsilon$ is constructed via an infinitesimal variation of the manifold element $\mathbf{\Lambda}$, on the rotation manifold ($\mathcal{G}^{orth+}$), i.e. where the lineaziation is done directly on the *manifold*.

- **indirect linearization form**, where $\mathbf{\Lambda}_\varepsilon$ is constructed via an infinitesimal variation of the rotational vector space element $\boldsymbol{\theta}$, into the rotational vector linear space, i.e. the linearization is done on the *linear space* of vectors which parameterize the rotation tensors.

For the **direct** linearization, the perturbed rotation $\mathbf{\Lambda}_\varepsilon$ can be constructed in two ways, known as spatial or material:

$$\begin{aligned}\mathbf{\Lambda}_\varepsilon &= \exp[\varepsilon\mathbf{W}_\delta]\mathbf{\Lambda} \qquad \text{with } \mathbf{W}_\delta \in T_{\mathbf{\Lambda}}\mathcal{G}^{orth+} \subset so(3) \text{ spatial},\\ \mathbf{\Lambda}_\varepsilon &= \mathbf{\Lambda}\exp[\varepsilon\mathbf{\Psi}_\delta] \qquad \text{with } \mathbf{\Psi}_\delta \in T_{\mathbf{I}}\mathcal{G}^{orth+} \subset so(3) \text{ material,}\end{aligned}$$

the detailed explanation of which is given in next sections. Note that these perturbed rotations respect the definition of the direct linearization form. They give rise to a tangent operator of the form

$$\delta\mathbf{\Lambda}[\mathbf{W}_\delta] = \lim_{\varepsilon\to 0}\frac{\exp[\varepsilon\mathbf{W}_\delta]\mathbf{\Lambda}-\mathbf{\Lambda}}{\varepsilon} \qquad \text{spatial,}$$

$$\delta\mathbf{\Lambda}[\mathbf{\Psi}_\delta] = \lim_{\varepsilon\to 0}\frac{\mathbf{\Lambda}\exp[\varepsilon\mathbf{\Psi}_\delta]-\mathbf{\Lambda}}{\varepsilon} \qquad \text{material,}$$

which is the particular case for the function $\mathbf{H} = \mathbf{I}$. It is fundamental to note that the point of linearization $\boldsymbol{\Lambda}$ and the direction of linearization $\mathbf{W}_\delta$ (or $\boldsymbol{\Psi}_\delta$) *belong to two different sets*, the first one to the manifold $\mathcal{G}^{orth+}$ while the second one to the linear space $so(3)$. It depends on the fact that we linearize directly on the manifold $\mathcal{G}^{orth+}$ which is a nonlinear manifold and consequently differs from its tangent space, $so(3)$.
For the **indirect** linearization, the perturbed rotation $\boldsymbol{\Lambda}_\varepsilon$ is constructed as

$$\boldsymbol{\Lambda}_\varepsilon = \boldsymbol{\Lambda}(\boldsymbol{\theta} + \varepsilon\delta\boldsymbol{\theta}) \equiv \exp[\boldsymbol{\Theta} + \varepsilon\delta\boldsymbol{\Theta}] \qquad \text{with} \quad \boldsymbol{\Theta}, \delta\boldsymbol{\Theta} \ \in so(3), \quad \boldsymbol{\theta}, \delta\boldsymbol{\theta} \ \in \mathcal{R}^3,$$

where $\boldsymbol{\Theta} = [\boldsymbol{\theta}\times]$. Note that this perturbed rotation respects the definition of the indirect linearization form. It gives rise to a tangent operator of the form

$$\delta\boldsymbol{\Lambda}(\boldsymbol{\Theta})[\delta\boldsymbol{\Theta}] = \lim_{\varepsilon\to 0} \frac{\exp[\boldsymbol{\Theta} + \varepsilon\delta\boldsymbol{\Theta}] - \exp[\boldsymbol{\Theta}]}{\varepsilon}.$$

In this case the point of linearization $\boldsymbol{\Theta}$ (or equivalently $\boldsymbol{\theta}$) and the direction of linearization $\delta\boldsymbol{\Theta}$ (or equivalently $\delta\boldsymbol{\theta}$) *belong to same space*, the rotational skew-tensor one (or rotational vector one). It depends on the fact that $\boldsymbol{\Lambda}$ is parametrized by the *linear space* of rotation vectors $\boldsymbol{\theta}$ and the linearization is taken into this space.

In the next paragraphs each of the two linearizations is deeply investigated. Figure (2.2) could be of help to understand the differences between the linearizations.

2.3.1 Direct linearization of rotation and spin tensors: spatial and material form

Spatial form. The quantity $\boldsymbol{\Lambda}_\varepsilon$ used in the spatial form of direct linearization

$$\boldsymbol{\Lambda}_\varepsilon = \exp[\varepsilon\mathbf{W}_\delta]\boldsymbol{\Lambda} \qquad \text{with} \ \ \mathbf{W}_\delta \in T_{\boldsymbol{\Lambda}}\mathcal{G}^{orth+} \subset so(3) \tag{2.3.3}$$

respects conditions (2.3.2), i.e. it is an infinitesimal perturbed configuration of $\boldsymbol{\Lambda}$. Since the exponential of a zero tensor is the identity tensor, from the expression of $\boldsymbol{\Lambda}_\varepsilon$ easily we have that

$$\boldsymbol{\Lambda}_\varepsilon = \boldsymbol{\Lambda} \qquad \text{for } \varepsilon = 0.$$

Moreover $\boldsymbol{\Lambda}_\varepsilon$ is a rotation since it is a product of two rotations. The order of product matrices into the expression of the perturbed rotation says that the new infinitesimal rotation $\exp[\varepsilon\mathbf{W}_\delta]$ is superimposed on the current rotation $\boldsymbol{\Lambda}$. It means that the skew tensor $\mathbf{W}_\delta$ lays in the space tangent to $\boldsymbol{\Lambda}$, or, more physically, that the rotation vector $\mathbf{w}_\delta$ of the infinitesimal rotation is a vector already affected by the rotation $\boldsymbol{\Lambda}$, i.e. it has been already rotated from its initial position to the actual one. This kind of rotation sequence, where the rotation vector of the second one is a vector that lives in the space rotated from the first one, is called by Argyris, [1], a compound of rotations around *follower axes*. For this reason $\mathbf{w}_\delta$ and the linearization form are called *spatial*.

The direct spatial linearization (or tangent operator) of the rotation tensor, $\delta\boldsymbol{\Lambda}$, is computed as stated in equation (2.3.1) by directional derivative as

$$\delta\boldsymbol{\Lambda} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\boldsymbol{\Lambda}_\varepsilon)\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\exp[\varepsilon\mathbf{W}_\delta])\Big|_{\varepsilon=0}\boldsymbol{\Lambda} = \mathbf{W}_\delta\exp[\varepsilon\mathbf{W}_\delta]\Big|_{\varepsilon=0}\boldsymbol{\Lambda}$$

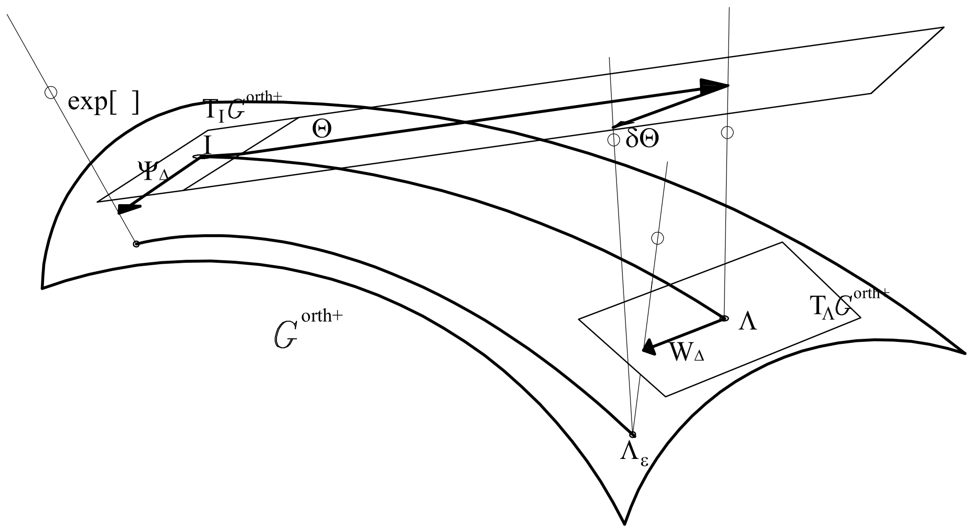


Figure 2.2: $\mathcal{G}^{orth+}$ manifold, its tangent spaces at current rotation, $\mathbf{\Lambda}$, and at $\mathbf{I}$, and the three parameterizations

where we have used the derivative of the exponential mapping as presented in section (). Evaluating the function for $\varepsilon = 0$ and recalling that $\exp[\mathbf{0}] = \mathbf{I}$ we obtain the *direct spatial rotation linearization*

$$\boxed{\delta\mathbf{\Lambda}[\mathbf{W}_\delta] = \mathbf{W}_\delta\mathbf{\Lambda}.} \tag{2.3.4}$$

The same result can be obtained in another elegant manner using directly the limit operation and the definition of exponential mapping. In fact the limit definition of $\delta\mathbf{\Lambda}$ can be written as

$$\delta\mathbf{\Lambda}[\mathbf{W}_\delta] = \lim_{\varepsilon\to 0}\frac{\exp[\varepsilon\mathbf{W}_\delta]\mathbf{\Lambda} - \mathbf{\Lambda}}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\big(\exp[\varepsilon\mathbf{W}_\delta] - \mathbf{I}\big)\mathbf{\Lambda}}{\varepsilon}.$$

Using the series expansion for the exponential of the skew-tensor $\varepsilon\mathbf{W}_\delta$ (see expression (2.2.20)), the last expression can be given as

$$\lim_{\varepsilon\to 0}\frac{\big(\exp[\varepsilon\mathbf{W}_\delta] - \mathbf{I}\big)\mathbf{\Lambda}}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{\big(\varepsilon\mathbf{W}_\delta + \varepsilon^2\mathbf{W}_\delta^2\backslash 2! + \varepsilon^3\mathbf{W}_\delta^3\backslash 3! + ...\big)\mathbf{\Lambda}}{\varepsilon}$$

which computing the limit and substituting yields to

$$\delta\mathbf{\Lambda}[\mathbf{W}_\delta] = \mathbf{W}_\delta\mathbf{\Lambda}.$$

Observe what we anticipated: the direct linearization depends on two measures, $\mathbf{\Lambda}$ and $\mathbf{W}_\delta$ which do not belong to the same space.

Focusing attention on the spin tensor $\mathbf{\Omega} = \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T$ and on its axial vector $\boldsymbol{\omega}$, its direct spatial linearization can be computed taking the directional derivative of the infinitesimal perturbed spin tensor $\mathbf{\Omega}_\varepsilon = \dot{\mathbf{\Lambda}}_\varepsilon\mathbf{\Lambda}_\varepsilon^T$ where $\mathbf{\Lambda}_\varepsilon$ is the direct spatial one. By substitution we get

$$\begin{aligned}\mathbf{\Omega}_\varepsilon &= \overline{\Big[\exp[\varepsilon\mathbf{W}_\delta]\mathbf{\Lambda}\Big]}^{\cdot}\Big[\exp[\varepsilon\mathbf{W}_\delta]\mathbf{\Lambda}\Big]^T = \\ &= \Big[\overline{\exp[\varepsilon\mathbf{W}_\delta]}^{\cdot}\mathbf{\Lambda} + \exp[\varepsilon\mathbf{W}_\delta]\dot{\mathbf{\Lambda}}\Big](\mathbf{\Lambda})^T\exp[-\varepsilon\mathbf{W}_\delta] = \\ &= \overline{\exp[\varepsilon\mathbf{W}_\delta]}^{\cdot}\exp[-\varepsilon\mathbf{W}_\delta] + \exp[\varepsilon\mathbf{W}_\delta]\mathbf{\Omega}\exp[-\varepsilon\mathbf{W}_\delta].\end{aligned} \tag{2.3.5}$$

where for convenience the derivative with respect to t has been explicitated. Consider the first addend $\overline{\exp[\varepsilon\mathbf{W}_\delta]}^{\cdot}\exp[-\varepsilon\mathbf{W}_\delta]$. As proved by Simo in [24], the term can be given in the form

$$\overline{\exp[\varepsilon\mathbf{W}_\delta]}^{\cdot}\exp[-\varepsilon\mathbf{W}_\delta] = \frac{2}{1+\parallel\boldsymbol{\rho}\parallel^2}\big(\dot{\mathbf{R}} + \mathbf{R}\dot{\mathbf{R}} - \dot{\mathbf{R}}\mathbf{R}\big)$$

with

$$\boldsymbol{\rho} = \mathrm{tang}\left(\frac{\parallel\varepsilon\mathbf{w}_\delta\parallel}{2}\right)\frac{\varepsilon\mathbf{w}_\delta}{\parallel\varepsilon\mathbf{w}_\delta\parallel}, \qquad \mathbf{R} = [\boldsymbol{\rho}\times].$$

Taking directional derivative with respect to ε evaluated at $\varepsilon = 0$, it can be proved that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\overline{\exp[\varepsilon\mathbf{W}_\delta]}^{\cdot}\exp[-\varepsilon\mathbf{W}_\delta]\Big)\Big|_{\varepsilon=0} = \dot{\mathbf{W}}_\delta. \tag{2.3.6}$$

Consider now the second addend of $\mathbf{\Omega}_\varepsilon$, $\exp[\varepsilon\mathbf{W}_\delta]\mathbf{\Omega}\exp[-\varepsilon\mathbf{W}_\delta]$. Recalling that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\exp[\varepsilon\mathbf{W}_\delta]\Big|_{\varepsilon=0} = \mathbf{W}_\delta \qquad \text{and} \qquad \exp[\mathbf{0}] = \mathbf{I},$$

using the derivative product rule we compute

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\exp[\varepsilon\mathbf{W}_\delta]\mathbf{\Omega}\exp[-\varepsilon\mathbf{W}_\delta]\Big)\Big|_{\varepsilon=0} = \mathbf{W}_\delta\mathbf{\Omega} - \mathbf{\Omega}\mathbf{W}_\delta. \tag{2.3.7}$$

Collecting the expressions of the two addend, the *direct spatial spin tensor linearization* follows

$$\delta\mathbf{\Omega} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{\Omega}_\varepsilon\Big|_{\varepsilon=0} = \dot{\mathbf{W}}_\delta + [\mathbf{W}_\delta\mathbf{\Omega} - \mathbf{\Omega}\mathbf{W}_\delta]. \tag{2.3.8}$$

Recognizing the Lie brackets (see section ()), the previous equation can be given in term of axial vectors $\delta\boldsymbol{\omega}$, $\mathbf{w}_\delta$ and $\boldsymbol{\omega}$ as

$$\boxed{\delta\boldsymbol{\omega}\times\mathbf{h} = \Big(\dot{\mathbf{w}}_\delta + \mathbf{w}_\delta\times\boldsymbol{\omega}\Big)\times\mathbf{h} \qquad \forall\,\mathbf{h}\in\mathcal{R}^3}, \tag{2.3.9}$$

which provides the *direct spatial spin vector* $\boldsymbol{\omega}$ *linearization.* This quantity will play a crucial role in the develop of the beam model.

The spatial linearization of the back spin vector $\boldsymbol{\omega}_r = \mathbf{\Lambda}^T\boldsymbol{\omega}$ will be computed directly later on in the beam kinematic.

Material form. The quantity $\mathbf{\Lambda}_\varepsilon$ used in the material form of direct linearization

$$\mathbf{\Lambda}_\varepsilon = \mathbf{\Lambda}\exp[\epsilon\mathbf{\Psi}_\delta] \qquad \text{with} \;\; \mathbf{\Psi}_\delta \in T_\mathbf{I}\mathcal{G}^{orth+} \subset so(3) \tag{2.3.10}$$

respects the conditions (2.3.2) to be an infinitesimal perturbed configuration of $\mathbf{\Lambda}$, as shown for the spatial case. The order of product matrices into the expression of the perturbed rotation says that the new infinitesimal rotation $\exp[\epsilon\mathbf{\Psi}_\delta]$ precedes in the sequence of rotations the actual one $\mathbf{\Lambda}$. It means that the skew tensor $\mathbf{\Psi}_\delta$ lays in the space tangent to the identity $\mathbf{I}$, or, more physically, that the rotation vector $\boldsymbol{\psi}_\delta$ of the infinitesimal rotation is a vector not affected by the rotation $\mathbf{\Lambda}$, i.e. it has not been already rotated. Moreover $\mathbf{\Lambda}$ continues to be the actual rotation as it was in spatial form, i.e. its rotation axis has not been rotated by previous rotations. For this reason $\boldsymbol{\psi}_\delta$ and the linearization form are called *material.* This kind of rotation sequence, where the rotation vectors do not suffer of previous rotations, is called by Argyris, [1], a compound of rotations around *fixed axes.* In that work the author proves that a sequence of two rotations around follower axes is equal to the inverted sequence around fixed axes and viceversa. This confirms that the spatial and material direct perturbed rotations $\mathbf{\Lambda}_\varepsilon$ are, indeed, equivalent.

The direct material linearization (or tangent operator) of the rotation tensor, $\delta\mathbf{\Lambda}$, is computed, as stated in equation (2.3.1), by directional derivative as

$$\delta\mathbf{\Lambda} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathbf{\Lambda}_\varepsilon)\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\mathbf{\Lambda}(\exp[\varepsilon\mathbf{\Psi}_\delta])\Big|_{\varepsilon=0} = \mathbf{\Lambda}\mathbf{\Psi}_\delta\exp[\varepsilon\mathbf{\Psi}_\delta]\Big|_{\varepsilon=0},$$

where we have used the derivative of the exponential mapping as presented in section (). Evaluating the function for $\varepsilon = 0$ and recalling the $\exp[\mathbf{0}] = \mathbf{I}$ we obtain the *direct material rotation linearization*

$$\boxed{\delta\mathbf{\Lambda}[\mathbf{\Psi}_\delta] = \mathbf{\Lambda}\mathbf{\Psi}_\delta}. \tag{2.3.11}$$

Here again note that the direct linearization depends on two measures, $\mathbf{\Lambda}$ and $\mathbf{\Psi}_\delta$ which do not belong to the same space.

We do not evaluate the spin tensor linearization in material form since we choose to use in the beam model the spatial direct form and not the material one.

An interesting analogy between directions of direct linearization and spin vectors. Explicitating the direction of linearization $\mathbf{W}_\delta$ with respect to the tangent operator $\delta\mathbf{\Lambda}$ (by reversing expression (2.3.11)) we obtain

$$\mathbf{W}_\delta = \delta\mathbf{\Lambda}\mathbf{\Lambda}^T \qquad \text{with} \quad \mathbf{W}_\delta \in so(3).$$

Exactly the same expression can be obtained formally linearizing the orthogonality condition $\mathbf{\Lambda}\mathbf{\Lambda}^T = I$.

We recall that the definition of the spin tensor

$$\mathbf{\Omega} = \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T \qquad \text{and} \quad \mathbf{\Omega} \in so(3)$$

has been obtained formally differentiating the same orthogonality condition with respect to the arbitrary parameter t from which $\mathbf{\Lambda}$ depends. Setting an analogy between the operation of linearization and of derivative with respect to an arbitrary parameter, we can see the perfect analogy between the spin tensor $\mathbf{\Omega}$ and the spatial direction of direct linearization $\mathbf{W}_\delta$. Moreover we have shown that when $\mathbf{\Lambda}$ is parameterized by its rotation vector $\boldsymbol{\theta}$ the spin vector $\boldsymbol{\omega}$ is related with t-derivative rotation vector $\dot{\boldsymbol{\theta}}$ by

$$\boldsymbol{\omega}(\boldsymbol{\theta}) = \mathbf{T}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}.$$

Because of the analogy just introduced, we can heuristically state that the same relation holds between the direction vector $\mathbf{w}_\delta$ and the linearization of rotation vector $\delta\boldsymbol{\theta}$, i.e.

$$\mathbf{w}_\delta = \mathbf{T}(\boldsymbol{\theta})\delta\boldsymbol{\theta}.$$

Considering the linearization and t-derivative of the other orthogonality condition, $\mathbf{\Lambda}^T\mathbf{\Lambda} = I$, the same analogy can be done for the material direction of direct linearization $\mathbf{\Psi}_\delta$ with the rotated back spin tensor $\mathbf{\Omega}_r$. As a consequence, considering the rotation vector parametrization case, we can state

$$\boldsymbol{\omega}_r = \mathbf{T}^T(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \rightarrow \boldsymbol{\psi}_\delta = \mathbf{T}^T(\boldsymbol{\theta})\delta\boldsymbol{\theta}.$$

To conclude the analogy, looking at the expressions relating $\mathbf{\Omega}$ with $\mathbf{\Omega}_r$ and $\boldsymbol{\omega}$ with $\boldsymbol{\omega}_r$, (2.1.8), (2.1.9) and (2.1.10), we can give the relations between spatial and material directions of direct linearization $\mathbf{W}_\delta$-$\mathbf{\Psi}_\delta$ and $\mathbf{w}_\delta$-$\boldsymbol{\psi}_\delta$

$$\mathbf{W}_\delta = \mathbf{\Lambda}\mathbf{\Psi}_\delta\mathbf{\Lambda}^T, \qquad \text{and} \qquad \mathbf{\Psi}_\delta = \mathbf{\Lambda}^T\mathbf{W}_\delta\mathbf{\Lambda},$$

$$\mathbf{w}_\delta = \mathbf{\Lambda}\boldsymbol{\psi}_\delta, \qquad \text{and} \qquad \boldsymbol{\psi}_\delta = \mathbf{\Lambda}^T\mathbf{w}_\delta.$$

All the relations presented in this paragraph have been *proved* explicitly by Ibrahimbegovic in [11].

2.3.2 Indirect linearization of rotation and spin tensors

The quantity $\mathbf{\Lambda}_\varepsilon$ used in the indirect linearization

$$\mathbf{\Lambda}_\varepsilon = \exp[\mathbf{\Theta} + \varepsilon\delta\mathbf{\Theta}] \qquad \text{with} \quad \mathbf{\Theta}, \delta\mathbf{\Theta} \ \in so(3) \tag{2.3.12}$$

respects conditions (2.3.2), i.e. it is an infinitesimal perturbed configuration of $\mathbf{\Lambda}$. It is trivial to see that $\mathbf{\Lambda}_\varepsilon = \exp[\mathbf{\Theta}]$ for $\varepsilon = 0$. Moreover $\mathbf{\Lambda}_\varepsilon \in \mathcal{G}^{orth+}$ because it is an exponential of a skew tensor. In this case the perturbed rotation is not anymore a rotation sequence since the linearization is not carried out into the group of rotation, $\mathcal{G}^{orth+}$. On the contrary, it is carried out into the *linear space* which parameterizes the rotation, i.e. the space of rotation vectors (or equivalently the space of skew-tensor associated with rotation vectors), via the chart which links $so(3)$ with $\mathcal{G}^{orth+}$, that is the *exponential map*[1]. Since the space of variation is linear, variations are is carried out on $\mathbf{\Theta}$ by usual *additive operations*, so the condition

$$\mathbf{\Theta}_\varepsilon \in so(3) \qquad \text{where} \ \ \mathbf{\Theta}_\varepsilon = \mathbf{\Theta} + \varepsilon\delta\mathbf{\Theta}$$

[1]we recall that since $so(3)$ is isomorphic with $\mathcal{R}^3$, see section (), we can refer equivalently either to the skew symmetric tensor $\mathbf{\Theta}$ or to its axial vector $\boldsymbol{\theta}$

is respected, together with $\mathbf{\Theta}_\varepsilon = \mathbf{\Theta}$ for $\varepsilon = 0$.

The indirect linearization or "tangent operator" of the rotation, $\delta\mathbf{\Lambda}$, is again calculated by the directional derivative

$$\delta\mathbf{\Lambda} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathbf{\Lambda}_\varepsilon)\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\exp[\mathbf{\Theta} + \varepsilon\delta\mathbf{\Theta}]\Big)\Big|_{\varepsilon=0}.$$

In this case we cannot use anymore the property of the exponential map derivative, as done in previous cases, since it refers to the case $\frac{\mathrm{d}}{\mathrm{d}t}(\exp[t\mathbf{A}])$ where t is a parameter and $\mathbf{A}$ a generical skew tensor, see section (). Instead we must substitute the perturbed $\mathbf{\Theta}_\varepsilon$ into the explicit expression of the exponential map (2.2.18) and carry out derivation. After some manipulations, the tangent operator $\delta\mathbf{\Lambda}$ results

$$\begin{aligned} \delta\mathbf{\Lambda}(\boldsymbol{\theta})[\delta\boldsymbol{\theta}] \quad = \quad & a_1(\theta)\delta\mathbf{\Theta} + a_2(\theta)(\delta\boldsymbol{\theta}\otimes\boldsymbol{\theta} + \boldsymbol{\theta}\otimes\delta\boldsymbol{\theta}) \\ & + b_0(\theta)(\boldsymbol{\theta}\cdot\delta\boldsymbol{\theta})\mathbf{I} + b_1(\theta)(\boldsymbol{\theta}\cdot\delta\boldsymbol{\theta})\mathbf{\Theta} + b_2(\theta)(\boldsymbol{\theta}\cdot\delta\boldsymbol{\theta})\boldsymbol{\theta}\otimes\boldsymbol{\theta} \end{aligned} \tag{2.3.13}$$

where a_i and b_i are the trigonometric functions given in section (2.2.1). Note that the indirect linearization depends on two measures, $\boldsymbol{\theta}$ and $\delta\boldsymbol{\theta}$, which belong to the *same space*. The tangent operator $\delta\mathbf{\Lambda}^T$ can be obtained from $\delta\mathbf{\Lambda}$ since $\delta\mathbf{\Lambda}^T = (\delta\mathbf{\Lambda})^T$. Hence $\delta\mathbf{\Lambda}^T$ can be obtained from $\delta\mathbf{\Lambda}$ just changing the signs to the skew tensor coefficients a_1 and b_1.

In the case of a product between $\delta\mathbf{\Lambda}$ (or $\delta\mathbf{\Lambda}^T$) and a general vector $\mathbf{a}\in\mathcal{R}^3$, if we highlight the linear (by definition) dependence on $\delta\boldsymbol{\theta}$, the result is clearly a product between an operator, function of $\mathbf{a}$ and $\boldsymbol{\theta}$, and the vector $\delta\boldsymbol{\theta}$, i.e.

$$\delta\mathbf{\Lambda}(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]\mathbf{a} = \Upsilon_{\delta\mathbf{\Lambda}}(\boldsymbol{\theta},\mathbf{a})\delta\boldsymbol{\theta} \qquad \forall\,\mathbf{a}\in\mathcal{R}^3 \tag{2.3.14}$$

$$\delta\mathbf{\Lambda}^T(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]\mathbf{a} = \Upsilon_{\delta\mathbf{\Lambda}^T}(\boldsymbol{\theta},\mathbf{a})\delta\boldsymbol{\theta} \qquad \forall\,\mathbf{a}\in\mathcal{R}^3. \tag{2.3.15}$$

As the operators Υ are quite useful for development of our beam theory using indirect linearization, we provide their expressions

$$\begin{aligned} \Upsilon_{\delta\mathbf{\Lambda}}(\boldsymbol{\theta},\mathbf{a}) = & -a_1(\theta)[\mathbf{a}\times] + a_2(\theta)(\boldsymbol{\theta}\cdot\mathbf{a})\mathbf{I} + a_2(\theta)\boldsymbol{\theta}\otimes\mathbf{a} + ... \\ & ... + b_0(\theta)\mathbf{a}\otimes\boldsymbol{\theta} + b_1(\theta)(\mathbf{\Theta}\mathbf{a}\otimes\boldsymbol{\theta}) + b_2(\theta)(\boldsymbol{\theta}\cdot\mathbf{a})\boldsymbol{\theta}\otimes\boldsymbol{\theta} \end{aligned} \tag{2.3.16}$$

$$\begin{aligned} \Upsilon_{\delta\mathbf{\Lambda}^T}(\boldsymbol{\theta},\mathbf{a}) = & \, a_1(\theta)[\mathbf{a}\times] + a_2(\theta)(\boldsymbol{\theta}\cdot\mathbf{a})\mathbf{I} + a_2(\theta)\boldsymbol{\theta}\otimes\mathbf{a} + ... \\ & ... + b_0(\theta)\mathbf{a}\otimes\boldsymbol{\theta} - b_1(\theta)(\mathbf{\Theta}\mathbf{a}\otimes\boldsymbol{\theta}) + b_2(\theta)(\boldsymbol{\theta}\cdot\mathbf{a})\boldsymbol{\theta}\otimes\boldsymbol{\theta}, \end{aligned} \tag{2.3.17}$$

where $[\mathbf{a}\times]$ is the skew tensor with $\mathbf{a}$ as axial vector.

Focusing attention on the spin tensors $\mathbf{\Omega} = \dot{\mathbf{\Lambda}}\mathbf{\Lambda}^T$ and $\mathbf{\Omega}_r = \mathbf{\Lambda}^T\dot{\mathbf{\Lambda}} = \mathbf{\Lambda}^T\mathbf{\Omega}\mathbf{\Lambda}$ and on their axial vectors, respectively, $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_r = \mathbf{\Lambda}^T\boldsymbol{\omega}$, we recall that their dependence on the rotation vector $\boldsymbol{\theta}$ is given by equations (2.2.26, 2.2.27)

$$\boldsymbol{\omega} = \mathbf{T}(\boldsymbol{\theta})\dot{\boldsymbol{\theta}} \qquad \text{and} \qquad \boldsymbol{\omega}_r = \mathbf{T}^T(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}.$$

The indirect linearizations of these vectors can be clearly computed taking linearization with respect to $\boldsymbol{\theta}$ of the previuos expressions, as follows

$$\delta\boldsymbol{\omega} = \delta\mathbf{T}\dot{\boldsymbol{\theta}} + \mathbf{T}\delta\dot{\boldsymbol{\theta}} \qquad \text{with} \qquad \delta\mathbf{T} = \delta\mathbf{T}(\boldsymbol{\theta})[\delta\boldsymbol{\theta}] \tag{2.3.18}$$

$$\delta\boldsymbol{\omega}_r = \delta\mathbf{T}^T\dot{\boldsymbol{\theta}} + \mathbf{T}^T\delta\dot{\boldsymbol{\theta}} \qquad \text{with} \qquad \delta\mathbf{T}^T = \delta\mathbf{T}^T(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]. \tag{2.3.19}$$

In order to compute them, we provide the expression of $\delta\mathbf{T}$, calculated taking directional derivative of $\mathbf{T}(\boldsymbol{\theta}+\varepsilon\delta\boldsymbol{\theta})$ with $\mathbf{T}$ from equation (2.2.28),

$$\begin{aligned}\delta\mathbf{T}(\boldsymbol{\theta})[\delta\boldsymbol{\theta}] &= a_2(\theta)\delta\boldsymbol{\Theta} + a_3(\theta)(\delta\boldsymbol{\theta}\otimes\boldsymbol{\theta}+\boldsymbol{\theta}\otimes\delta\boldsymbol{\theta}) \\ &\quad + b_1(\theta)(\boldsymbol{\theta}\cdot\delta\boldsymbol{\theta})\mathbf{I} + b_2(\theta)(\boldsymbol{\theta}\cdot\delta\boldsymbol{\theta})\boldsymbol{\Theta} + b_3(\theta)(\boldsymbol{\theta}\cdot\delta\boldsymbol{\theta})\boldsymbol{\theta}\otimes\boldsymbol{\theta}. \qquad (2.3.20)\end{aligned}$$

Note that this expression is equal to the indirect linearization $\delta\boldsymbol{\Lambda}$ (2.3.13), where the indices of a_i and b_i are incrmentated of one unit. The tangent operator $\delta\mathbf{T}^T$ can be obtained from $\delta\mathbf{T}$ since $\delta\mathbf{T}^T = (\delta\mathbf{T})^T$. Hence $\delta\mathbf{T}^T$ can be obtained from $\delta\mathbf{T}$ just changing the signs to the skew tensor coefficients a_2 and b_2.

Here again, in case of a product between $\delta\mathbf{T}$ (or $\delta\mathbf{T}^T$) and a general vector $\mathbf{a}\in\mathcal{R}^3$, it is useful to highlight dependence on $\delta\boldsymbol{\theta}$, which is done defining the operators $\Upsilon_{\delta\mathbf{T}}(\boldsymbol{\theta},\mathbf{a})$ and $\Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta},\mathbf{a})$

$$\delta\mathbf{T}(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]\mathbf{a} = \Upsilon_{\delta\mathbf{T}}(\boldsymbol{\theta},\mathbf{a})\delta\boldsymbol{\theta} \qquad \forall\ \mathbf{a}\in\mathcal{R}^3 \qquad (2.3.21)$$

$$\delta\mathbf{T}^T(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]\mathbf{a} = \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta},\mathbf{a})\delta\boldsymbol{\theta} \qquad \forall\ \mathbf{a}\in\mathcal{R}^3, \qquad (2.3.22)$$

with

$$\begin{aligned}\Upsilon_{\delta\mathbf{T}}(\boldsymbol{\theta},\mathbf{a}) &= -a_2(\theta)[\mathbf{a}\times] + a_3(\theta)(\boldsymbol{\theta}\cdot\mathbf{a})\mathbf{I} + a_3(\theta)\boldsymbol{\theta}\otimes\mathbf{a} + ... \\ &\quad ... + b_1(\theta)\mathbf{a}\otimes\boldsymbol{\theta} + b_2(\theta)(\boldsymbol{\Theta}\mathbf{a}\otimes\boldsymbol{\theta}) + b_3(\theta)(\boldsymbol{\theta}\cdot\mathbf{a})\boldsymbol{\theta}\otimes\boldsymbol{\theta} \qquad (2.3.23) \\ \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta},\mathbf{a}) &= a_2(\theta)[\mathbf{a}\times] + a_3(\theta)(\boldsymbol{\theta}\cdot\mathbf{a})\mathbf{I} + a_3(\theta)\boldsymbol{\theta}\otimes\mathbf{a} + ... \\ &\quad ... + b_1(\theta)\mathbf{a}\otimes\boldsymbol{\theta} - b_2(\theta)(\boldsymbol{\Theta}\mathbf{a}\otimes\boldsymbol{\theta}) + b_3(\theta)(\boldsymbol{\theta}\cdot\mathbf{a})\boldsymbol{\theta}\otimes\boldsymbol{\theta}, \qquad (2.3.24)\end{aligned}$$

where $[\mathbf{a}\times]$ is the skew tensor with $\mathbf{a}$ as axial vector.
An extensive treatment about the indirect linearization can be found in [11] and [21].

Chapter 3

Three-dimensional finite-deformation small-strain beam theory

In this chapter we illustrate the beam theory formulation. As we said in the introduction, the formulation is based on the beam equilibrium equations obtained from the three-dimensional *principle of virtual work* in which beam deformation and strain measures have been introduced.

Therefore section 3.1 is devoted to the computation of beam deformation and strain measures and of their variations, needed to exploit the principle of virtual work. Both direct and indirect forms of rotation variations are computed in order to present later the respective associated forms of principle of virtual. In this section we introduce an interesting decomposition of the deformation gradient which leads to a very compact formulation of kinematics and, later, of equilibrium allowing us to individuate clearly the beam strain resultants.

Section 3.2 deals with integration over the beam cross-section of three-dimensional principle of virtual work using the internal work in term of both *second Piola-Kirchhof*, **S**, and *first Piola-Kirchhof*, **P**, stress tensors. The integration leads to the one-dimensional principles where beam stress resultants and variation of beam strain resultants are identified.

In section 3.3 we introduce the small-strain hypothesis, which consists of neglecting into the Green-Lagrange strain tensor, **E**, a term quadratic in the beam strain vector. A first consequence is a simplified form of the stress resultants in the principle of virtual work. A second consequence is that postulating a linear elastic constitutive relation between **S** and the simplified form of **E**, we prove that linear elastic constitutive equations in term of strain and simplified stress resultants follow. In particular we obtain the same result postulated by Reissner [18, 19] and Simo [23, 24] in their pioneering works. Only in Simo's work [25] we have found a justification of the one-dimensional constitutive equations, while the work of other authors do not pay much attention to the subject. It must pointed out that the clear identification of the term to be neglected in $\mathbf{E}$ in order to get a small-strain theory follows from the decomposition of $\mathbf{F}$. The chapter ends with the linearization of finite-deformations small-strains model equations using the direct form of rotation linearization.

A schematical presentation of the model can be found in a number of works, as [6, 11, 13, 15, 21] and many others, which pay more attention to computational aspects than to the formulation of basic model equations. An extensive treatment of the latter can instead be found in works by Reissner [18, 19] and Simo [23, 24] where anyway the equations are obtained in a different way than in the present work. In particular Reissner follows a *direct approach*, i.e he considers the equilibrium on an infinitesimal beam element with forces and moments at extreme cross-sections and a distribuited load along the element, obtaining the one-dimensional differential equilibrium. Hence, having external loads in term of internal resultants, he directly substitutes them into the one-dimensional beam principle of virtual work to obtain the expression of beam virtual strain measures in term of kinematic quantities. Finally, by some manipulations, he find the strain-displacement relations. Instead Simo in [23] first defines the beam deformation map and the beam stress resultants as the integral over the cross-section of the first Piola-Kirchhof stress tensor. Then, introducing the map into the principle of inner power ($\mathbf{P} : \delta\dot{\mathbf{F}}$) and isolating stress resultants, he is able to recognize beam strain resultants. At the same time, introducing the stress resultant expression into the three-dimensional differential equilibrium he gets the differential one-dimensional beam equilibrium. In a later work, [24], he gets the weak form of equilibrium by testing the differential form and integrating by parts, using the direct form of rotation variations.

3.1 Kinematics

3.1.1 Geometrical description and motion of the beam

In the three-dimensional space the geometrical beam structure is described by a set of plane elements, called *cross-sections*, and by a curve which connects the cross-section centroids, called *line of centroids*. For simplicity, we assume the line of centroids to be a *straight line* at the reference configuration and the cross section to have area and shape constant along the line of centroids.

Consider a cartesian reference system $\{O;\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\}$ with coordinates $\{X_1, X_2, X_3\}$. At the reference configuration, the line of centroid is parallel to the third axis $\mathbf{E}_3$ and has length L. Moreover cross-sections are parallel to the plane identified by the axes $\mathbf{E}_1, \mathbf{E}_2$ and each cross section occupies a region denoted by $\mathcal{A}_0 \subset \mathcal{R}^2$. As a consequence, the general subset $\Omega_0 \subset \mathcal{R}^3$, which identifies a material body in three-dimensional continuum at the reference configuration, here takes the special form $\Omega_0 = \mathcal{A}_0 \times [0, L]$. The *position vector*, $\mathbf{X}$, of a beam material point *at the reference configuration* then becomes $\mathbf{X} \in \mathcal{A}_0 \times [0, L]$ and takes the form

$$\mathbf{X} = X_\alpha \mathbf{E}_\alpha + X_3 \mathbf{E}_3, \qquad \text{with} \quad (X_1, X_2) \in \mathcal{A}_0 \quad \text{and} \quad X_3 \in [0, L] \tag{3.1.1}$$

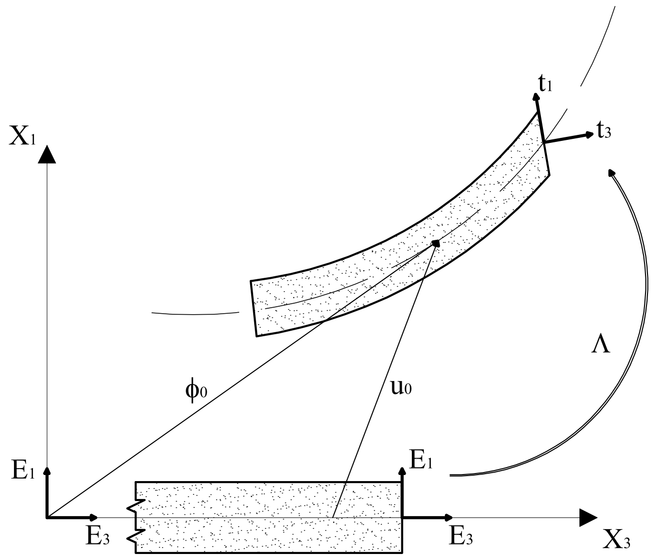


Figure 3.1: Geometric description of beam deformation. Definition of reference frame, $\mathbf{E}_I$, and moving frame, $\mathbf{t}_i$

Deformation map. To describe the change of configuration for the beam we introduce

- a vector field, $\boldsymbol{\phi}_0 = \boldsymbol{\phi}_0(X_3)$, which defines the position of the line of centroids in the three-dimensional space;

- an orthonormal frame, $\mathbf{t}_I = \mathbf{t}_I(X_3)$, such that $\mathbf{t}_1$ and $\mathbf{t}_2$ define the unit vectors laying on each cross section in the current configuration and $\mathbf{t}_3$ defines the unit vector normal to each cross section in the current configuration.

The orthonormal frame is called *moving* or *intrisic frame*. The orientation of the components $\mathbf{t}_\alpha$ with respect to the components $\mathbf{E}_\alpha$ of the reference system defines the section orientation. At the reference configuration $\boldsymbol{\phi}_0 \equiv X_3\mathbf{E}_3$ and $\mathbf{t}_I = \mathbf{E}_I$.
The current configuration of the beam, $\boldsymbol{x} \in \mathcal{R}^3$, is defined through the *deformation map function* $\boldsymbol{\phi}(X_1, X_2, X_3)$ as

$$\boxed{\boldsymbol{x} \equiv \boldsymbol{\phi}(X_1, X_2, X_3) = \boldsymbol{\phi}_0(X_3) + X_\alpha \mathbf{t}_\alpha(X_3).} \tag{3.1.2}$$

Since the moving frame $\mathbf{t}_I$ is orthonormal and function only of X_3, there exists a one-parameter orthogonal tensor $\boldsymbol{\Lambda}(X_3) \in \mathcal{G}^{orth+}$which maps uniquely the reference frame $\mathbf{E}_I$

into the moving frame $\mathbf{t}_I$ as

$$\mathbf{t}_I(X_3) = \mathbf{\Lambda}(X_3)\mathbf{E}_I. \tag{3.1.3}$$

We call $\mathbf{\Lambda}$ *cross-section rotation tensor*. Using the rotation tensor, the deformation map in (3.1.2) can be recast as

$$\boxed{\boldsymbol{x} \equiv \boldsymbol{\phi}(X_1, X_2, X_3) = \boldsymbol{\phi}_0(X_3) + X_\alpha \mathbf{\Lambda}(X_3)\mathbf{E}_\alpha.} \tag{3.1.4}$$

From (3.1.4) we note that $\boldsymbol{x}$ has a fundamental property: it is uniquely defined by the functions $\boldsymbol{\phi}_0$ and $\mathbf{\Lambda}$, since X_α is an independent field. Moreover we remark that $\boldsymbol{\phi}_0 = \boldsymbol{\phi}_0(X_3)$ and $\mathbf{\Lambda} = \mathbf{\Lambda}(X_3)$, i.e. they depend only on X_3. Hence, the *set of beam configuration spaces* $\mathcal{C}$ takes the form

$$\mathcal{C} = \{(\boldsymbol{\phi}_0, \mathbf{\Lambda}) : X_3 \rightarrow \mathcal{R}^3 \times \mathcal{G}^{orth+}\}. \tag{3.1.5}$$

The displacement field $\mathbf{u} = \boldsymbol{x} - \mathbf{X}$ following from equations (3.1.1) and(3.1.2) is

$$\mathbf{u} = \boldsymbol{\phi}_0 + X_\alpha \mathbf{t}_\alpha - X_I \mathbf{E}_I \qquad \text{with} \quad \mathbf{u} = \mathbf{u}(X_1, X_2, X_3). \tag{3.1.6}$$

Within a beam theory, expression (3.1.2) includes some important physical hypotheses, here summarized

- cross-sections remain plane in the current configuration;
- cross sections remain undeformed in their plane in the current configuration;
- the unit vector $\mathbf{t}_3$ is normal to the cross section but not necessarily tangent to the line of centroids.

The first two hypotheses follow from the fact that the deformation map is independent of any function of the section coordinates X_1, X_2. In [25], Simo and Vu-Quoc introduce a generalization of this including in the deformation map a warping phenomenon. The last hypothesis sets the independence in the current configuration between the tangent to the centroid line, $\boldsymbol{\phi}_{0,3}$, and the normal to the cross-section $\mathbf{t}_3$. This means that the theory takes into account the *shear strain* effect. The analogy with the Timoschenko hypothesis for small displacement theory is clear.

To conclude, we give some interesting properties of the cross-section rotation matrix $\mathbf{\Lambda}$. Since both the frames $\mathbf{t}_I$ and $\mathbf{E}_I$ are orthonormal, by equation (3.1.3) it can be proved that

$$\mathbf{\Lambda}(X_3) = \mathbf{t}_I(X_3) \otimes \mathbf{E}_I \tag{3.1.7}$$

if $\mathbf{E}_I$ is the standard basis in $\mathcal{R}^3$. Equation (3.1.7) points out that $\mathbf{\Lambda}$ is a *two point tensor*.

$\mathbf{\Lambda}$ transforms a vector $\mathbf{a} \in \mathcal{R}^3$ as follows

$$\mathbf{\Lambda}\mathbf{a} = a_i \mathbf{t}_i \qquad \text{with} \quad \mathbf{a} = a_I \mathbf{E}_I$$

The expression above can be obtained directly from (3.1.3) or from (3.1.7) using the tensor product definition (see [9] page 4). We see that $\mathbf{\Lambda}$ maps $\mathbf{a}$ into a vector which has the same components of $\mathbf{a}$ in the basis $\mathbf{t}_i$.

From a computational point of view, it is interesting to note that taking $\mathbf{E}_I$ as the standard basis in $\mathcal{R}^3$ the column vectors constituting $\mathbf{\Lambda}$ are respectively $\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3$ in the standard basis.

The definition of a beam deformation map is the basic step to build a beam theory from a three-dimensional continuum theory using a principle of virtual work [1]. By equation (3.1.2) we have imposed *mathematical constrains* on the three-dimensional continuum expression of the deformation map, i.e we have done a kinematical hypothesis.

3.1.2 Deformation gradient and strain measures

Before proceeding with the computation of deformation gradient, we recall that the derivative of a general one-parameter rotation matrix is $\dot{\mathbf{\Lambda}}(t) = \mathbf{\Omega}(t)\mathbf{\Lambda}(t)$ where $\mathbf{\Omega}(t)$ is the skew-symmetric spin tensor, as demonstrated in section (2.1.2). Considering that also here we deal with a one-parameter rotation, the cross-section rotation $\mathbf{\Lambda}(X_3)$, we can naturally write

$$\frac{d}{dX_3}\mathbf{\Lambda} \equiv \mathbf{\Lambda}_{,3} = \mathbf{\Omega}\mathbf{\Lambda}, \qquad \text{with} \quad \mathbf{\Omega} = \mathbf{\Omega}(X_3), \quad \mathbf{\Omega} \in so(3). \tag{3.1.8}$$

From the three-dimensional theory of nonlinear continuum mechanics the deformation gradient $\mathbf{F}$ is defined in term of the deformation map $\boldsymbol{\phi}$ as

$$\mathbf{F} = \frac{\partial \boldsymbol{\phi}}{\partial \mathbf{X}} \qquad \text{or} \qquad F_{iJ} = \frac{\partial x_i}{\partial X_J}. \tag{3.1.9}$$

Substituting into the deformation gradient definition the beam deformation map expression (3.1.2), $\mathbf{F}$ is computed as

$$\mathbf{F} = \mathbf{t}_1 \otimes \mathbf{E}_1 + \mathbf{t}_2 \otimes \mathbf{E}_2 + (\boldsymbol{\phi}_{0,3} + X_\alpha \mathbf{\Lambda}_{,3}\mathbf{E}_\alpha) \otimes \mathbf{E}_3.$$

Using the spin tensor $\mathbf{\Omega}$ to express the cross-section rotation derivative $\mathbf{\Lambda}_{,3}$, the term $X_\alpha \mathbf{\Lambda}_{,3}\mathbf{E}_\alpha$ in the previous expression can be rearranged as

$$X_\alpha \mathbf{\Lambda}_{,3}\mathbf{E}_\alpha = X_\alpha \mathbf{\Omega}\mathbf{\Lambda}\mathbf{E}_\alpha = X_\alpha \mathbf{\Omega}\mathbf{t}_\alpha,$$

where in the second equality we have used equation (3.1.3). By substitution of this term, the deformation gradient becomes

$$\mathbf{F} = \mathbf{t}_\alpha \otimes \mathbf{E}_\alpha + (\boldsymbol{\phi}_{0,3} + X_\alpha \mathbf{\Omega}\mathbf{t}_\alpha) \otimes \mathbf{E}_3.$$

Adding and subtracting the tensor $\mathbf{t}_3 \otimes \mathbf{E}_3$ into the right-hand-side, we obtain

$$\mathbf{F} = \mathbf{\Lambda} + [(\boldsymbol{\phi}_{0.3} - \mathbf{t}_3) + X_\alpha \mathbf{\Omega}\mathbf{t}_\alpha] \otimes \mathbf{E}_3,$$

[1] If we use another variational principle, as a mixed one for example, we should make hypotheses also on other quantities besides the deformation map

where we used that $\mathbf{\Lambda} = \mathbf{t_I} \otimes \mathbf{E_I}$ when $\mathbf{E_I}$ is the standard basis of $\mathcal{R}^3$. Finally, collecting rotation $\mathbf{\Lambda}$, the deformation gradient expression becomes

$$\boxed{\mathbf{F} = \mathbf{\Lambda}\Big[\mathbf{I} + [\mathbf{\Lambda}^T(\boldsymbol{\phi}_{0.3} - \mathbf{t}_3) + \mathbf{\Lambda}^T X_\alpha \mathbf{\Omega}\mathbf{t}_\alpha] \otimes \mathbf{E}_3\Big],} \tag{3.1.10}$$

or in more compact form

$$\boxed{\mathbf{F} = \mathbf{\Lambda}\mathbf{A},} \tag{3.1.11}$$

where

$$\mathbf{A} = \mathbf{I} + \mathbf{a} \otimes \mathbf{E}_3, \tag{3.1.12}$$

$$\mathbf{a} = \boldsymbol{\gamma}_r + X_\alpha \boldsymbol{\kappa}_\alpha^r, \tag{3.1.13}$$

and

$$\boldsymbol{\gamma}_r = \mathbf{\Lambda}^T \boldsymbol{\gamma} \qquad \text{with} \quad \boldsymbol{\gamma} = \boldsymbol{\phi}_{0,3} - \mathbf{t}_3, \tag{3.1.14}$$

$$\boldsymbol{\kappa}_\alpha^r = \mathbf{\Lambda}^T \boldsymbol{\kappa}_\alpha \qquad \text{with} \quad \boldsymbol{\kappa}_\alpha = \mathbf{\Omega}\mathbf{t}_\alpha = \boldsymbol{\omega} \times \mathbf{t}_\alpha = \mathbf{t}_{\alpha,3}. \tag{3.1.15}$$

We observe that

$$\boldsymbol{\gamma}_r = \mathbf{\Lambda}^T(\boldsymbol{\phi}_{0.3} - \mathbf{t}_3) = \mathbf{\Lambda}^T(\boldsymbol{\phi}_{0.3} - \mathbf{\Lambda}\mathbf{E}_3) = \mathbf{\Lambda}^T\boldsymbol{\phi}_{0.3} - \mathbf{E}_3, \tag{3.1.16}$$

$$\boldsymbol{\kappa}_\alpha^r = \mathbf{\Lambda}^T(\boldsymbol{\omega} \times \mathbf{t}_\alpha) = \mathbf{\Lambda}^T(\boldsymbol{\omega} \times \mathbf{\Lambda}\mathbf{E}_\alpha) = \boldsymbol{\omega}_r \times \mathbf{E}_\alpha, \tag{3.1.17}$$

$$\text{with} \qquad \boldsymbol{\omega}_r = \mathbf{\Lambda}^T\boldsymbol{\omega}. \tag{3.1.18}$$

In the last equality of equation (3.1.17) we have used the distributivity of cross product respect to a rotation tensor (see section).

The decomposition of the deformation gradient presented in equation (3.1.11) is a **left extended polar decomposition** of $\mathbf{F}$, [14, 28], because $\mathbf{\Lambda}$ is a pure rotation tensor and $\mathbf{A}$ is a pure stretch tensor and an upper triangular matrix.[2] We have proved in appendix that $\mathbf{A}$ is a measure of pure stretch, i.e.

$$\mathbf{A} = \mathbf{I} \qquad \text{for a rigid body motion.}$$

Consequently, the tensor $\mathbf{L}$ defined as

$$\mathbf{L} = \mathbf{A} - \mathbf{I}. \tag{3.1.19}$$

is a measure of *pure strain*, since $\mathbf{L} = \mathbf{0}$ for a rigid body motion. Recalling $\mathbf{A}$ from equations (3.1.12) we obtain

$$\mathbf{L} = \mathbf{a} \otimes \mathbf{E}_3. \tag{3.1.20}$$

The matrix representation of $\mathbf{A}$ and $\mathbf{L}$ in term of $\mathbf{a}$ is

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & a_1 \\ 0 & 1 & a_2 \\ 0 & 0 & 1 + a_3 \end{bmatrix} \qquad \mathbf{L} = \begin{bmatrix} 0 & 0 & a_1 \\ 0 & 0 & a_2 \\ 0 & 0 & a_3 \end{bmatrix} \tag{3.1.21}$$

[2] Pay attention to the fact that this tensor, $\mathbf{A}$, is not the usual tensor, $\mathbf{U}$, of the polar decomposition because the latter is symmetric (and positive definite) while $\mathbf{A}$ is not symmetric

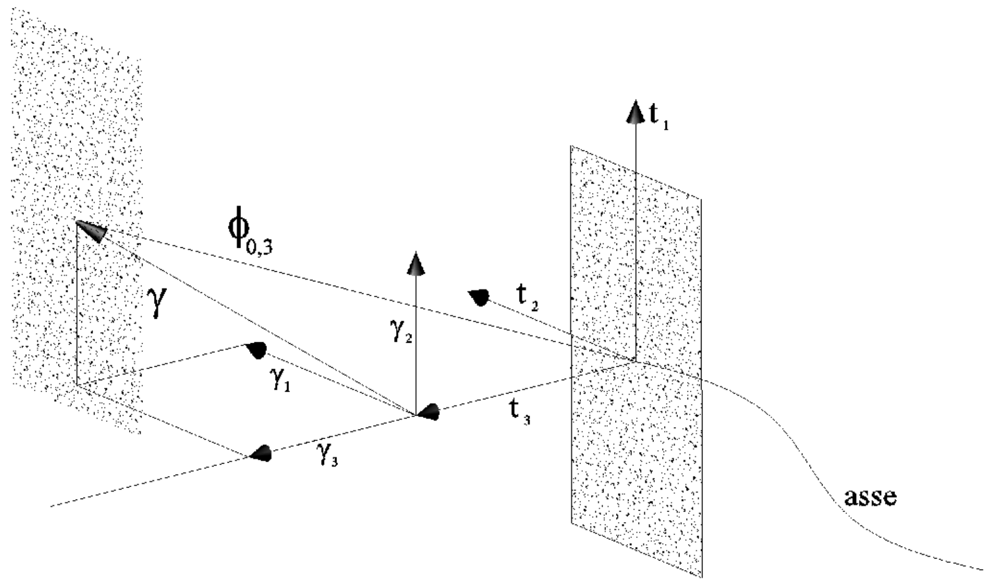


Figure 3.2: Beam strain measure $\boldsymbol{\gamma}$: physical meaning and components in the current configuration $\mathbf{t}_I$

Since $\mathbf{L}$ is a pure strain tensor, relation (3.1.20) states that all the beam strain measures are contained in the vector $\mathbf{a}$. From (3.1.13) with (3.1.14) and (3.1.15) we see that $\mathbf{a}$ depends on $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}_\alpha$ through $\boldsymbol{\gamma}_r$ and $\boldsymbol{\kappa}_\alpha^r$, hence we are interested in understanding their physical meaning.

Consider first $\boldsymbol{\gamma} = \boldsymbol{\phi}_{0,3} - \mathbf{t}_3$ and $\boldsymbol{\gamma}_r = \boldsymbol{\Lambda}^T\boldsymbol{\gamma}$. $\boldsymbol{\gamma}$ is the difference between the vector tangent to the current line of centroids ($\boldsymbol{\phi}_{0,3}$) and the unit vector orthogonal to the cross-section in the current configuration (see Figure 3.2). It is clear that the components of $\boldsymbol{\gamma}$ with respect to the current moving frame $\mathbf{t}_i$ can be interpreted as follow.

- γ_1 and γ_2, i.e. the components in directions $\mathbf{t}_1$ and $\mathbf{t}_2$ respectively, represent the shear flow between sections;

- γ_3, i.e. the component in direction $\mathbf{t}_3$, represents the elongation, or shrinkage, of an infinitesimal fiber in the direction of the cross-section normal.

It means that these components are the physical *true shear and axial strain measures.*

The vector $\boldsymbol{\gamma}_r = \boldsymbol{\Lambda}^T\boldsymbol{\gamma}$ is $\boldsymbol{\gamma}$ rotated back from the current to the reference configuration. Since, by definition, a rotation tensor $\mathbf{R}$ maps a vector, with components measured with respect to a reference system $\mathbf{m}_i$, into another vector with the same components measured with respect to the rotated reference system $\mathbf{R}\mathbf{m}_i$, it means that the components of $\boldsymbol{\gamma}$ are numerically the same of $\boldsymbol{\gamma}_r$ measured in the reference system $\boldsymbol{\Lambda}^T\mathbf{t}_I = \mathbf{E}_I$. Since we refer the beam equations to the reference configuration, i.e. $\mathbf{E}_I$ is our basis, the *shear and axial strain model measures* will be the components γ_1^r, γ_2^r, γ_3^r. This statement will be later confirmed by the equilibrium equations.

Consider now $\boldsymbol{\kappa}_\alpha = \boldsymbol{\omega} \times \mathbf{t}_\alpha$, $\boldsymbol{\omega}_r = \boldsymbol{\Lambda}^T\boldsymbol{\omega}$ and $\boldsymbol{\kappa}_\alpha^r = \boldsymbol{\Lambda}^T\boldsymbol{\kappa}_\alpha$. Since $\boldsymbol{\omega}$ is a spin vector with respect to the direction $\mathbf{t}_3$, we aspect that it controls the variation of rotation in this direction. In fact consider the case when $\boldsymbol{\omega}$ lines up with $\mathbf{t}_3$ (see Figure 3.3). The cross products $\boldsymbol{\omega} \times \mathbf{t}_1$ and $\boldsymbol{\omega} \times \mathbf{t}_2$ are vectors laying in plane $\mathbf{t}_1 - \mathbf{t}_2$ and turning around $\mathbf{t}_3$. It means that they represent the physical variation of rotation around $\mathbf{t}_3$, i.e. the cross section torsion. Similar observation can be done in the case when $\boldsymbol{\omega}$ lines up with $\mathbf{t}_1$ or $\mathbf{t}_2$. In these case the respectively cross products represent the variation of rotation around $\mathbf{t}_1$ and $\mathbf{t}_2$, i.e. the physical cross-section bending around $\mathbf{t}_1$ and $\mathbf{t}_2$. For the exposed reasons we can say that the components of $\boldsymbol{\omega}$ represent the true *bending and torsional strain model measures.* The vector $\boldsymbol{\kappa}_\alpha$ is a combination of all these variations of rotations, i.e. it represents the global curvature of the beam.

For the same reasons explained in the case of $\boldsymbol{\gamma}_r$, the components of $\boldsymbol{\omega}_r = \boldsymbol{\Lambda}^T\boldsymbol{\omega}$ will be *bending and torsional strain model measures.* Also this statement will be confirmed by the equilibrium equations.

The matrix form of $\mathbf{F}$ which arises directly from equation (3.1.9) is

$$\mathbf{F} = \begin{bmatrix} \Lambda_{11} & \Lambda_{12} & \phi_{01,3} + X_\alpha \Lambda_{1\alpha,3} \\ \Lambda_{21} & \Lambda_{22} & \phi_{02,3} + X_\alpha \Lambda_{2\alpha,3} \\ \Lambda_{31} & \Lambda_{32} & \phi_{03,3} + X_\alpha \Lambda_{3\alpha,3}. \end{bmatrix} \tag{3.1.22}$$

Note that $\mathbf{t}_I = \boldsymbol{\Lambda}_I$, as follows from (3.1.3), where $\boldsymbol{\Lambda}_\mathbf{I}$ is the *Ith* column of $\boldsymbol{\Lambda}$. We adopt $\mathbf{t}_I$ each time we want to point out the physical meaning of an expression, while we adopt $\boldsymbol{\Lambda}_I$ each time we want to point out the dependence of an expression from $\boldsymbol{\Lambda}$. For example, in $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}_\alpha$ we have used $\mathbf{t}_I$. In the matrix form of $\mathbf{F}$ we have chosen to use $\boldsymbol{\Lambda}_I$. So for the components of $\mathbf{C}$ and $\mathbf{E}$ we will use $\mathbf{t}_I$ to understand to physical meaning of their terms.

The right Cauchy-Green deformation tensor, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$, can be computed from the left extended polar decomposition of $\mathbf{F}$ as

$$\mathbf{C} = (\boldsymbol{\Lambda}\mathbf{A})^T(\boldsymbol{\Lambda}\mathbf{A}) = \mathbf{A}^T\mathbf{A} = \mathbf{I} + \mathbf{E}_3 \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{E}_3 + (\mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3. \tag{3.1.23}$$

As a consequence the Green-Lagrange strain tensor, $\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I})$, is computed as

$$\boxed{\mathbf{E} = \frac{1}{2}(\mathbf{E}_3 \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{E}_3 + (\mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3),} \tag{3.1.24}$$

or in term of $\mathbf{L}$

$$\boxed{\mathbf{E} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T + \mathbf{L}^T\mathbf{L})}. \tag{3.1.25}$$

It must be emphasized that $\mathbf{E}$, which is by definition a pure strain measure, is quadratic in the strain vector $\mathbf{a}$.

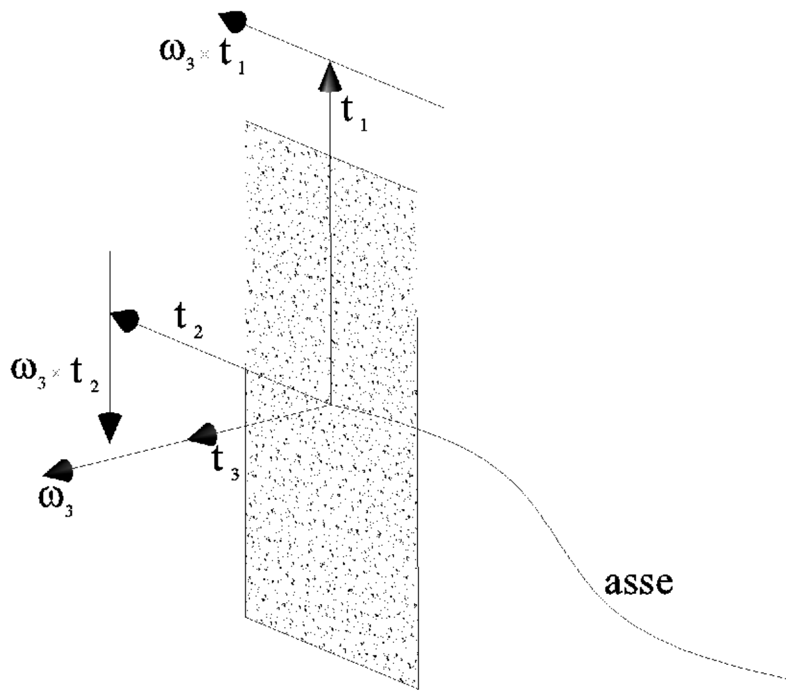


Figure 3.3: Beam strain measure $\boldsymbol{\omega}$: physical meaning and components in the current configuration $\mathbf{t}_I$

Using the extended decomposition of $\mathbf{F}$ as in (3.1.11) and the quantities afterward introduced, the expression of $\mathbf{E}$ takes a form extremely compact and at the same time understandable.

Finally we provide the matrix form of $\mathbf{C}$ and $\mathbf{E}$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & C_{13} \\ 0 & 1 & C_{23} \\ C_{31} & C_{32} & C_{33}, \end{bmatrix} \tag{3.1.26}$$

where
$C_{13} = C_{31} = \mathbf{t}_1 \cdot \boldsymbol{\phi}_{0,3} + \mathbf{t}_1 \cdot X_\alpha \mathbf{t}_{\alpha,3}$,
$C_{23} = C_{32} = \mathbf{t}_2 \cdot \boldsymbol{\phi}_{0,3} + \mathbf{t}_2 \cdot X_\alpha \mathbf{t}_{\alpha,3}$,
$C_{33} = \boldsymbol{\phi}_{0,3} \cdot \boldsymbol{\phi}_{0,3} + 2\boldsymbol{\phi}_{0,3} \cdot X_\alpha \mathbf{t}_{\alpha,3} + X_\alpha \mathbf{t}_{\alpha,3} \cdot X_\beta \mathbf{t}_{\beta,3}$,

and

$$\mathbf{E} = \begin{bmatrix} 0 & 0 & E_{13} \\ 0 & 0 & E_{23} \\ E_{31} & E_{32} & E_{33}, \end{bmatrix} \tag{3.1.27}$$

where
$E_{13} = E_{31} = \mathbf{t}_1 \cdot \boldsymbol{\phi}_{0,3} + \mathbf{t}_1 \cdot X_\alpha \mathbf{t}_{\alpha,3}$,
$E_{23} = E_{32} = \mathbf{t}_2 \cdot \boldsymbol{\phi}_{0,3} + \mathbf{t}_2 \cdot X_\alpha \mathbf{t}_{\alpha,3}$,
$E_{33} = \boldsymbol{\phi}_{0,3} \cdot \boldsymbol{\phi}_{0,3} + 2\boldsymbol{\phi}_{0,3} \cdot X_\alpha \mathbf{t}_{\alpha,3} + X_\alpha \mathbf{t}_{\alpha,3} \cdot X_\beta \mathbf{t}_{\beta,3} - 1.$

These expression arise directly from the matrix expression of $\mathbf{F}$.

Some interesting observations about E and C. From the three-dimensional continuum theory we know that $\mathbf{E}$ can be expressed also as

$$\mathbf{E} = \frac{1}{2}\Big(\nabla_{\mathbf{X}}\mathbf{u} + \nabla_{\mathbf{X}}^T\mathbf{u} + \nabla_{\mathbf{X}}^T\mathbf{u}\nabla_{\mathbf{X}}\mathbf{u}\Big), \tag{3.1.28}$$

which has the same structure as expression (3.1.25). Consider the equations (3.1.25) and (3.1.28) in the rearranged form

$$\mathbf{E} = \mathbf{L}^s + \frac{1}{2}\mathbf{L}^T\mathbf{L}, \tag{3.1.29}$$

$$\mathbf{E} = \boldsymbol{\varepsilon} + \frac{1}{2}\nabla_{\mathbf{X}}^T\mathbf{u}\nabla_{\mathbf{X}}\mathbf{u}, \tag{3.1.30}$$

where $\mathbf{L}^s = \mathrm{sym}[\mathbf{L}]$ and $\boldsymbol{\varepsilon} = \mathrm{sym}[\nabla_{\mathbf{X}}\mathbf{u}]$. The two expressions are obviously equivalent, but we remark that

$$\mathbf{L}^T\mathbf{L} \neq \nabla_{\mathbf{X}}^T\mathbf{u}\nabla_{\mathbf{X}}\mathbf{u} \qquad \text{and} \qquad \mathbf{L}^s \neq \boldsymbol{\varepsilon}. \tag{3.1.31}$$

The material displacement gradient, $\nabla_{\mathbf{X}}\mathbf{u} = \mathbf{F} - \mathbf{I}$, can be easily computed in term of $\mathbf{L}$ from (3.1.11) and (3.1.19) as

$$\nabla_{\mathbf{X}}\mathbf{u} = \boldsymbol{\Lambda} + \boldsymbol{\Lambda}\mathbf{L} - \mathbf{I}, \tag{3.1.32}$$

which gives

$$\nabla_{\mathbf{X}}^T\mathbf{u}\nabla_{\mathbf{X}}\mathbf{u} = 2\mathbf{I} + \mathbf{L} + \mathbf{L}^T - \boldsymbol{\Lambda} - \boldsymbol{\Lambda}^T - \boldsymbol{\Lambda}\mathbf{L} - \mathbf{L}^T\boldsymbol{\Lambda}^T + \mathbf{L}^T\mathbf{L}. \tag{3.1.33}$$

We see that the right-hand-side includes more terms than just $\mathbf{L}^T\mathbf{L}$, in particular it includes rigid rotations and linear strains $\mathbf{L}$. Accordingly, if we can simplify the theory for example neglecting quadratic terms in strain (small strain hypothesis) we however could not at all neglect $\nabla_{\mathbf{X}}^T\mathbf{u}\nabla_{\mathbf{X}}\mathbf{u}$, because we would neglect more than just quadratic strain but also rigid rotations and linear strains. This consideration will be recalled in section (3.3).

Moreover in this finite-deformation model $\boldsymbol{\varepsilon}$ turns to not be a pure strain measure. Using equation (3.1.32) we compute

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla_{\mathbf{X}}\mathbf{u} + \nabla_{\mathbf{X}}^T\mathbf{u}) = \frac{1}{2}(\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T\mathbf{L} - 2\mathbf{I} + \boldsymbol{\Lambda}^T + \boldsymbol{\Lambda}^T\mathbf{L}^T),$$

which for rigid body motion becomes

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\boldsymbol{\Lambda} + \boldsymbol{\Lambda}^T - 2\mathbf{I}) = \mathrm{sym}[\boldsymbol{\Lambda}] - \mathbf{I} \neq \mathbf{0}. \tag{3.1.34}$$

Instead, in the small rotation model, where $\boldsymbol{\Lambda} \approx \mathbf{I} + \boldsymbol{\Theta}$ with $\boldsymbol{\Theta}$ skew tensor, $\boldsymbol{\varepsilon}$ reduces to

$$\boldsymbol{\varepsilon} = \mathrm{sym}[\mathbf{I} + \boldsymbol{\Theta}] = \mathrm{sym}[\boldsymbol{\Theta}] + \mathbf{I} - \mathbf{I} = \mathbf{0}.$$

It confirms that in the small rotation regime $\boldsymbol{\varepsilon} = \mathrm{sym}[\nabla_{\mathbf{X}}\mathbf{u}]$ is a measure of pure strain.[3].

From the expression of $\mathbf{C}$ in (3.1.23) we can compute the measure

$$\lambda^2 = \lambda^2(\mathbf{N}) = \| \mathbf{F}\mathbf{N} \|^2 = \mathbf{C}\mathbf{N} \cdot \mathbf{N}$$

In a three-dimensional solid λ^2 is the stretching, or ratio of elongation, of a fiber passing from the reference to the current configuration and originally orientated in direction $\mathbf{N}$. Particularly easy and useful is the computation of λ^2 in directions $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$. In fact from equation (3.1.23) we get

$$\begin{aligned} \lambda^2(\mathbf{E}_1) &= \mathbf{C}\mathbf{E}_1 \cdot \mathbf{E}_1 = 1, \\ \lambda^2(\mathbf{E}_2) &= \mathbf{C}\mathbf{E}_2 \cdot \mathbf{E}_2 = 1, \\ \lambda^2(\mathbf{E}_3) &= \mathbf{C}\mathbf{E}_3 \cdot \mathbf{E}_3 = (1 + a_3)^2. \end{aligned}$$

It can be observed that

- $\lambda^2(\mathbf{E}_1)$, $\lambda^2(\mathbf{E}_2)$, $\lambda^2(\mathbf{E}_3)$ are the squares of the eigenvalues of $\mathbf{A}$,
- any fiber, originally orthogonal to the cross-section, stretch out along *its final direction* by coefficient $(1 + a_3)$,
- $\lambda(\mathbf{E}_3)$ linearly depends on the coordinate section X_α, as can be seen from the expression of a_3.

The last statement confirms that section remains plane in the current configuration.

Properties of A. Since $\mathbf{A}$ is an upper triangular matrix, its eigenvalues are the element of the principal diagonal. Then the *eigenvalues* of $\mathbf{A}$ are

$$\lambda_1 = 1, \qquad \lambda_2 = 1, \qquad \lambda_3 = 1 + a_3. \tag{3.1.35}$$

The respectively associated *eigenvectors* are

$$\mathbf{n}_1 = \left\{ \begin{array}{c} c_1 \\ c_2 \\ 0 \end{array} \right\} \qquad \mathbf{n}_2 = \left\{ \begin{array}{c} c_3 \\ c_4 \\ 0 \end{array} \right\} \qquad \mathbf{n}_3 = \left\{ \begin{array}{c} \frac{a_1}{a_3} c_5 \\ \frac{a_2}{a_3} c_5 \\ c_5 \end{array} \right\} \tag{3.1.36}$$

[3]For further information on the subject see ([26])

where $\{c_1, c_2, c_3, c_4, c_5\}$ are arbitrary constants[4]. Equations $(3.1.36)_1$ and $(3.1.36)_2$ show that the eigenvectors associated with the eigenvalue $\lambda = 1$ necessarily lay on the reference cross-section $\mathcal{A}_0$. Moreover (3.1.36) show that the eigenvectors cannot form an orthogonal triad unless to reduce the third one to the null vector.
We choose $\{c_1 = c_4 = 1, c_2 = c_3 = 0\}$ such that

$$\mathbf{n}_1 = \mathbf{E}_1, \qquad \mathbf{n}_2 = \mathbf{E}_2, \tag{3.1.37}$$

and $c_5 = a_3$ such that

$$\mathbf{n}_3 = \mathbf{a}. \tag{3.1.38}$$

Note that the vector $\mathbf{a}$ which defines $\mathbf{A}$ in (3.1.12) is also an eigenvector of $\mathbf{A}$. From the definition of eigenvectors [5]we obtain

$$\mathbf{A}\mathbf{E}_1 = \mathbf{E}_1, \qquad \mathbf{A}\mathbf{E}_2 = \mathbf{E}_2, \qquad \mathbf{A}\mathbf{a} = (1 + a_3)\mathbf{a} \tag{3.1.39}$$

which is confirmed by explicit computation. Since a cross section in the reference configuration is parameterized as $\mathcal{A}_0 = X_\alpha \mathbf{E}_\alpha$, equations $(3.1.39)_1$ and $(3.1.39)_2$ show that $\mathbf{A}$ maps the cross-section into itself.
$\mathbf{A}$ can be diagonalized. The condition for a matrix $\mathbf{H}$ to be diagonalized is

$$p - r[\mathbf{H} - \lambda_i \mathbf{I}] = m[\lambda_i]$$

where p is the matrix dimension, $r[\cdot]$ stands for the rank and $m[\lambda_i]$ stands for the molteplicity of the eigenvalue λ_i. For $\mathbf{A}$ we have that

$$\text{for} \quad \lambda = 1 \longrightarrow r[\mathbf{A} - \mathbf{I}] = 1 \quad n - r = 2 \qquad m[\lambda] = 2, \tag{3.1.40}$$

$$\text{for} \quad \lambda = 1 + a_3 \longrightarrow r[\mathbf{A} - (1 + a_3)\mathbf{I}] = 2 \quad n - r = 1 \qquad m[\lambda] = 1, \tag{3.1.41}$$

which confirm that $\mathbf{A}$ can be diagonalized.

3.1.3 Virtual variations of deformation and strain quantities

To exploit the principle of virtual work using as internal work both the form with the first Piola-Kirkhhof stress tensor ($\mathbf{P} : \delta\mathbf{F}$) and with the second Piola-Kirkhhof stress tensor ($\mathbf{S} : \delta\mathbf{E}$), we clearly need to compute

$$\delta\mathbf{F} \qquad \text{and} \qquad \delta\mathbf{E}.$$

With the introduced left extended polar decomposition of the deformation gradient $\mathbf{F}$, we are going to show that the quantities needed to compute those expressions are $\delta\boldsymbol{\Lambda}$, $\delta\boldsymbol{\phi}_{0,3}$

[4]The eigenvector $\mathbf{n}_k$ is computed resolving the system $\Big(\mathbf{A} - \lambda_k \mathbf{I}\Big)\mathbf{n}_k = \mathbf{0}$.

[5]Given a matrix $\mathbf{H}$ with eigenvalue α, the eigenvector of $\mathbf{H}$ associated to the eigenvalue α is the vector $\mathbf{g}$ such that $\mathbf{H}\mathbf{g} = \alpha\mathbf{g}$

and $\delta\boldsymbol{\omega}$.

First of all we recall that

$$\mathbf{E} = \frac{1}{2}(\mathbf{E}_3 \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{E}_3 + (\mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3)$$

and

$$\mathbf{F} = \boldsymbol{\Lambda}\mathbf{A} \qquad \text{with} \quad \mathbf{A} = \mathbf{a} \otimes \mathbf{E}_3.$$

Hence taking their linearization, $\delta\mathbf{E}$ and $\delta\mathbf{F}$ formally take the form

$$\delta\mathbf{E} = \frac{1}{2}(\mathbf{E}_3 \otimes \delta\mathbf{a} + \delta\mathbf{a} \otimes \mathbf{E}_3 + 2(\delta\mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3), \tag{3.1.42}$$

$$\delta\mathbf{F} = \delta\boldsymbol{\Lambda}\mathbf{A} + \boldsymbol{\Lambda}\delta\mathbf{A} \qquad \text{with} \quad \delta\mathbf{A} = \delta\mathbf{a} \otimes \mathbf{E}_3. \tag{3.1.43}$$

According to these expressions, we need to compute $\delta\boldsymbol{\Lambda}$ and $\delta\mathbf{a}$. We will deal with the first one below. Here we point out that

$$\begin{aligned} \mathbf{a} &= \boldsymbol{\gamma}_r + X_\alpha \boldsymbol{\kappa}_\alpha^r, \\ \text{with} \quad \boldsymbol{\gamma}_r &= \boldsymbol{\Lambda}^T(\boldsymbol{\phi}_{0,3} - \mathbf{t}_3) = \boldsymbol{\Lambda}^T \boldsymbol{\phi}_{0,3} - \mathbf{E}_3, \\ \boldsymbol{\kappa}_\alpha^r &= \boldsymbol{\omega}_r \times \mathbf{E}_\alpha. \end{aligned}$$

Hence the variation $\delta\mathbf{a}$ can be evaluated as

$$\delta\mathbf{a} = \delta\boldsymbol{\gamma}_r + \delta X_\alpha \boldsymbol{\kappa}_\alpha^r, \tag{3.1.44}$$

where, taking linearization of $\boldsymbol{\gamma}_r$ and $\boldsymbol{\kappa}_r$, we have

$$\delta\boldsymbol{\gamma}_r = \delta\boldsymbol{\Lambda}^T \boldsymbol{\phi}_{0,3} + \boldsymbol{\Lambda}^T \delta\boldsymbol{\phi}_{0,3}, \tag{3.1.45}$$

$$\delta\boldsymbol{\kappa}_\alpha^r = \delta\boldsymbol{\omega}_r \times \mathbf{E}_\alpha. \tag{3.1.46}$$

Since $\boldsymbol{\omega}_r = \boldsymbol{\Lambda}^T \boldsymbol{\omega}$, we finally note that the independent linearized measures we need in order to get $\delta\mathbf{F}$ and $\delta\mathbf{E}$ are $\delta\boldsymbol{\phi}_{0,3}$, $\delta\boldsymbol{\Lambda}$ and $\delta\boldsymbol{\omega}$, as stated at the beginning.
Since $\boldsymbol{\phi}_0$ is a vectorial field, its linearization is straightforward, resulting from the directional derivative of the perturbed field $\boldsymbol{\phi}_{0\varepsilon} = \boldsymbol{\phi}_0 + \varepsilon\delta\boldsymbol{\phi}_0$

$$\delta\boldsymbol{\phi}_0 = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\boldsymbol{\phi}_{0\varepsilon})\Big|_{\varepsilon=0} = \delta\boldsymbol{\phi}_0. \tag{3.1.47}$$

From derivative linearity it naturally follows that the linearization of $\boldsymbol{\phi}_{0,3}$ is $\delta\boldsymbol{\phi}_{0,3}$.
The crucial linearization of the rotation tensor $\boldsymbol{\Lambda}$ and of the spin vector $\boldsymbol{\omega}$ can be evaluated both in the direct and indirect form, as explained in section 2.3. Using both of them we will explicitly evaluate the variations $\delta\boldsymbol{\gamma}_r$ and $\delta\boldsymbol{\omega}_r$, which result to be fundamental in the equilibrium equations.

Direct linearization form of rotation. Using this linearization form we recall that $\delta\mathbf{\Lambda}$ is

$$\delta\mathbf{\Lambda} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\mathbf{\Lambda}_\varepsilon)\Big|_{\varepsilon=0} = \frac{\mathrm{d}}{\mathrm{d}\varepsilon}(\exp[\varepsilon\mathbf{W}_\delta])\Big|_{\varepsilon=0}\mathbf{\Lambda} = \mathbf{W}_\delta\exp[\varepsilon\mathbf{W}_\delta]\Big|_{\varepsilon=0}\mathbf{\Lambda} = \mathbf{W}_\delta\mathbf{\Lambda},$$

which entails

$$\delta\mathbf{\Lambda}^T = -\mathbf{\Lambda}^T\mathbf{W}_\delta$$

since $\mathbf{W}_\delta$ is a skew-symmetric tensor. We recall also that the linearization of the spin vector, $\delta\boldsymbol{\omega}$, is

$$\delta\boldsymbol{\omega}\times\mathbf{h} = \Big(\mathbf{w}_{\delta,3} + \mathbf{w}_\delta\times\boldsymbol{\omega}\Big)\times\mathbf{h} \qquad \forall\,\mathbf{h}\in\mathcal{R}^3.$$

With these quantities in hand we can explicit the direct form of $\delta\boldsymbol{\gamma}_r$ and $\delta\boldsymbol{\omega}_r$. By substitution into (3.1.45), the first one becomes

$$\delta\boldsymbol{\gamma}_r = -\mathbf{\Lambda}^T\mathbf{W}_\delta\boldsymbol{\phi}_{0,3} + \mathbf{\Lambda}^T\delta\boldsymbol{\phi}_{0,3}.$$

Collecting $\mathbf{\Lambda}^T$ and using the axial vector $\mathbf{w}_\delta$ of $\mathbf{W}_\delta$, the equation is recast in the final form

$$\boxed{\delta\boldsymbol{\gamma}_r = \mathbf{\Lambda}^T[\delta\boldsymbol{\phi}_{0,3} - \mathbf{w}_\delta\times\boldsymbol{\phi}_{0,3}]}. \tag{3.1.48}$$

The second linearization, $\delta\boldsymbol{\omega}_r$, can be compute as

$$\begin{aligned}\delta\boldsymbol{\omega}_r = \delta[\mathbf{\Lambda}^T\boldsymbol{\omega}] &= \delta\mathbf{\Lambda}^T\boldsymbol{\omega} + \mathbf{\Lambda}^T\delta\boldsymbol{\omega}\\ &= -\mathbf{\Lambda}^T\mathbf{W}_\delta\boldsymbol{\omega} + \mathbf{\Lambda}^T\Big(\mathbf{w}_{\delta,3} + \mathbf{w}_\delta\times\boldsymbol{\omega}\Big).\end{aligned}$$

Using the vector notation for the first addend in the last expression, $-\mathbf{\Lambda}^T\mathbf{W}_\delta\boldsymbol{\omega} = -\mathbf{\Lambda}^T(\mathbf{w}_\delta\times\boldsymbol{\omega})$ and splitting the second term, the equation can be rearranged as

$$\delta\boldsymbol{\omega}_r = -\mathbf{\Lambda}^T(\mathbf{w}_\delta\times\boldsymbol{\omega}) + \mathbf{\Lambda}^T(\mathbf{w}_\delta\times\boldsymbol{\omega}) + \mathbf{\Lambda}^T\mathbf{w}_{\delta,3},$$

which gives the final form

$$\boxed{\delta\boldsymbol{\omega}_r = \mathbf{\Lambda}^T\mathbf{w}_{\delta,3}.} \tag{3.1.49}$$

Now we have all the needed quantities to calculate $\delta\boldsymbol{\kappa}^r_\alpha$ and subsequently $\delta\mathbf{a}$ and finally $\delta\mathbf{F}$ and $\delta\mathbf{E}$. Anyway this operation will not be carried out explicitly since, as anticipated, the variations which will appear into the principles of virtual work will be directly $\delta\boldsymbol{\gamma}_r$ and $\delta\boldsymbol{\omega}_r$.

Indirect linearization form of rotation. Using this linearization form we briefly recall that the rotation tensor is directly parameterized by its rotation vector $\boldsymbol{\theta}$, $\mathbf{\Lambda} = \mathbf{\Lambda}(\boldsymbol{\theta})$, hence the linearization of the rotation is carried out with respect to the *vector space* $\boldsymbol{\theta}$

$$\delta\mathbf{\Lambda} = \delta\mathbf{\Lambda}(\hat{\boldsymbol{\theta}})[\delta\boldsymbol{\theta}]$$

where the hat specifies the point of linearization and in the following will be omitted. The linearized expression is quite long, as can be seen in section (2.3.2), and indeed not so useful for our purposes, which are to find the expressions for $\delta\boldsymbol{\gamma}_r$ and $\delta\boldsymbol{\omega}_r$. Therefore we directly focus attention on the latter ones.
In this section we computed that

$$\delta\boldsymbol{\gamma}_r = \delta\boldsymbol{\Lambda}^T\boldsymbol{\phi}_{0,3} + \boldsymbol{\Lambda}^T\delta\boldsymbol{\phi}_{0,3}.$$

First addend of the right-hand-side, $\delta\boldsymbol{\Lambda}^T\boldsymbol{\phi}_{0,3}$, using the full notation become

$$\delta\boldsymbol{\Lambda}^T\boldsymbol{\phi}_{0,3} \equiv \delta\boldsymbol{\Lambda}^T(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]\boldsymbol{\phi}_{0,3}.$$

This matrix-vector product is by definition linear in $\delta\boldsymbol{\theta}$. Noting that such product expression has been already provided in section (2.3.2) for a general vector $\mathbf{a} \in \mathcal{R}^3$, by substitution of $\mathbf{a}$ with $\boldsymbol{\phi}_{0,3}$, we obtain

$$\delta\boldsymbol{\Lambda}^T(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]\boldsymbol{\phi}_{0,3} = \Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\theta}, \boldsymbol{\phi}_{0,3})\delta\boldsymbol{\theta}. \tag{3.1.50}$$

Since it is obvious that the operator in the last equation is function of $\boldsymbol{\theta}$, for notation simplicity this dependency will be omitted in the following, hence $\Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\theta}, \boldsymbol{\phi}_{0,3}) \equiv \Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\phi}_{0,3})$. The expression of the operator is

$$\begin{aligned}\Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\phi}_{0,3}) \quad &= \quad a_1(\theta)[\boldsymbol{\phi}_{0,3}\times] + a_2(\theta)(\boldsymbol{\theta}\cdot\boldsymbol{\phi}_{0,3})\mathbf{I} + a_2(\theta)\boldsymbol{\theta}\otimes\boldsymbol{\phi}_{0,3}...\\ &... \quad +b_0(\theta)\boldsymbol{\phi}_{0,3}\otimes\boldsymbol{\theta} - b_1(\theta)[(\boldsymbol{\theta}\times\boldsymbol{\phi}_{0,3})\otimes\boldsymbol{\theta}]...\\ &... \quad +b_2(\theta)(\boldsymbol{\theta}\cdot\boldsymbol{\phi}_{0,3})\boldsymbol{\theta}\otimes\boldsymbol{\theta}.\end{aligned} \tag{3.1.51}$$

where the coefficient a_i, and b_i has been introduced in section (2.2.1). Substituting results of equations (3.1.50) and (3.1.51) into $\delta\boldsymbol{\gamma}_r$ we finally obtain

$$\boxed{\delta\boldsymbol{\gamma}_r = \Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\phi}_{0,3})\delta\boldsymbol{\theta} + \boldsymbol{\Lambda}^T\delta\boldsymbol{\phi}_{0,3}} \tag{3.1.52}$$

Let us consider now the variation $\delta\boldsymbol{\omega}_r$. In section (2.2.1) we introduced the relation $\boldsymbol{\omega}_r = \mathbf{T}^T(\boldsymbol{\theta})\dot{\boldsymbol{\theta}}$ which can be here clearly specified for the derivative with respect to the parameter X_3 as

$$\boldsymbol{\omega}_r = \mathbf{T}^T(\boldsymbol{\theta})\boldsymbol{\theta}_{,3}. \tag{3.1.53}$$

Linearizing this expression we obtain

$$\delta\boldsymbol{\omega}_r = \delta\mathbf{T}^T\boldsymbol{\theta}_{,3} + \mathbf{T}^T\delta\boldsymbol{\theta}_{,3}, \tag{3.1.54}$$

where the first addend on the right-hand-side, $\delta\mathbf{T}^T\boldsymbol{\theta}_{,3}$, indicated in full notation becomes

$$\delta\mathbf{T}^T\boldsymbol{\theta}_{,3} \equiv \delta\mathbf{T}^T(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]\boldsymbol{\theta}_{,3}.$$

This matrix-vector product is by definition linear in $\delta\boldsymbol{\theta}$. Noting that the expression of such a product has been already provided in section (2.3.2) for a general vector $\mathbf{a} \in \mathcal{R}^3$, by substitution of $\mathbf{a}$ with $\boldsymbol{\theta}_{,3}$, we obtain

$$\delta\mathbf{T}^T(\boldsymbol{\theta})[\delta\boldsymbol{\theta}]\boldsymbol{\theta}_{,3} = \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta}, \boldsymbol{\theta}_3)\delta\boldsymbol{\theta}. \tag{3.1.55}$$

Here again we simplify the notation by $\Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta},\boldsymbol{\theta}_{,3}) \equiv \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta}_{,3})$. The last operator has the expression

$$\begin{aligned}\Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta}_{,3}) &= a_2(\theta)[\boldsymbol{\theta}_{,3}\times] + a_3(\theta)(\boldsymbol{\theta}\cdot\boldsymbol{\theta}_{,3})\mathbf{I} + a_3(\theta)\boldsymbol{\theta}\otimes\boldsymbol{\theta}_{,3}...\\ ... &\quad +b_1(\theta)\boldsymbol{\theta}_{,3}\otimes\boldsymbol{\theta} - b_2(\theta)[(\boldsymbol{\theta}\times\boldsymbol{\theta}_{,3})\otimes\boldsymbol{\theta}]...\\ ... &\quad +b_3(\theta)(\boldsymbol{\theta}\cdot\boldsymbol{\theta}_{,3})\boldsymbol{\theta}\otimes\boldsymbol{\theta}. \end{aligned} \tag{3.1.56}$$

Substituting results of equations (3.1.55) and (3.1.56) into the expressions for $\delta\boldsymbol{\omega}_r$, (3.1.54), we obtain

$$\boxed{\delta\boldsymbol{\omega}_r = \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta}_{,3})\delta\boldsymbol{\theta} + \mathbf{T}^T\delta\boldsymbol{\theta}_{,3}} \tag{3.1.57}$$

As already noted in section (2.3.2), the operator $\Upsilon_{\delta\mathbf{T}^T}(\mathbf{a})$ is equal to $\Upsilon_{\delta\boldsymbol{\Lambda}^T}(\mathbf{a})$ once changing a_i with a_{i+1} and b_i with b_{i+1}.
Here again, as for the other linerization form, we do not calculate explicitly the variation $\delta\mathbf{F}$ and $\delta\mathbf{E}$.

3.2 Equilibrium: principle of virtual work

In the present section we consider the *three-dimensional principle of virtual work* in the form of integral equation *over the reference configuration*, for which the internal virtual work can be expressed equivalently both in term of tensors $\mathbf{P}$ and $\delta\mathbf{F}$ and in term of tensors $\mathbf{S}$ and $\delta\mathbf{E}$, as shown in section 1.3.

We first introduce the beam deformation and strain variations into the internal work expressions. Than we collect the terms depending on the area section and we split the volume integral into a line integral over the beam reference axis and surface integrals over beam reference cross-section. Recognizing the surface integrals as the stress and couple section resultants, it appears a line integral of the scalar product between the stress-couple resultants and the variation of beam strain resultants. This line integral is the searched *one-dimensional* beam internal work. Finally we show that the equivalence between the two forms of three-dimensional internal virtual work is preserved also between their associated one-dimensional forms.

For the moment we give the final equilibrium equations in term of the beam strain variations, $\delta\boldsymbol{\gamma}_r$ and $\delta\boldsymbol{\omega}_r$, without explicit direct or indirect rotation variations. In fact, the following computations hold for both direct and indirect forms of rotation variations, and hence there is no need to specialize the theory for one of them at the moment. Moreover the same final equations hold for a finite-strain theory since small-strain hypothesis is not yet introduced.

The three-dimensional principle of virtual work we consider is

$$\delta L = \delta L_{int} - \delta L_{ext} = 0 \quad \forall \quad \delta\mathbf{u}, \tag{3.2.1}$$

where

$$\delta L_{int} = \int_{\Omega_0}\Big[\mathbf{S}:\delta\mathbf{E}\Big]dv_0 \qquad \text{or} \qquad \delta L_{int} = \int_{\Omega_0}\Big[\mathbf{P}:\delta\mathbf{F}\Big]dv_0, \tag{3.2.2}$$

and

$$\delta L_{ext} = \int_{\Omega_0} \Big[\mathbf{b}_0 \cdot \delta\mathbf{u}\Big] dv_0 + \int_{\partial\Omega_0} \Big[\bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\Big] \mathrm{da}_0. \tag{3.2.3}$$

The explicit expression of external work is not presented in this section but in 3.4 since it depends on the choice of direct or indirect form of rotation variations. In that section we also present the strong forms of equilibrium and the Neumann boundary condition obtained by integrating by part the weak equilibrium equations.

3.2.1 Internal virtual work using $\mathbf{P}$ and $\delta\mathbf{F}$

Consider the internal virtual work

$$\delta L_{int} = \int_{\Omega_0} \Big[\mathbf{P} : \delta\mathbf{F}\Big] dv_0.$$

Taking linearization of extended left polar decomposition of deformation gradient, $\mathbf{F} = \mathbf{\Lambda}\mathbf{A}$, the expression of deformation gradient variation, $\delta\mathbf{F}$, becomes

$$\delta\mathbf{F} = \delta\mathbf{\Lambda}\mathbf{A} + \mathbf{\Lambda}\delta\mathbf{A}. \tag{3.2.4}$$

Knowing from section 2.1.2 that the linearization of a rotation tensor, $\delta\mathbf{\Lambda}$, can always be expressed as a product between a skew tensor, $\mathbf{W}_\delta$, and the rotation, i.e. $\delta\mathbf{\Lambda} = \mathbf{W}_\delta\mathbf{\Lambda}$, substituting this expression into the previuos equation we get

$$\delta\mathbf{F} = (\mathbf{W}_\delta\mathbf{\Lambda})\mathbf{A} + \mathbf{\Lambda}\delta\mathbf{A} = \mathbf{W}_\delta\mathbf{F} + \mathbf{\Lambda}\delta\mathbf{A}.$$

Introducing this expression into that of the internal work, δL_{int} becomes

$$\delta L_{int} = \int_{\Omega_0} \Big[\mathbf{P} : \mathbf{W}_\delta\mathbf{F}\Big] dv_0 + \int_{\Omega_0} \Big[\mathbf{P} : \mathbf{\Lambda}\delta\mathbf{A}\Big] dv_0. \tag{3.2.5}$$

Let us consider the first integral, $\int_{\Omega_0} \Big[\mathbf{P} : \mathbf{W}_\delta\mathbf{F}\Big] dv_0$. Its argument can be recast as

$$\mathbf{P} : \mathbf{W}_\delta\mathbf{F} = \mathbf{P}\mathbf{F}^T : \mathbf{W}_\delta.$$

Recalling from the three-dimensional equilibrium that $\mathbf{P}\mathbf{F}^T$ is a symmetric tensor, ($\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T$), since $\mathbf{W}_\delta$ is a skew-symmetric tensor their double contraction is zero. Then

$$\int_{\Omega_0} \Big[\mathbf{P} : \delta\mathbf{W}\mathbf{F}\Big] dv_0 = 0. \tag{3.2.6}$$

Let us consider now the second integral, $\int_{\Omega_0} \Big[\mathbf{P} : \mathbf{\Lambda}\delta\mathbf{A}\Big] dv_0$. Using equation (3.1.43), the expression $\mathbf{\Lambda}\delta\mathbf{A}$ can be recast in the form

$$\mathbf{\Lambda}\delta\mathbf{A} = \mathbf{\Lambda}(\delta\mathbf{a} \otimes \mathbf{E}_3) = (\mathbf{\Lambda}\delta\mathbf{a}) \otimes \mathbf{E}_3,$$

where the second equality can be easily verified exploiting index notation. Developing the double contraction, the integral becomes

$$\int_{\Omega_0} \mathbf{P} : \mathbf{\Lambda}\delta\mathbf{A}\, dv_0 = \int_{\Omega_0} \mathbf{P} : (\mathbf{\Lambda}\delta\mathbf{a} \otimes \mathbf{E}_3)\, dv_0 = \int_{\Omega_0} \mathbf{P}\mathbf{E}_3 \cdot \mathbf{\Lambda}\delta\mathbf{a}\, dv_0 = \int_{\Omega_0} \mathbf{P}_3 \cdot \mathbf{\Lambda}\delta\mathbf{a}\, dv_0, \tag{3.2.7}$$

where the second and third equalities can be computed easily exploiting index notation.

Collecting the results for the first and second integral, the internal virtual work becomes

$$\delta L_{int} = \int_{\Omega_0} [\mathbf{P}_3 \cdot \mathbf{\Lambda}\delta\mathbf{a}] dv_0. \tag{3.2.8}$$

Recalling that $\delta\mathbf{a} = \delta\boldsymbol{\gamma}_r + X_\alpha \delta\boldsymbol{\kappa}^r_\alpha$ and that $\delta\boldsymbol{\kappa}^r_\alpha = \delta\boldsymbol{\omega}_r \times \mathbf{E}_\alpha$, equation (3.2.8) becomes by substitution

$$\delta L_{int} = \int_{\Omega_0} \mathbf{P}_3 \cdot \mathbf{\Lambda}\delta\boldsymbol{\gamma}_r + \mathbf{P}_3 \cdot \mathbf{\Lambda}(\delta\boldsymbol{\omega}_r \times X_\alpha \mathbf{E}_\alpha)\;\; dv_0. \tag{3.2.9}$$

Let us focus our attention on the second addend. Using the distributivity of cross product with respect to a rotation tensor, (0.0.5), and recalling $\mathbf{\Lambda}\mathbf{E}_\alpha = \mathbf{t}_\alpha$, we get

$$\mathbf{P}_3 \cdot \mathbf{\Lambda}(\delta\boldsymbol{\omega}_r \times X_\alpha \mathbf{E}_\alpha) = \mathbf{P}_3 \cdot (\mathbf{\Lambda}\delta\boldsymbol{\omega}_r \times X_\alpha \mathbf{t}_\alpha).$$

The right-hand-side can then be rearranged using the mixed product rule as

$$\mathbf{P}_3 \cdot (\mathbf{\Lambda}\delta\boldsymbol{\omega}_r \times X_\alpha \mathbf{t}_\alpha) = \mathbf{\Lambda}\delta\boldsymbol{\omega}_r \cdot (X_\alpha \mathbf{t}_\alpha \times \mathbf{P}_3).$$

With this expression of the second addend, the internal work becomes by substitution

$$\delta L_{int} = \int_{\Omega_0} \mathbf{P}_3 \cdot \mathbf{\Lambda}\delta\boldsymbol{\gamma}_r + \mathbf{\Lambda}\delta\boldsymbol{\omega}_r \cdot (X_\alpha \mathbf{t}_\alpha \times \mathbf{P}_3)\;\; dv_0. \tag{3.2.10}$$

Noting that $\mathbf{\Lambda}\delta\boldsymbol{\gamma}_r$ and $\mathbf{\Lambda}\delta\boldsymbol{\omega}_r$ are independent of area coordinates, X_1 and X_2, the volume integral in (3.2.9) can be split into a line integral along the beam reference length and two surface integrals over the reference area sections

$$\delta L_{int} = \int_{L_0} \left[\mathbf{\Lambda}\delta\boldsymbol{\gamma}_r \cdot \int_{A_0} \mathbf{P}_3 da_0 + \mathbf{\Lambda}\delta\boldsymbol{\omega}_r \cdot \int_{A_0} [X_\alpha \mathbf{t}_\alpha \times \mathbf{P}_3] da_0 \right] dl_0. \tag{3.2.11}$$

From the first surface integral we define the cross-section *stress resultant*, $\mathbf{f}$,

$$\mathbf{f} = \int_{A_0} \mathbf{P}_3 da_0. \tag{3.2.12}$$

From the second surface integral we define the cross-section *couple resultant*, $\mathbf{m}$,

$$\mathbf{m} = \int_{A_0} [X_\alpha \mathbf{t}_\alpha \times \mathbf{P}_3] da_0. \tag{3.2.13}$$

With this notation in hand, the internal virtual work along the beam reference length takes the form

$$\boxed{\delta L_{int} = \int_{L_0} \Big[\mathbf{\Lambda} \delta \boldsymbol{\gamma}_r \cdot \mathbf{f} + \mathbf{\Lambda} \delta \boldsymbol{\omega}_r \cdot \mathbf{m} \Big] dl_0.} \tag{3.2.14}$$

We refer to this equation as the *spatial* form of the internal beam virtual work.
Expressions of $\mathbf{f}$ and $\mathbf{m}$ have a clear physical meaning. As shown in chapter (1.3), $\mathbf{P}\mathbf{n}_0 da_0 = \boldsymbol{\sigma}\mathbf{n} da$ where $\mathbf{n}_0$ and $\mathbf{n}$ are the unit normal to an internal plane respectively in reference and current configuration. Considering beam cross-sections, $\mathbf{n}_0 = \mathbf{E}_3$ and $\mathbf{n} = \mathbf{t}_3$. By substitution of the last expression into the three-dimensional ones we obtain

$$\mathbf{P}\mathbf{E}_3 da_0 = \boldsymbol{\sigma}\mathbf{t}_3 da \quad \rightarrow \quad \mathbf{P}_3 da_0 = \boldsymbol{\sigma}_3 da,$$

where $\boldsymbol{\sigma}_3$ is the vector of Cauchy's tensor components acting on the cross-section in the current configuration. Consequently

$$\int_{A_0} \mathbf{P}_3 da_0 = \int_A \boldsymbol{\sigma}_3 da,$$

which shows, recalling equation (3.2.12) that the $\mathbf{f}$ components are the integral over the current cross-section of the respective $\boldsymbol{\sigma}_3$ components, i.e. they are the stress resultant in the current configuration. Noting that $X_\alpha \mathbf{t}_\alpha$ which appears in $\mathbf{m}$ represents the distance between a section point and the centroid in the current configuration, then $\mathbf{m}$ expression shows that the components of $\mathbf{m}$ are bending and torsional moments acting on the current cross-section.

3.2.2 Internal virtual work using S and δE

Consider the expression of Green-Lagrange stain tensor variation, $\delta \mathbf{E}$, equation (3.1.42)

$$\delta \mathbf{E} = \frac{1}{2}(\mathbf{E}_3 \otimes \delta \mathbf{a} + \delta \mathbf{a} \otimes \mathbf{E}_3 + 2(\delta \mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3). \tag{3.2.15}$$

The internal virtual work

$$\delta L_{int} = \int_{\Omega_0} \Big[\mathbf{S} : \delta \mathbf{E} \Big] dv_0$$

can be computed developing the double contraction as follows. First we split $\delta \mathbf{E}$ and obtain

$$\delta L_{int} = \int_{\Omega_0} \Big[\mathbf{S} : \frac{1}{2}(\mathbf{E}_3 \otimes \delta \mathbf{a} + \delta \mathbf{a} \otimes \mathbf{E}_3) + \mathbf{S} : (\delta \mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3 \Big] dv_0.$$

The first double contraction can be computed by index notation using the symmetry of $\mathbf{S}$

$$\begin{aligned}
\mathbf{S} : \frac{1}{2}(\mathbf{E}_3 \otimes \delta \mathbf{a} + \delta \mathbf{a} \otimes \mathbf{E}_3) &= \frac{1}{2} S_{ij} E_{3i} \delta a_j + \frac{1}{2} S_{ij} E_{3j} \delta a_i = \\
&= \frac{1}{2} S_{ji} E_{3i} \delta a_j + \frac{1}{2} S_{ij} E_{3j} \delta a_i = \\
&= 2(\frac{1}{2} S_{ij} E_{3j} \delta a_i) = \\
&= \delta \mathbf{a} \cdot \mathbf{S}\mathbf{E}_3.
\end{aligned}$$

The second one easily follows by index notation computation as

$$\mathbf{S} : (\delta\mathbf{a}\cdot\mathbf{a})\mathbf{E}_3\otimes\mathbf{E}_3 = (\delta\mathbf{a}\cdot\mathbf{a})\mathbf{E}_3\cdot\mathbf{S}\mathbf{E}_3.$$

Substituting these expressions into the internal work δL_{int}, we obtain

$$\delta L_{int} = \int_{\Omega_0}\Big[\delta\mathbf{a}\cdot\mathbf{S}\mathbf{E}_3 + (\delta\mathbf{a}\cdot\mathbf{a})\mathbf{E}_3\cdot\mathbf{S}\mathbf{E}_3\Big]dv_0 \tag{3.2.16}$$

and then, collecting $\delta\mathbf{a}$, we get

$$\delta L_{int} = \int_{\Omega_0}\Big[\delta\mathbf{a}\cdot(\mathbf{S}_3 + \mathbf{a}S_{33})\Big]dv_0. \tag{3.2.17}$$

Recalling that $\delta\mathbf{a} = \delta\boldsymbol{\gamma}_r + X_\alpha\delta\boldsymbol{\kappa}_\alpha^r$ and that $\delta\boldsymbol{\kappa}_\alpha^r = \delta\boldsymbol{\omega}_r\times\mathbf{E}_\alpha$, equation (3.2.17) becomes by substitution

$$\delta L_{int} = \int_{\Omega_0}\delta\boldsymbol{\gamma}_r\cdot(\mathbf{S}_3 + \mathbf{a}S_{33}) + (\delta\boldsymbol{\omega}_r\times X_\alpha\mathbf{E}_\alpha)\cdot(\mathbf{S}_3 + \mathbf{a}S_{33})\ \ dv_0.$$

Using the mixed product rule the second addend can be rearranged in order to isolate $\delta\boldsymbol{\omega}_r$, hence the previous equation takes the form

$$\delta L_{int} = \int_{\Omega_0}\delta\boldsymbol{\gamma}_r\cdot(\mathbf{S}_3 + \mathbf{a}S_{33}) + \delta\boldsymbol{\omega}_r\cdot[X_\alpha\mathbf{E}_\alpha\times(\mathbf{S}_3 + \mathbf{a}S_{33})]\ \ dv_0.$$

Noting that $\delta\boldsymbol{\gamma}_r$ and $\delta\boldsymbol{\omega}_r$ are independent of section coordinates, X_1 and X_2, the volume integral in previous equation can be split into a line integral along the beam reference length and two surface integrals over the reference area sections, yielding

$$\delta L_{int} = \int_{L_0}\left[\delta\boldsymbol{\gamma}_r\cdot\int_{A_0}[\mathbf{S}_3 + \mathbf{a}S_{33}]da_0 + \delta\boldsymbol{\omega}_r\cdot\int_{A_0}[X_\alpha\mathbf{E}_\alpha\times(\mathbf{S}_3 + \mathbf{a}S_{33})]da_0\right]dl_0. \tag{3.2.18}$$

From the first surface integral we define the cross-section *stress resultant*, $\mathbf{f}_r$,

$$\mathbf{f}_r \equiv \int_{A_0}[\mathbf{S}_3 + \mathbf{a}S_{33}]da_0. \tag{3.2.19}$$

From the second surface integral we define the cross *couple resultant*, $\mathbf{m}_r$,

$$\mathbf{m}_r \equiv \int_{A_0}[X_\alpha\mathbf{E}_\alpha\times(\mathbf{S}_3 + \mathbf{a}S_{33})]da_0. \tag{3.2.20}$$

With this notation in hand, the internal virtual work along the beam reference length takes the form

$$\boxed{\delta L_{int} = \int_{L_0}\Big[\delta\boldsymbol{\gamma}_r\cdot\mathbf{f}_r + \delta\boldsymbol{\omega}_r\cdot\mathbf{m}_r\Big]dl_0.} \tag{3.2.21}$$

We refer to this equation as the *material* form of the internal beam virtual work.
The physical meaning of $\mathbf{f}_r$ and $\mathbf{m}_r$ does not appear so clear looking at their integral

definition. In the next section we show that they are the rotated-back expression of $\mathbf{f}$ and $\mathbf{m}$. Note that the strain measure $\mathbf{a}$ appears in the integral expression of force.

Observe that in the internal work only the components of $\mathbf{S}$ acting on the reference cross section, i.e. S_{33}, S_{13}, S_{23}, appears. It means that $S_{11}, S_{22}, S_{12} = S_{21}$ naturally do not come into the internal work, without the necessity to enforce any hypothesis on them. The reason for this derives from the fact that $\delta E_{11}, \delta E_{22}, \delta E_{12} = \delta E_{21}$ are zero (see matrix expression of $\mathbf{E}$ in (3.1.27)), as a consequence of the particular deformation map assumed. Hence it is not correct to say that we assume components of stress to be null on the planes parallel to axis of beam.

Finally we note that

$$\mathbf{S}_3 + \mathbf{a} S_{33} = (\mathbf{I} + \mathbf{a} \otimes \mathbf{E}_3)\mathbf{S}_3 = \mathbf{A}\mathbf{S}_3. \tag{3.2.22}$$

By this equality, equations (3.2.17), (3.2.19), (3.2.20) can be respectively recast as

$$\delta L_{int} = \int_{\Omega_0} \Big[\delta \mathbf{a} \cdot \mathbf{A}\mathbf{S}_3\Big] dv_0, \tag{3.2.23}$$

$$\mathbf{f}_r = \int_{A_0} [\mathbf{A}\mathbf{S}_3] da_0, \tag{3.2.24}$$

$$\mathbf{m}_r = \int_{A_0} [X_\alpha \mathbf{E}_\alpha \times \mathbf{A}\mathbf{S}_3] da_0. \tag{3.2.25}$$

3.2.3 Map between material and spatial beam internal work

In this section we refer to *spatial* internal virtual work, equation (3.2.14), with the symbol δL^P_{int} and to the *material* internal virtual work, equation(3.2.21), with the symbol δL^S_{int}. Observing the spatial work in equation (3.2.14) we note that stress and couple resultants, $\mathbf{f}$ and $\mathbf{m}$, work respectively for $\mathbf{\Lambda}\delta\boldsymbol{\gamma}_r$ and $\mathbf{\Lambda}\delta\boldsymbol{\omega}_r$, which are the virtual variations, $\delta\boldsymbol{\gamma}_r$, $\delta\boldsymbol{\omega}_r$ *rotated-forward* from the reference to the current configuration. This observation encourages us to investigate which relation links $\mathbf{f}$ and $\mathbf{m}$ with $\mathbf{f}_r$ and $\mathbf{m}_r$, the stress and couple resultants work conjugate with $\delta\boldsymbol{\gamma}_r$ and $\delta\boldsymbol{\omega}_r$.

Recalling the equation relating the first and the second Piola-Kirchhoff stress tensors, $\mathbf{P} = \mathbf{F}\mathbf{S}$, it follows that

$$\mathbf{P}_3 = \mathbf{P}\mathbf{E}_3 = \mathbf{F}\mathbf{S}\mathbf{E}_3 = \mathbf{F}\mathbf{S}_3.$$

Using extend polar decomposition of deformation gradient ($\mathbf{F} = \mathbf{\Lambda}\mathbf{A}$), the surface integrals $\mathbf{f} = \int_{A_0} \mathbf{P}_3 da_0$ and $\mathbf{m} = \int_{A_0} [X_\alpha \mathbf{t}_\alpha \times \mathbf{P}_3] da_0$ can be written, substituting $\mathbf{P}_3$, as

$$\mathbf{f} = \int_{A_0} \mathbf{\Lambda}\mathbf{A}\mathbf{S}_3 da_0, \qquad \mathbf{m} = \int_{A_0} [X_\alpha \mathbf{t}_\alpha \times \mathbf{\Lambda}\mathbf{A}\mathbf{S}_3] da_0.$$

Since $\mathbf{\Lambda}$ can be taken out of the area integral, we finally get

$$\mathbf{f} = \mathbf{\Lambda} \int_{A_0} \mathbf{A}\mathbf{S}_3 da_0 = \mathbf{\Lambda}\mathbf{f}_r,$$

$$\mathbf{m} = \int_{A_0} [\mathbf{\Lambda}(X_\alpha \mathbf{E}_\alpha \times \mathbf{A}\mathbf{S}_3)] da_0 = \mathbf{\Lambda} \int_{A_0} [X_\alpha \mathbf{E}_\alpha \times \mathbf{A}\mathbf{S}_3] da_0 = \mathbf{\Lambda}\mathbf{m}_r.$$

The obtained map

$$\boxed{\mathbf{f} = \mathbf{\Lambda}\mathbf{f}_r} \qquad \boxed{\mathbf{m} = \mathbf{\Lambda}\mathbf{m}_r} \tag{3.2.26}$$

states that stress and couple resultants, $\mathbf{f}$ and $\mathbf{m}$, in the spatial internal virtual work are the *rotated-forward* stress and couple resultants, $\mathbf{f}_r$ and $\mathbf{m}_r$, of the material internal virtual work.

This result, even if obtained by simple manipulations, is very interesting. Using it into equation (3.2.14), the spatial work δL^P_{int} takes the form

$$\delta L^P_{int} = \int_{L_0} \Big[\mathbf{\Lambda}\delta\boldsymbol{\gamma}_r \cdot \mathbf{\Lambda}\mathbf{f}_r + \mathbf{\Lambda}\delta\boldsymbol{\omega}_r \cdot \mathbf{\Lambda}\mathbf{m}_r\Big] dl_0, \tag{3.2.27}$$

which, compared with the material internal work, δL^S_{int}

$$\delta L^S_{int} = \int_{L_0} \Big[\delta\boldsymbol{\gamma}_r \cdot \mathbf{f}_r + \delta\boldsymbol{\omega}_r \cdot \mathbf{m}_r\Big] dl_0,$$

provides the following two important results

1. the resultants and their conjugate virtual variations in the *spatial* internal virtual work are the resultants and the conjugate virtual variations of the *material* internal virtual work both *rotated-forward* by the actual rotation $\mathbf{\Lambda}$;

2. the two forms of the internal beam work are confirmed to be equal, since from the definition of an orthogonal tensor it follows that

   $$\mathbf{\Lambda}\delta\boldsymbol{\gamma}_r \cdot \mathbf{\Lambda}\mathbf{f}_r = \delta\boldsymbol{\gamma}_r \cdot \mathbf{f}_r, \qquad \mathbf{\Lambda}(\mathbf{\Lambda}^T\delta\mathbf{w}_{,3}) \cdot \mathbf{\Lambda}\mathbf{m}_r = \mathbf{\Lambda}^T\delta\mathbf{w}_{,3} \cdot \mathbf{m}_r,$$

   where we recall that $\mathbf{\Lambda}$ is a rotation tensor.

Finally, it is of interest to deeply investigate what represent the expressions $\mathbf{\Lambda}\delta\boldsymbol{\gamma}_r$ and $\mathbf{\Lambda}\delta\boldsymbol{\omega}_r$ conjugate, in the spatial work, respectively with $\mathbf{f}$ and $\mathbf{m}$. Recalling the expression of $\boldsymbol{\gamma}_r$ and $\boldsymbol{\omega}_r$

$$\boldsymbol{\gamma}_r = \mathbf{\Lambda}^T\boldsymbol{\gamma}, \qquad \boldsymbol{\omega}_r = \mathbf{\Lambda}^T\boldsymbol{\omega},$$

by a simple substitution we can write

$$\mathbf{\Lambda}\delta\boldsymbol{\gamma}_r = \mathbf{\Lambda}\delta(\mathbf{\Lambda}^T\boldsymbol{\gamma}), \qquad \mathbf{\Lambda}\delta\boldsymbol{\omega}_r = \mathbf{\Lambda}\delta(\mathbf{\Lambda}^T\boldsymbol{\omega}).$$

The last equations state that $\boldsymbol{\gamma}$ and $\boldsymbol{\omega}$ are first rotated back to the reference configuration, than linearized, and finally rotated-forward again to the current configuration, i.e. $\mathbf{\Lambda}\delta\boldsymbol{\gamma}_r$ and $\mathbf{\Lambda}\delta\boldsymbol{\omega}_r$ are a *Lie derivative* of respectively $\boldsymbol{\gamma}$ and $\boldsymbol{\omega}$ with the rotation tensor $\mathbf{\Lambda}$ being the map for pull-back and push-forward. Hence, the internal work δL^P_{int} can also be given in the form

$$\delta L^P_{int} = \int_{L_0} \Big[\delta_{\mathbf{\Lambda}}\boldsymbol{\gamma} \cdot \mathbf{f} + \delta_{\mathbf{\Lambda}}\boldsymbol{\omega} \cdot \mathbf{m}\Big] dl_0, \tag{3.2.28}$$

where the symbol $\delta_{\mathbf{\Lambda}}$ stands for the Lie derivative.

3.3 Small-strain hypothesis and constitutive equations

This section is devoted to the introduction and to the explanation of the small-strain hypothesis which consists of a reduction of the full expression of the Green-Lagrange strain tensor $\mathbf{E}$. Attention will be given to the consequent reduced expression of the internal virtual work and to the constitutive equations that can be obtained in the context of this hypothesis. The relation between constitutive equations and the small-strain hypothesis will be in particular investigated since in literature it results to be not so clear a point while in our presentation it results to be straightforward.
Consider the expression of $\mathbf{E}$ in term of the pure strain tensor $\mathbf{L}$

$$\mathbf{E} = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T + \mathbf{L}^T\mathbf{L}).$$

or, substituting the expression $\mathbf{L} = \mathbf{a} \otimes \mathbf{E}_3$ which is useful to make evident the strain vector $\mathbf{a}$,

$$\mathbf{E} = \frac{1}{2}(\mathbf{E}_3 \otimes \mathbf{a} + \mathbf{a} \otimes \mathbf{E}_3 + (\mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3).$$

The *small strain hypothesis* consists of approximating $\mathbf{E}$ by neglecting the term $\mathbf{L}^T\mathbf{L} = (\mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3$, which is quadratic in the pure strain tensor $\mathbf{L}$ or in the pure strain vector $\mathbf{a}$. Therefore the approximated Green-Lagrange strain tensor $\mathbf{E}^*$ is

$$\mathbf{E} \approx \mathbf{E}^* = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) = \mathbf{L}^s, \qquad \text{or} \qquad \mathbf{E} \approx \mathbf{E}^* = (\mathbf{E}_3 \otimes \mathbf{a})^s. \tag{3.3.1}$$

The exact identification of the term quadratic in pure strain follows from the parametrization of Green-Lagrange strain tensor in term of $\mathbf{L}$ which is a direct consequence of introduction of the left extended polar decomposition of the deformation gradient. The only reference where we have found a similar development of the small-strain hypothesis is [16], which anyway deals only with plane problems.
Before investigating the consequences of the small-strain assumption, it must be pointed out that neglecting $\mathbf{L}^T\mathbf{L}$ in $\mathbf{E} = \mathbf{E}(\mathbf{L})$ is very different than neglecting $\nabla_X^T\mathbf{u}\nabla_X\mathbf{u}$ in $\mathbf{E} = \mathbf{E}(\nabla_X\mathbf{u})$, i.e.

$$\mathbf{E}^* = \frac{1}{2}(\mathbf{L} + \mathbf{L}^T) \quad \neq \quad \boldsymbol{\varepsilon} = \mathrm{sym}[\nabla_{\mathbf{x}}\mathbf{u}].$$

In fact, the quadratic term in $\nabla_X\mathbf{u}$ contains rigid rotations and linear term in $\mathbf{L}$ which cannot be neglected in a large displacement theory. Moreover $\boldsymbol{\varepsilon}^* = \nabla_X^s\mathbf{u}$ is not a pure strain measure in a large displacement beam theory, i.e. $\boldsymbol{\varepsilon}^* \neq \mathbf{0}$ for rigid body motion. For a detailed description see section (3.1).

3.3.1 Internal work with small-strain hypothesis

The linearization $\mathbf{E}^*$, can be easily computed from equation (3.3.1) as

$$\delta\mathbf{E}^* = \frac{1}{2}(\mathbf{E}_3 \otimes \delta\mathbf{a} + \delta\mathbf{a} \otimes \mathbf{E}_3). \tag{3.3.2}$$

Consider the approximated internal virtual work

$$\delta L_{int} \approx \delta L^*_{int} = \int_{\Omega_0} \mathbf{S} : \delta \mathbf{E}^* dv_0.$$

Substituting expression of $\delta \mathbf{E}^*$ and than referring to the computations done in section (3.2.2), it follows that

$$\delta L^*_{int} = \int_{\Omega_0} \mathbf{S} : \frac{1}{2}(\mathbf{E}_3 \otimes \delta \mathbf{a} + \delta \mathbf{a} \otimes \mathbf{E}_3)\, dv_0 = \int_{\Omega_0} \delta \mathbf{a} \cdot \mathbf{S}_3 dv_0. \tag{3.3.3}$$

Comparing this expression with the complete expression of δL_{int} we note that the term $\mathbf{S} : (\delta \mathbf{a} \cdot \mathbf{a})\mathbf{E}_3 \otimes \mathbf{E}_3$ does not appear in the reduced internal work. The term is in fact the quadratic one in the strain vector $\mathbf{a}$. Finally the small-strain internal work, δL^*_{int}, can indeed be obtained by the approximation of the finite-strain internal work as follows

$$\delta L_{int} = \int_{\Omega_0} [\delta \mathbf{a} \cdot (\mathbf{S}_3 + \mathbf{a} S_{33})]\, dv_0 \quad \approx \quad \delta L^*_{int} = \int_{\Omega_0} [\delta \mathbf{a} \cdot \mathbf{S}_3]\, dv_0,$$

or, since $\mathbf{S}_3 + \mathbf{a} S_{33} = \mathbf{A}\mathbf{S}_3$, as

$$\delta L_{int} = \int_{\Omega_0} [\delta \mathbf{a} \cdot \mathbf{A}\mathbf{S}_3]\, dv_0 \quad \approx \quad \delta L^*_{int} = \int_{\Omega_0} [\delta \mathbf{a} \cdot \mathbf{S}_3]\, dv_0.$$

Clearly the stress and couple resultant $\mathbf{f}_r$ and $\mathbf{m}_r$ identified in the finite-strain internal work, in the small-strain theory get the approximated form

$$\mathbf{f}_r = \int_{A_0} \mathbf{A}\mathbf{S}_3\, da_0 \quad \approx \quad \mathbf{f}^*_r = \int_{A_0} \mathbf{S}_3,\ da_0 \tag{3.3.4}$$

$$\mathbf{m}_r = \int_{A_0} (X_\alpha \mathbf{E}_\alpha \times \mathbf{A}\mathbf{S}_3)\, da_0 \quad \approx \quad \mathbf{m}^*_r = \int_{A_0} (X_\alpha \mathbf{E}_\alpha \times \mathbf{S}_3)\, da_0. \tag{3.3.5}$$

It must be pointed out that the small-strain hypothesis cannot be imposed in a compact form on the three-dimensional internal virtual work in term of $\mathbf{P}$ and $\delta \mathbf{F}$. The reason can be understood comparing the two equivalent internal works

$$\int_{\Omega_0} \Big[\mathbf{S} : \delta \mathbf{E}\Big]\, dv_0 \ \text{ and } \ \int_{\Omega_0} \Big[\mathbf{P} : \delta \mathbf{F}\Big]\, dv_0 = \int_{\Omega_0} \Big[\mathbf{F}\mathbf{S} : \delta \mathbf{F}\Big]\, dv_0,$$

where in the second one we used $\mathbf{P} = \mathbf{F}\mathbf{S}$. In the first one all the deformation is into $\delta \mathbf{E}$, and therefore we have a quadratic term in the pure strain $\mathbf{a}$. Differently, in the second one, part of the deformation is "hidden" into $\mathbf{P}$ which in fact is double contracted with a linear local measure of deformation $\delta \mathbf{F}$. For this reason it is not possible to explicit a small-strain hypothesis when using the internal work in term of $\mathbf{P}$.

Anyway we have demonstrated in section 3.2.3 that the two beam internal work expressions, material and spatial, are fully equivalent so here and in the following we will refer to the material one. We remark that the linearization procedure gets quite different using material rather than spatial work, since in the second expression every vector is multiplied by $\mathbf{\Lambda}$. However in section 3.4 we will show explicitly that, in the small-strain hypothesis and after the introduction of constitutive equations, the two forms are not only equivalent but formally equal.

3.3.2 Constitutive equations

Hypotheses and three-dimensional equations

We assume the three-dimensional linear elastic constitutive equation, $\mathbf{S}_3 = \mathbf{S}_3(\mathbf{E}_3^*)$, as

$$\mathbf{S}_3 = \mathbf{C}\mathbf{E}_3^* \qquad \text{where} \quad \mathbf{C} = \text{diag}[2G, 2G, E], \tag{3.3.6}$$

which by components takes form

$$S_{13}(= S_{31}) = 2GE_{13}^*; \qquad S_{23}(= S_{32}) = 2GE_{23}^*; \qquad S_{33} = EE_{33}^*,$$

where E is the *Young's modulus* and G is the *shear modulus.* Note that this relation does not refer to components $S_{11}, S_{22}, S_{12} = S_{21}$, since they do not appear into the principle of virtual work. Moreover it is imposed on the *approximated* form of the Green-Lagrange strain tensor $\mathbf{E}^*$, so it holds only in the context of small-strains.

These kind of constitutive equations, which are standard for an elastic beam in small displacements, completely neglects the effects of strain E_{ij} in directions $k \neq i$, i.e. the stress tensor components depend only on the strain components in the same direction and the Poisson coefficient, ν, doesn't appear in the law. All of this is in agreement with the kinematic hypothesis of no section deformability.

Note that the constitutive equation is posed on three-dimensional stress and strain tensors, which is unusual in large displacement beam theories[6], and not on the beam stress-couple resultants,

Since $\mathbf{E}^* = \text{sym}[\mathbf{a} \otimes \mathbf{E}_3]$, $\mathbf{E}_3^*$ can be easily computed as

$$\mathbf{E}_3^* = \frac{1}{2}(\mathbf{a} \otimes \mathbf{E}_3 + \mathbf{E}_3 \otimes \mathbf{a})\mathbf{E}_3 = \mathbf{N}\mathbf{a}, \qquad \text{where} \quad \mathbf{N} = \text{diag}[\frac{1}{2}, \frac{1}{2}, 1].$$

By substituting the previous equation in the constitutive equation (3.3.6), the latter becomes

$$\mathbf{S}_3 = \mathbf{D}\mathbf{a} \qquad \text{where} \quad \mathbf{D} = \mathbf{C}\mathbf{N} = \text{diag}[G, G, E]. \tag{3.3.7}$$

Integration of 3d constitutive equations over the cross-section

We are interested in analyzing the assumed expressions of material stress and couple resultants, $\mathbf{f}_r^*$ and $\mathbf{m}_r^*$, when the integration over the reference area section is carried out using the constitutive equation (3.3.7). The integration is done with respect to the *centroidal reference system.*

Consider the stress resultant $\mathbf{f}_r^* = \int_{A_0} \mathbf{S}_3 \, da_0$ and the constitutive law $\mathbf{S}_3 = \mathbf{D}\mathbf{a}$ with $\mathbf{D}$ constant. Substituting the latter into the stress resultant expression and recalling that $\mathbf{a} = \boldsymbol{\gamma}_r + \boldsymbol{\kappa}_r$, $\mathbf{f}_r^*$ can be given as

$$\mathbf{f}_r^* = \int_{A_0} \mathbf{D}\boldsymbol{\gamma}_r da_0 + \int_{A_0} \mathbf{D}\boldsymbol{\kappa}_r da_0,$$

[6] In [25] can be found a three-dimensional approach for the constitutive law of a large displacements beam

where $\boldsymbol{\kappa}_r = \boldsymbol{\omega}_r \times X_\alpha \mathbf{E}_\alpha$ with $\boldsymbol{\omega}_r = \boldsymbol{\omega}_r(X_3)$. The second integral vanishes because it is linear in the area coordinate X_α; hence, since $\boldsymbol{\gamma}_r$ is independent on the area coordinate, the resultant stress $\mathbf{f}_r^*$ is integrated as

$$\mathbf{f}_r^* = A_0 \mathbf{D} \boldsymbol{\gamma}_r = \mathbf{C}_f \boldsymbol{\gamma}_r, \qquad \text{where} \quad \mathbf{C}_f = \text{diag}[A_0 G, A_0 G, A_0 E], \tag{3.3.8}$$

where A_0 is the area of the cross-section.

Consider the couple resultant $\mathbf{m}_r^* = \int_{A_0} X_\alpha \mathbf{E}_\alpha \times \mathbf{S}_3 \, da_0$. Substituting the constitutive law into the expression, here again we can split the integral into two integrals

$$\mathbf{m}_r^* = \int_{A_0} X_\alpha \mathbf{E}_\alpha \times \mathbf{D} \boldsymbol{\gamma}_r \, da_0 + \int_{A_0} X_\alpha \mathbf{E}_\alpha \times \mathbf{D} \boldsymbol{\kappa}_r \, da_0.$$

Since $\boldsymbol{\gamma}_r$ is independent on the area coordinate, the first integral vanishes because linear in the area coordinate X_α. Using the expression of $\boldsymbol{\kappa}_r$, $\mathbf{m}_r^*$ can therefore be given as

$$\mathbf{m}_r^* = \int_{A_0} X_\alpha \mathbf{E}_\alpha \times \mathbf{D}[\boldsymbol{\omega}_r \times X_\beta \mathbf{E}_\beta] da_0.$$

The integration follows. For notation simplicity, let us pose $\boldsymbol{\omega}_r = \mathbf{g}$. Since $X_\alpha \mathbf{E}_\alpha = \{X_1, X_2, 0\}$, the cross product $\mathbf{g} \times X_\alpha \mathbf{E}_\alpha$ becomes

$$\mathbf{g} \times X_\alpha \mathbf{E}_\alpha = -g_3 X_2 \mathbf{E}_1 + g_3 X_1 \mathbf{E}_2 + (g_1 X_2 - g_2 X_1) \mathbf{E}_3 = \left\{ \begin{array}{c} -g_3 X_2 \\ g_3 X_1 \\ g_1 X_2 - g_2 X_1 \end{array} \right\}.$$

The argument of $\mathbf{m}_r^*$ integral can be explicitly computed as

$$X_\alpha \mathbf{E}_\alpha \times \mathbf{D}[\mathbf{g} \times X_\alpha \mathbf{E}_\alpha] = G(X_2^2 + X_1^2) g_3 \mathbf{E}_3 + E X_2^2 g_1 \mathbf{E}_1 - E X_1 X_2 g_1 \mathbf{E}_2 + E X_1^2 g_2 \mathbf{E}_2 - E X_1 X_2 g_2 \mathbf{E}_1.$$

In matrix notation, the previous equation becomes

$$X_\alpha \mathbf{E}_\alpha \times \mathbf{D}[\mathbf{g} \times X_\alpha \mathbf{E}_\alpha] = \mathbf{M}\mathbf{g}, \qquad \text{where} \quad \mathbf{M} = \begin{bmatrix} E X_2^2 & -E X_1 X_2 & 0 \\ -E X_1 X_2 & E X_1^2 & 0 \\ 0 & 0 & G(X_2^2 + X_1^2) \end{bmatrix}.$$

Then, reintroducing $\boldsymbol{\omega}_r$ in place of $\mathbf{g}$, $\mathbf{m}_r^*$ can be integrated using the following equation

$$\mathbf{m}_r^* = \int_{A_0} \mathbf{M} \boldsymbol{\omega}_r \, da_0 = \int_{A_0} \mathbf{M} \, da_0 \;\; \boldsymbol{\omega}_r,$$

where we have used the independence of $\boldsymbol{\omega}_r$ from the area. The integration leads to the equation

$$\mathbf{m}_r^* = \mathbf{C}_m \boldsymbol{\omega}_r, \qquad \text{where} \quad \mathbf{C}_m = \begin{bmatrix} E J_1 & -E J_{12} & 0 \\ -E J_{21} & E J_2 & 0 \\ 0 & 0 & G J_t \end{bmatrix}. \tag{3.3.9}$$

The constitutive equations in term of small-strain material stress and couple beam resultants, $\mathbf{f}_r^*$ and $\mathbf{m}_r^*$, are here summarized:

$$\boxed{\mathbf{f}_r^* = \mathbf{C}_f \boldsymbol{\gamma}_r}, \qquad \text{where} \quad \mathbf{C}_f = \text{diag}[A_0 G, A_0 G, A_0 E]; \tag{3.3.10}$$

$$\boxed{\mathbf{m}_r^* = \mathbf{C}_m \boldsymbol{\omega}_r} \qquad \text{where} \quad \mathbf{C}_m = \begin{bmatrix} EJ_1 & -EJ_{12} & 0 \\ -EJ_{21} & EJ_2 & 0 \\ 0 & 0 & GJ_t \end{bmatrix} \tag{3.3.11}$$

where A_0 is the area section and $J_1, J_2, J_{12} = J_{21}, J_t$ are the moments of inertia, defined in table (3.3.2). These equations assume the matrix form

$$\begin{Bmatrix} \mathbf{f}_r^* \\ \mathbf{m}_r^* \end{Bmatrix} = \mathbf{C} \begin{Bmatrix} \gamma_r \\ \boldsymbol{\omega}_r \end{Bmatrix} \qquad \text{where} \quad \mathbf{C} = \begin{bmatrix} \mathbf{C}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_m \end{bmatrix} \tag{3.3.12}$$

Comments about the relation between the linearity of constitutive equations and the small-strain hypothesis

Since in literature it is not clear the relation holding between the hypotheses of linear elastic constitutive equation and of small-strains, we comment here on the subject with the help of our formulation.

First of all the small strain hypothesis, as we have defined it, is an hypothesis in the structure of the Green-Lagrange strain tensor. Generally this hypothesis is not related with the one of a linear elastic relation between three-dimensional full nonlinear stress and strain tensors. It means that we can state a linear elastic relation between for example **S** and **E** and impose them on the finite-strain beam principle of virtual work. Clearly we will obtain a linear elastic theory in finite strain.[7]

Different is the case when a linear elastic constitutive relation is imposed between the three-dimensional *small-strain approximated* strain tensor, $\mathbf{E}^*$, and the stress tensor, **S**. We have demonstrated that this procedure *entails* a linear elastic constitutive relation between the *beam* stress and strain *resultants*. Therefore if we obtain equilibrium equations in term of stress and strain resultants without imposing a small-strain hypothesis, as in section (3.2), and then we *postulate* a linear elastic relation between the same resultants we are entailing under the counter the small-strain hypothesis and reducing the general finite-strain to a small-strain theory.

This fact do not appear clearly in literature. Most of authors do not pay attention to the subject. Simo in [23] and [24] calls oddly his theory "finite-strain" even thought he postulates a linear elastic relation between beam stress and strain resultants. Anyway in a later work [25] Simo leaves the terminology "finite-strain" for *geometrically exact* beam theory. In that work he presents a justification of the linear elastic relation between resultants starting from a linear elastic relation between approximated forms of $\boldsymbol{\sigma}$ and **E**.

[7] Clearly it can be pointed out that these hypotheses have few physical meaning, but it not of our interest now

J_1	J_2	$J_{12} = J_{21}$	J_t
$\int_{A_0} X_2^2 da_0$	$\int_{A_0} X_1^2 da_0$	$\int_{A_0} X_1 X_2 da_0$	$\int_{A_0} X_2^2 + X_1^2 da_0$

Table 3.3.1: Moments of inertia

3.4 Finite-deformation small-strain model: solution equations

In this section we are going to recover the final and complete form of the principle of virtual work in the finite-deformation small-strain hypothesis. With reference to the internal work, we introduce in it the constitutive equations and the explicit expressions of strain variations in both direct and indirect forms. Moreover, we state explicitly the external virtual work, introducing in it the expression of the displacement field variation using both direct and indirect forms for rotation variations. For the complete principle of virtual work using the direct form we recover the strong equilibrium equations and compare them with those presented by Simo in [23].

Having already proved the equivalence between the beam internal material and spatial work, we refer to the material one. Therefore, the equilibrium equation we are going to consider is

$$\delta L = \delta L_{int} - \delta L_{ext} = 0 \qquad \forall\, \delta\boldsymbol{\phi}_0, \mathbf{w}_\delta \quad \text{(direct form)}$$
$$\text{or} \quad \forall\, \delta\boldsymbol{\phi}_0, \delta\boldsymbol{\theta} \quad \text{(indirect form)}$$

where

$$\delta L_{int} = \int_{L_0} \left[\delta\boldsymbol{\gamma}_r \cdot \mathbf{f}_r^* + \delta\boldsymbol{\omega}_r \cdot \mathbf{m}_r^*\right] dl_0 \qquad \text{with} \quad \begin{Bmatrix} \mathbf{f}_r^* \\ \mathbf{m}_r^* \end{Bmatrix} = \mathbf{C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix}$$

and

$$\delta L_{ext} = \int_{\Omega_0} \left[\mathbf{b}_0 \cdot \delta\mathbf{u}\right] dv_0 + \int_{\partial\Omega_0} \left[\bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\right] da_0.$$

The external boundary virtual work $\int_{\partial\Omega_0} \left[\bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\right] da_0$ is given in section (1.3) as

$$\int_{\partial\Omega_0} \left[\bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\right] da_0 = \int_{\partial\Omega_0} \left[\bar{\mathbf{P}}\mathbf{n}_0 \cdot \delta\mathbf{u}\right] da_0,$$

where $\mathbf{n}_0$ is the unit vector normal to the boundary region in the *reference* configuration. In the case of our beam model, the boundary regions are the cross-sections $A_0 = A\,|_{X_3=0}$ and $A_L = A\,|_{X_3=L}$ and the beam lateral surface. The only boundary region we consider is $A_0 \bigcup A_L$, since we assume the beam lateral surface to be *unloaded*. Therefore in this case

$$\begin{aligned} \mathbf{n}_0 &= -\mathbf{E}_3 && \text{for } A_0, \\ \mathbf{n}_0 &= \mathbf{E}_3 && \text{for } A_L. \end{aligned} \tag{3.4.1}$$

.

3.4.1 Direct form

Internal virtual work. The direct variations of $\boldsymbol{\gamma}_r$ and $\boldsymbol{\omega}_r$ have been already computed in section (3.1.3) as

$$\begin{aligned} \delta\boldsymbol{\gamma}_r &= \boldsymbol{\Lambda}^T[\delta\boldsymbol{\phi}_{0,3} - \mathbf{w}_\delta \times \boldsymbol{\phi}_{0,3}], \\ \delta\boldsymbol{\omega}_r &= \boldsymbol{\Lambda}^T\mathbf{w}_{\delta,3} \end{aligned}$$

In order to rearrange them into a matrix form we introduce the $[6 \times 6]$ matrices $\mathbf{\Xi}$, $\mathbf{I}_{,3}$ and $\mathbf{\Pi}$, defined as

$$\mathbf{\Xi} = \begin{bmatrix} \mathbf{I}_{,3} & \mathbf{0} \\ -[\boldsymbol{\phi}_{0,3}\times] & \mathbf{I}_{,3} \end{bmatrix} \qquad \text{with} \quad \mathbf{I}_{,3} = \frac{\mathrm{d}(\cdot)}{\mathrm{d}X_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{3.4.2}$$

$$\mathbf{\Pi} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda} \end{bmatrix}. \tag{3.4.3}$$

The matrix operator $\mathbf{I}_{,3}$ always works on right-multiplied vectors. Noting that

$$\mathbf{\Xi}^T = \begin{bmatrix} \mathbf{I}_{,3} & [\boldsymbol{\phi}_{0,3}\times] \\ \mathbf{0} & \mathbf{I}_{,3} \end{bmatrix},$$

since $(-[\boldsymbol{\phi}_{0,3}\times])^T = [\boldsymbol{\phi}_{0,3}\times]$ (it is a skew tensor), it is easy to show that the vector $\{\delta\boldsymbol{\gamma}_r, \delta\boldsymbol{\omega}_r\}^T$ can be written as the matrix product

$$\begin{Bmatrix} \delta\boldsymbol{\gamma}_r \\ \delta\boldsymbol{\omega}_r \end{Bmatrix} = \mathbf{\Pi}^T \mathbf{\Xi}^T \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}. \tag{3.4.4}$$

Recalling again the constitutive equation

$$\begin{Bmatrix} \mathbf{f}_r^* \\ \mathbf{m}_r^* \end{Bmatrix} = \mathbf{C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix},$$

the internal virtual work , δL_{int}, can be recast in the matrix form

$$\begin{aligned} \delta L_{int} &= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\gamma}_r \\ \delta\boldsymbol{\omega}_r \end{Bmatrix} \cdot \begin{Bmatrix} \mathbf{f}_r^* \\ \mathbf{m}_r^* \end{Bmatrix} dl_0 = \\ &= \int_{L_0} \mathbf{\Pi}^T \mathbf{\Xi}^T \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix} \cdot \mathbf{C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} dlo_0, \end{aligned}$$

which in compact notation becomes

$$\boxed{\delta L_{int} = \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \mathbf{\Xi\Pi C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} dl_0}. \tag{3.4.5}$$

External virtual work. To compute the external virtual work we need the displacement field variation, $\delta\mathbf{u}$, which has not yet been presented. Recalling the expression of the displacement field $\mathbf{u}$, (3.1.6),

$$\mathbf{u} = \boldsymbol{\phi}_0 + X_\alpha \mathbf{\Lambda} \mathbf{E}_\alpha - X_I \mathbf{E}_I,$$

taking its direct linearization we get

$$\begin{aligned} \delta\mathbf{u} &= \delta\boldsymbol{\phi}_0 + X_\alpha \delta\mathbf{\Lambda} \mathbf{E}_\alpha \\ &= \delta\boldsymbol{\phi}_0 + X_\alpha \mathbf{W}_\delta \mathbf{\Lambda} \mathbf{E}_\alpha. \end{aligned}$$

Introducing the axial vector of $\mathbf{W}_\delta$, $\mathbf{w}_\delta$, and substituting the equality $\mathbf{\Lambda E}_\alpha = \mathbf{t}_\alpha$, the displacement variation $\delta\mathbf{u}$ in the direct form becomes

$$\delta\mathbf{u} = \delta\boldsymbol{\phi}_0 + \mathbf{w}_\delta \times X_\alpha \mathbf{t}_\alpha. \tag{3.4.6}$$

Substituting this expression into the external virtual body force work $\int_{\Omega_0}\Big[\mathbf{b}_0 \cdot \delta\mathbf{u}\Big]dv_0$ we obtain

$$\begin{aligned}\int_{\Omega_0}\Big[\mathbf{b}_0 \cdot \delta\mathbf{u}\Big]\,dv_0 &= \int_{\Omega_0}\Big[\mathbf{b}_0 \cdot \delta\boldsymbol{\phi}_0 + \mathbf{b}_0 \cdot \Big(\mathbf{w}_\delta \times X_\alpha\mathbf{t}_\alpha\Big)\Big]dv_0 = \\ &= \int_{\Omega_0}\Big[\mathbf{b}_0 \cdot \delta\boldsymbol{\phi}_0 + \mathbf{w}_\delta \cdot \Big(X_\alpha\mathbf{t}_\alpha \times \mathbf{b}_0\Big)\Big]dv_0,\end{aligned}$$

where in the last equality we have used the mixed product rule. Considering that $\delta\boldsymbol{\phi}_0$ and $\mathbf{w}_\delta$ are independent of the area section, the integral over the reference beam volume can be split into one over the surface reference cross-section and one along the reference beam axis as

$$\int_{\Omega_0}\Big[\mathbf{b}_0 \cdot \delta\mathbf{u}\Big]\,dv_0 = \int_{L_0}\Big[\delta\boldsymbol{\phi}_0 \cdot \int_{A_0}(\mathbf{b}_0)\,da_0 + \mathbf{w}_\delta \cdot \int_{A_0}(X_\alpha\mathbf{t}_\alpha \times \mathbf{b}_0)\,da_0\Big]\,dl_0.$$

Defining the *cross-section body stress and couple resultants* $\hat{\mathbf{f}}$ and $\hat{\mathbf{m}}$ as

$$\hat{\mathbf{f}} \equiv \int_{A_0}\mathbf{b}_0\,da_0, \tag{3.4.7}$$

$$\hat{\mathbf{m}} \equiv \int_{A_0}(X_\alpha\mathbf{t}_\alpha \times \mathbf{b}_0)\,da_0, \tag{3.4.8}$$

the *external virtual body force work* finally becomes

$$\boxed{\int_{\Omega_0}\Big[\mathbf{b}_0 \cdot \delta\mathbf{u}\Big]\,dv_0 = \int_{L_0}\Big[\hat{\mathbf{f}} \cdot \delta\boldsymbol{\phi}_0 + \hat{\mathbf{m}} \cdot \mathbf{w}_\delta\Big]\,dl_0}. \tag{3.4.9}$$

Note that the external resultants are valued in the current configuration, even if integrated on the reference area section, as the moment radius $\mathbf{t}_\alpha$ clearly shows. This is in full agreement with the three-dimensional continuum theory.

Substituting the definition of $\mathbf{n}_0$ and the displacement virtual variation $\delta\mathbf{u}$, (3.4.6), into the external boundary virtual work $\int_{\partial\Omega_0}\Big[\bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\Big]\,da_0$ we obtain

$$\begin{aligned}\int_{\partial\Omega_0}\Big[\bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\Big]\,da_0 &= \int_{\partial\Omega_0}\Big[\bar{\mathbf{P}}\mathbf{n}_0 \cdot \delta\mathbf{u}\Big]\,da_0 = \\ &= \int_{A_L}\Big[\bar{\mathbf{P}}_3 \cdot \delta\mathbf{u}\Big]\,da_0 - \int_{A_0}\Big[\bar{\mathbf{P}}_3 \cdot \delta\mathbf{u}\Big]\,da_0 = \\ &= \int_{A_L}\Big[\bar{\mathbf{P}}_3 \cdot \delta\boldsymbol{\phi}_0 + (X_\alpha\mathbf{t}_\alpha \times \bar{\mathbf{P}}_3) \cdot \mathbf{w}_\delta\Big]\,da_0 - ... \\ &\quad ... - \int_{A_0}\Big[\bar{\mathbf{P}}_3 \cdot \delta\boldsymbol{\phi}_0 + (X_\alpha\mathbf{t}_\alpha \times \bar{\mathbf{P}}_3) \cdot \mathbf{w}_\delta\Big]\,da_0,\end{aligned} \tag{3.4.10}$$

where in the last step we have used the mixed product rule. Defining the *boundary stress and couple resultants* $\bar{\mathbf{f}}_0$, $\bar{\mathbf{m}}_0$, $\bar{\mathbf{f}}_L$ and $\bar{\mathbf{m}}_L$

$$\bar{\mathbf{m}}_0 = \int_{A_0} [X_\alpha \mathbf{t}_\alpha \times \bar{\mathbf{P}}_3] \, da_0, \qquad \bar{\mathbf{f}}_0 = \int_{A_0} [\bar{\mathbf{P}}_3] \, da_0, \tag{3.4.11}$$

$$\bar{\mathbf{m}}_L = \int_{A_L} [X_\alpha \mathbf{t}_\alpha \times \bar{\mathbf{P}}_3] \, da_0, \qquad \bar{\mathbf{f}}_L = \int_{A_L} [\bar{\mathbf{P}}_3] \, da_0, \tag{3.4.12}$$

$$\tag{3.4.13}$$

the *boundary virtual work* takes the form

$$\boxed{\int_{\partial\Omega_0} \left[\bar{\mathbf{t}}_{\mathbf{n}_0} \cdot \delta\mathbf{u}\right] da_0 = \Big(\bar{\mathbf{f}}_L \cdot \delta\boldsymbol{\phi}_0(L) + \bar{\mathbf{m}}_L \cdot \mathbf{w}_\delta(L)\Big) - \Big(\bar{\mathbf{f}}_0 \cdot \delta\boldsymbol{\phi}_0(0) + \bar{\mathbf{m}}_0 \cdot \mathbf{w}_\delta(0)\Big)}. \tag{3.4.14}$$

Using equations (3.4.9) and (3.4.14) the external work δL_{ext} in the direct form can be computed. Note that the external work is not affected by the small-strain hypothesis, even though it is presented in the small-strain context.

With both internal and external virtual works, the finite-displacement small-strain direct weak equilibrium can be computed.

Strong form of equilibrium; Simo equations.

The internal virtual work (3.4.21) can be rearranged with some manipulation in the form

$$\begin{aligned} \delta L_{int} &= \int_{L_0} \boldsymbol{\Xi}^T \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix} \cdot \boldsymbol{\Pi}\mathbf{C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} dl_0 \\ &= \int_{L_0} \boldsymbol{\Xi}^T \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix} \cdot \begin{Bmatrix} \mathbf{f}^* \\ \mathbf{m}^* \end{Bmatrix} dl_0, \end{aligned}$$

where $\mathbf{f}^* = \boldsymbol{\Lambda}\mathbf{f}^*_r$ and $\mathbf{m}^* = \boldsymbol{\Lambda}\mathbf{m}^*_r$. Introducing the expression of $\boldsymbol{\Xi}^T$, the internal work becomes

$$\delta L_{int} = \int_{L_0} \Big[\mathbf{m}^* \cdot \mathbf{w}_{\delta,3} + \mathbf{f}^* \cdot (\delta\boldsymbol{\phi}_{0,3} - \mathbf{w}_\delta \times \boldsymbol{\phi}_{0,3})\Big] dl_0, \tag{3.4.15}$$

and hence, recollecting internal and external work, the complete principle of virtual work in the direct form is

$$\begin{aligned} \delta L \;=\; & \int_{L_0} \Big[\mathbf{m}^* \cdot \mathbf{w}_{\delta,3} + \mathbf{f}^* \cdot \Big[\delta\boldsymbol{\phi}_{0,3} - \mathbf{w}_\delta \times \boldsymbol{\phi}_{0,3}\Big]\Big] dl_0 - \\ & - \int_{L_0} \Big[\hat{\mathbf{f}} \cdot \delta\boldsymbol{\phi}_0 + \hat{\mathbf{m}} \cdot \mathbf{w}_\delta\Big] dl_0 - \\ & - \Big(\bar{\mathbf{f}}_{(L)} \cdot \delta\boldsymbol{\phi}_0(L) + \bar{\mathbf{m}}_{(L)} \cdot \mathbf{w}_\delta(L)\Big) + \Big(\bar{\mathbf{f}}_{(0)} \cdot \delta\boldsymbol{\phi}_0(0) + \bar{\mathbf{m}}_{(0)} \cdot \mathbf{w}_\delta(0)\Big) = 0 \qquad \forall \delta\boldsymbol{\phi}_0, \mathbf{w}_\delta. \end{aligned}$$

Integrating by parts the addend $\int_{L_0}(\mathbf{f}^* \cdot \delta\boldsymbol{\phi}_{0,3})\, dl_0$ and $\int_{L_0}(\mathbf{m}^* \cdot \mathbf{w}_{\delta,3})\, dl_0$ of the internal work, we obtain

$$\int_{L_0} \mathbf{f}^* \cdot \delta\boldsymbol{\phi}_{0,3}\, dl_0 = -\int_{L_0} \Big[\mathbf{f}^*_{,3} \cdot \delta\boldsymbol{\phi}_0\Big] dl_0 + \Big[\mathbf{f}^* \cdot \delta\boldsymbol{\phi}_0\Big]_{X_3\in[0,L]}$$
$$\int_{L_0} \mathbf{m}^* \cdot \mathbf{w}_{\delta,3}\, dl_0 = -\int_{L_0} \Big[\mathbf{m}^*_{,3} \cdot \mathbf{w}_\delta\Big] dl_0 + \Big[\mathbf{m}^* \cdot \mathbf{w}_\delta\Big]_{X_3\in[0,L]}.$$

Introducing these equations into the virtual work and collecting the virtual variations, we finally get

$$\begin{aligned}\delta L_{int} \quad = \quad & -\int_{L_0} \Big[\Big(\mathbf{m}^*_{,3} + (\boldsymbol{\phi}_{0,3} \times \mathbf{f}^*) + \hat{\mathbf{m}}\Big) \cdot \mathbf{w}_\delta + \Big(\mathbf{f}^*_{,3} + \hat{\mathbf{f}}\Big) \cdot \delta\boldsymbol{\phi}_0\Big] dl_0 + \\ & + (\mathbf{f}_{(0)} + \bar{\mathbf{f}}_{(0)}) \cdot \delta\boldsymbol{\phi}_0(0) + (\mathbf{m}_{(0)} + \bar{\mathbf{m}}_{(0)}) \cdot \mathbf{w}_\delta(0) + \\ & + (\mathbf{f}_{(L)} - \bar{\mathbf{f}}_{(L)}) \cdot \delta\boldsymbol{\phi}_0(L) + (\mathbf{m}_{(L)} - \hat{\mathbf{m}}_{(L)}) \cdot \mathbf{w}_\delta(L) = 0 \qquad \forall \quad \delta\boldsymbol{\phi}_0, \mathbf{w}_\delta.\end{aligned} \tag{3.4.16}$$

From the *Fundamental Lemma of variational calculus* the *strong equilibrium equations* and *the Neumann boundary conditions*, it follows that

$$\mathbf{m}^*_{,3} + \boldsymbol{\phi}_{0,3} \times \mathbf{f}^* + \hat{\mathbf{m}} = \mathbf{0}, \tag{3.4.17}$$
$$\mathbf{f}^*_{,3} + \hat{\mathbf{f}} = \mathbf{0}, \tag{3.4.18}$$

$$\begin{aligned}\mathbf{f}_{(0)} + \bar{\mathbf{f}}_{(0)} = \mathbf{0} \qquad \mathbf{m}_{(0)} + \bar{\mathbf{m}}_{(0)} = \mathbf{0} \\ \mathbf{f}_{(L)} - \bar{\mathbf{f}}_{(L)} = \mathbf{0} \qquad \mathbf{m}_{(L)} - \hat{\mathbf{m}}_{(L)} = \mathbf{0}\end{aligned} \tag{3.4.19}$$

The first one differential equation is the rotational equilibrium and the second one is the translational equilibrium. These equations *are not* exactly those of Simo in [23], as for Simo the stress and couple resultants are

$$\mathbf{f} = \int_{A_0} \mathbf{P}_3 da_0, \qquad \mathbf{m} = \int_{A_0} [X_\alpha \mathbf{t}_\alpha \times \mathbf{P}_3] da_0,$$

while in our equations they are

$$\mathbf{f}^* = \boldsymbol{\Lambda} \int_{A_0} \mathbf{S}_3 da_0, \qquad \mathbf{m}^* = \boldsymbol{\Lambda} \int_{A_0} [X_\alpha \mathbf{E}_\alpha \times \mathbf{S}_3] da_0.$$

The reason of the difference is that we derived the differential equilibrium from a virtual work on which small-strain hypotheses have *already* been imposed. If we perform the same computation starting from the finite-strain virtual work (equation (3.2.14)) we get exactly Simo's equations. We pointed out this fact in order to emphasize that Simo in [23] deals with finite-strain equilibrium equations but then he does not impose explicitly the small-strain hypothesis while he imposes it covertly, by postulating a linear elastic relation between stress and couple resultants.

3.4.2 Indirect form

Internal virtual work. The indirect variations of $\boldsymbol{\gamma}_r$ and $\boldsymbol{\omega}_r$ have been already computed in section (3.1.3) as

$$\begin{aligned}\delta\boldsymbol{\gamma}_r &= \Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\phi}_{0,3})\delta\boldsymbol{\theta} + \boldsymbol{\Lambda}^T\delta\boldsymbol{\phi}_{0,3},\\ \delta\boldsymbol{\omega}_r &= \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta}_{,3})\delta\boldsymbol{\theta} + \mathbf{T}^T\delta\boldsymbol{\theta}_{,3}.\end{aligned}$$

It is easy to rearrange them in matrix form as

$$\begin{Bmatrix}\delta\boldsymbol{\gamma}_r\\ \delta\boldsymbol{\omega}_r\end{Bmatrix} = \begin{bmatrix}\boldsymbol{\Lambda}^T & \Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\phi}_{0,3}) & \mathbf{0}\\ \mathbf{0} & \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta}_{,3}) & \mathbf{T}^T\end{bmatrix}\begin{bmatrix}\mathbf{I}_{,3} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{I}_{,3}\end{bmatrix}\begin{Bmatrix}\delta\boldsymbol{\phi}_0\\ \delta\boldsymbol{\theta}\end{Bmatrix}. \tag{3.4.20}$$

Recalling again the constitutive equation

$$\begin{Bmatrix}\mathbf{f}_r^*\\ \mathbf{m}_r^*\end{Bmatrix} = \mathbf{C}\begin{Bmatrix}\boldsymbol{\gamma}_r\\ \boldsymbol{\omega}_r\end{Bmatrix},$$

the internal virtual work , δL_{int}, can be recast in matrix form as

$$\begin{aligned}\delta L_{int} &= \int_{L_0}\begin{Bmatrix}\delta\boldsymbol{\gamma}_r\\ \delta\boldsymbol{\omega}_r\end{Bmatrix}\cdot\begin{Bmatrix}\mathbf{f}_r^*\\ \mathbf{m}_r^*\end{Bmatrix} dl_0 =\\ &= \int_{L_0}\begin{bmatrix}\boldsymbol{\Lambda}^T & \Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\phi}_{0,3}) & \mathbf{0}\\ \mathbf{0} & \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta}_{,3}) & \mathbf{T}^T\end{bmatrix}\begin{bmatrix}\mathbf{I}_{,3} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}\\ \mathbf{0} & \mathbf{I}_{,3}\end{bmatrix}\begin{Bmatrix}\delta\boldsymbol{\phi}_0\\ \delta\boldsymbol{\theta}\end{Bmatrix}\cdot\mathbf{C}\begin{Bmatrix}\boldsymbol{\gamma}_r\\ \boldsymbol{\omega}_r\end{Bmatrix} dl_0,\end{aligned}$$

which in compact notation becomes

$$\boxed{\delta L_{int} = \int_{L_0}\begin{Bmatrix}\delta\boldsymbol{\phi}_0\\ \delta\boldsymbol{\theta}\end{Bmatrix}^T\begin{bmatrix}\mathbf{I}_{,3} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I} & \mathbf{I}_{,3}\end{bmatrix}\begin{bmatrix}\boldsymbol{\Lambda}^T & \Upsilon_{\delta\boldsymbol{\Lambda}^T}(\boldsymbol{\phi}_{0,3}) & \mathbf{0}\\ \mathbf{0} & \Upsilon_{\delta\mathbf{T}^T}(\boldsymbol{\theta}_{,3}) & \mathbf{T}^T\end{bmatrix}^T\mathbf{C}\begin{Bmatrix}\boldsymbol{\gamma}_r\\ \boldsymbol{\omega}_r\end{Bmatrix} dl_0}. \tag{3.4.21}$$

External virtual work. The external virtual work using indirect form for rotation variation can be quickly evaluated. Recalling from section (2.2.1) that

$$\mathbf{w}_\delta = \mathbf{T}(\boldsymbol{\theta})\delta\boldsymbol{\theta},$$

we have just to substitute this expression into the linearization of displacement, $\delta\mathbf{u}$, and make exactly the same calculations as for the direct case. It appears clearly that the external indirect work is equal to the direct one changing in the final expression $\mathbf{w}_\delta$ with $\mathbf{T}(\boldsymbol{\theta})\delta\boldsymbol{\theta}$.

3.5 Linearization of virtual work which uses direct variation of rotation

The virtual work equations for the finite-deformation small-strain beam model are highly *nonlinear* in the deformation functions, $\boldsymbol{\phi}_0$ and $\boldsymbol{\Lambda}$. For this reason to solve them we will employ a Newton-Rapson's approach, which needs the equation linearization. Since we are going to implement only the virtual work which uses direct variation of rotation, we deal with linearization only of this one. For further information on linearization of virtual work which uses indirect variation of rotation, references [11, 12, 15, 21] can be consulted. For our case of linearization, some information can be found also in [24].

Denoting by $\mathcal{L}[\delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})]$ the linear part of the virtual work $\delta L(\boldsymbol{\phi}_0, \boldsymbol{\Lambda})$ at the configuration $(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})$, by definition we have

$$\mathcal{L}[\delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})] = \delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}}) + \Delta[\delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})], \tag{3.5.1}$$

where

- $\delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})$ is the work evaluated at point $(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})$ and supplies the unbalanced force at that point, yielding the so-called *residual* vector;
- $\Delta[\delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})]$ is the work part linearly depending of configuration increments, yielding the so-called *tangent stiffness* matrix.

We recall that the functional we are going to linearize (direct variation form of rotation) is

$$\delta L = \delta L_{int} - \delta L_{ext} = 0 \qquad \forall\, \delta\boldsymbol{\phi}_0, \mathbf{w}_\delta,$$

where

$$\delta L_{int} = \int_{L_0} \Big[\begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \boldsymbol{\Xi}\boldsymbol{\Pi}\mathbf{C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} \Big] dl_0$$

and

$$\delta L_{ext} = \int_{L_0} \Big[\begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \begin{Bmatrix} \hat{\mathbf{f}} \\ \hat{\mathbf{m}} \end{Bmatrix} \Big] dl_0 + \delta L_{bound}.$$

Internal work linearization. Looking at the expression of the internal work δL_{int}, we see that in order to compute its linearization $\Delta[\delta L_{int}]$ we need to evaluate the linearization of $\boldsymbol{\Xi}$, $\boldsymbol{\Pi}$ and $\{\boldsymbol{\gamma}_r \quad \boldsymbol{\omega}_r\}$, respectively $\Delta\boldsymbol{\Xi}$, $\Delta\boldsymbol{\Pi}$ and $\Delta\{\boldsymbol{\gamma}_r, \boldsymbol{\omega}_r\}$.[8] In analogy with variation procedure, we trivially have that $\Delta\boldsymbol{\phi}_0 = \Delta\boldsymbol{\phi}_0$, obtained from the directional derivative of $\boldsymbol{\phi}_{0\varepsilon} = \boldsymbol{\phi}_0 + \varepsilon\Delta\boldsymbol{\phi}_0$, while, for the direct form,

$$\Delta\boldsymbol{\Lambda} = \mathbf{W}_\Delta \boldsymbol{\Lambda}, \tag{3.5.2}$$

[8] $\mathbf{C}$ is constant

obtained from directional derivative of $\mathbf{\Lambda}_\varepsilon = \exp[\varepsilon \mathbf{W}_\Delta]\mathbf{\Lambda}$. In analogy with variations, the linearization of strain measures, i.e. the incremental strain $\Delta\boldsymbol{\gamma}_r$ and $\Delta\boldsymbol{\omega}_r$, are

$$\Delta\boldsymbol{\gamma}_r = \mathbf{\Lambda}^T[\Delta\boldsymbol{\phi}_{0,3} - \mathbf{w}_\Delta \times \boldsymbol{\phi}_{0,3}] = \mathbf{\Lambda}^T[\Delta\boldsymbol{\phi}_{0,3} + \boldsymbol{\phi}_{0,3} \times \mathbf{w}_\Delta]; \tag{3.5.3}$$

$$\Delta\boldsymbol{\omega}_r = \mathbf{\Lambda}^T \mathbf{w}_{\Delta,3} \tag{3.5.4}$$

which using the matrices $\mathbf{\Xi}$ and $\mathbf{\Pi}$, can be rearranged in the matrix form

$$\Delta \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} = \mathbf{\Pi}^T \mathbf{\Xi}^T \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{Bmatrix}. \tag{3.5.5}$$

$\Delta\mathbf{\Xi}$ is computed by directional derivative as

$$\Delta\mathbf{\Xi} = \frac{d}{d\varepsilon}\mathbf{\Xi}_\varepsilon\Big|_{\varepsilon=0} \qquad \text{with} \quad \mathbf{\Xi}_\varepsilon = \begin{bmatrix} \mathbf{I}_{,3} & \mathbf{0} \\ -[(\boldsymbol{\phi}_{0,3} + \varepsilon\Delta\boldsymbol{\phi}_{0,3})\times] & \mathbf{I}_{,3} \end{bmatrix},$$

and becomes

$$\Delta\mathbf{\Xi} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -[\Delta\boldsymbol{\phi}_{0,3})\times] & \mathbf{0} \end{bmatrix}. \tag{3.5.6}$$

$\Delta\mathbf{\Pi}$, with $\mathbf{\Pi} = \begin{bmatrix} \mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda} \end{bmatrix}$, is easily computed using equation (3.5.2) as

$$\Delta\mathbf{\Pi} = \begin{bmatrix} \mathbf{W}_\Delta\mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_\Delta\mathbf{\Lambda} \end{bmatrix}. \tag{3.5.7}$$

So that the linearization of the internal virtual work is

$$\Delta[\delta L_{int}] = \int_{L_0} \left[\begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \Delta[\mathbf{\Xi}\mathbf{\Pi}]\, \mathbf{C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} \right] \mathrm{d}l_0 + \int_{L_0} \left[\begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \mathbf{\Xi}\mathbf{\Pi}\mathbf{C}\, \Delta \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} \right] \mathrm{d}l_0. \tag{3.5.8}$$

The second addend yields the so-called *material* part of the stiffness matrix, since it is the linearization of internal stress and couple resultants, given in term of strain resultants by the constitutive relation . The first addend instead yields the so-called *geometric* part of the stiffness matrix, since it is the linearization of virtual strain variations $\delta\boldsymbol{\gamma}_r$ and $\delta\boldsymbol{\omega}_r$.

Let us consider the second addend. Substituting into it the linearization of strain measures (3.5.5) we obtain

$$\begin{aligned} \int_{L_0} \left[\begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \mathbf{\Xi}\mathbf{\Pi}\mathbf{C}\, \Delta \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} \right] \mathrm{d}l_0 &= \int_{L_0} \left[\begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \mathbf{\Xi}\mathbf{\Pi}\mathbf{C}\mathbf{\Pi}^T\mathbf{\Xi}^T \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{Bmatrix} \right] \mathrm{d}l_0 = \\ &= \int_{L_0} \left[\begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \mathbf{S} \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{Bmatrix} \right] \mathrm{d}l_0, \end{aligned} \tag{3.5.9}$$

where

$$\mathbf{S} = \mathbf{\Xi}\mathbf{\Pi}\mathbf{C}\mathbf{\Pi}^T\mathbf{\Xi}^T \tag{3.5.10}$$

is the *material* part of the tangent matrix. Note that the matrix is symmetric and it has the typical structure of constitutive linear elastic models.

Let us now consider the first addend of (3.5.8). The term $\Delta[\mathbf{\Xi\Pi}]$ can be evaluated using the expressions of $\Delta\mathbf{\Xi}$ and $\Delta\mathbf{\Pi}$, respectively (3.5.6) and (3.5.7), as

$$\Delta[\mathbf{\Xi\Pi}] = \Delta[\mathbf{\Xi}]\mathbf{\Pi} + \mathbf{\Xi}\Delta[\mathbf{\Pi}] = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -[\Delta\boldsymbol{\phi}_{0,3})\times] & \mathbf{0} \end{bmatrix} \mathbf{\Pi} + \mathbf{\Xi} \begin{bmatrix} \mathbf{W}_\Delta\mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_\Delta\mathbf{\Lambda} \end{bmatrix}.$$

By substitution, the integral becomes

$$\begin{aligned}
&\int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \Delta[\mathbf{\Xi\Pi}]\mathbf{C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -[\Delta\boldsymbol{\phi}_{0,3}\times] & \mathbf{0} \end{bmatrix} \mathbf{\Pi} + \mathbf{\Xi} \begin{bmatrix} \mathbf{W}_\Delta\mathbf{\Lambda} & \mathbf{0} \\ \mathbf{0} & \mathbf{W}_\Delta\mathbf{\Lambda} \end{bmatrix} \right) \mathbf{C} \begin{Bmatrix} \boldsymbol{\gamma}_r \\ \boldsymbol{\omega}_r \end{Bmatrix} \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \left(\begin{Bmatrix} \mathbf{0} \\ -[\Delta\boldsymbol{\phi}_{0,3}\times]\mathbf{\Lambda}\mathbf{C}_f\boldsymbol{\gamma}_r \end{Bmatrix} + \mathbf{\Xi} \begin{Bmatrix} \mathbf{W}_\Delta\mathbf{\Lambda}\mathbf{C}_f\boldsymbol{\gamma}_r \\ \mathbf{W}_\Delta\mathbf{\Lambda}\mathbf{C}_m\boldsymbol{\omega}_r \end{Bmatrix} \right) \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \left(\begin{Bmatrix} \mathbf{0} \\ -[\Delta\boldsymbol{\phi}_{0,3}\times]\mathbf{f} \end{Bmatrix} + \mathbf{\Xi} \begin{Bmatrix} \mathbf{W}_\Delta\mathbf{f} \\ \mathbf{W}_\Delta\mathbf{m} \end{Bmatrix} \right) \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \left(\begin{Bmatrix} \mathbf{0} \\ -\Delta\boldsymbol{\phi}_{0,3} \times \mathbf{f} \end{Bmatrix} + \mathbf{\Xi} \begin{Bmatrix} \mathbf{w}_\Delta \times \mathbf{f} \\ \mathbf{w}_\Delta \times \mathbf{m} \end{Bmatrix} \right) \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \left(\begin{Bmatrix} \mathbf{0} \\ \mathbf{f} \times \Delta\boldsymbol{\phi}_{0,3} \end{Bmatrix} - \mathbf{\Xi} \begin{Bmatrix} \mathbf{f} \times \mathbf{w}_\Delta \\ \mathbf{m} \times \mathbf{w}_\Delta \end{Bmatrix} \right) \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{f}\times & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\phi}_{0,3} \\ \mathbf{0} \end{Bmatrix} - \mathbf{\Xi} \begin{bmatrix} \mathbf{0} & [\mathbf{f}\times] \\ \mathbf{0} & [\mathbf{m}\times] \end{bmatrix} \begin{Bmatrix} \mathbf{0} \\ \mathbf{w}_\Delta \end{Bmatrix} \right) \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \left(\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ (\mathbf{f}\times)\mathbf{I}_{,3} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{0} \end{Bmatrix} - \begin{bmatrix} \mathbf{0} & \mathbf{I}_{,3}[\mathbf{f}\times] \\ \mathbf{0} & -[\boldsymbol{\phi}_{0,3}\times][\mathbf{f}\times] + \mathbf{I}_{,3}[\mathbf{m}\times] \end{bmatrix} \begin{Bmatrix} \mathbf{0} \\ \mathbf{w}_\Delta \end{Bmatrix} \right) \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \begin{bmatrix} \mathbf{0} & -\mathbf{I}_{,3}[\mathbf{f}\times] \\ (\mathbf{f}\times)\mathbf{I}_{,3} & [\boldsymbol{\phi}_{0,3}\times][\mathbf{f}\times] - \mathbf{I}_{,3}[\mathbf{m}\times] \end{bmatrix} \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{Bmatrix} \mathrm{d}l_0 = \\
&= \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \mathbf{T} \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{Bmatrix} \mathrm{d}l_0 \qquad (3.5.11)
\end{aligned}$$

$$\text{where} \quad \mathbf{T} = \begin{bmatrix} \mathbf{0} & -\mathbf{I}_{,3}[\mathbf{f}\times] \\ (\mathbf{f}\times)\mathbf{I}_{,3} & [\boldsymbol{\phi}_{0,3}\times][\mathbf{f}\times] - \mathbf{I}_{,3}[\mathbf{m}\times] \end{bmatrix}. \qquad (3.5.12)$$

The matrix $\mathbf{T}$ is the *geometric* part of the tangent matrix.

Finally recollecting the first and second integral of equation (3.5.8), the linearization of the internal work takes the compact form

$$\Delta[\delta L_{int}] = \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \mathbf{K} \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{Bmatrix} \mathrm{d}l_0, \qquad (3.5.13)$$

with

$$\mathbf{K} = \mathbf{S} + \mathbf{T} \qquad \text{tangent matrix} \tag{3.5.14}$$

$$\mathbf{S} = \boldsymbol{\Xi}\boldsymbol{\Pi}\mathbf{C}\boldsymbol{\Pi}^T\boldsymbol{\Xi}^T \qquad \text{material part} \tag{3.5.15}$$

$$\mathbf{T} = \begin{bmatrix} \mathbf{0} & -\mathbf{I}_{,3}[\mathbf{f}\times] \\ (\mathbf{f}\times)\mathbf{I}_{,3} & \mathbf{f}\otimes\boldsymbol{\phi}_{0,3} - (\mathbf{f}\cdot\boldsymbol{\phi}_{0,3})\mathbf{I} - \mathbf{I}_{,3}[\mathbf{m}\times] \end{bmatrix} \qquad \text{geometric part,} \tag{3.5.16}$$

where we have used in $\mathbf{T}$ the equality $[\boldsymbol{\phi}_{0,3}\times][\mathbf{f}\times] = \mathbf{f}\otimes\boldsymbol{\phi}_{0,3} - (\mathbf{f}\cdot\boldsymbol{\phi}_{0,3})\mathbf{I}$. We recall that in small-displacement beam theories, i.e. in geometrically linear theories, the geometric tangent is zero, i.e. $\mathbf{T} = \mathbf{0}$. The reason is that in small-displacement theories the strain variations depend on the displacement variations and not directly on displacements, as the "geometrically linear" terminology states.

External work linearization The expression of the external work is

$$\delta L_{ext} = \int_{L_0} \Big[\left\{ \begin{matrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{matrix} \right\}^T \left\{ \begin{matrix} \hat{\mathbf{f}} \\ \hat{\mathbf{m}} \end{matrix} \right\} \Big] dl_0 + \delta L_{boun},$$

with

$$\hat{\mathbf{m}} \equiv \int_{A_0} [X_\alpha \mathbf{t}_\alpha \times \hat{\mathbf{b}}] da_0 \qquad \hat{\mathbf{f}} \equiv \int_{A_0} [\mathbf{b}_0] da_0.$$

We assume that $\hat{\mathbf{m}} = \mathbf{0}$. Since $\hat{\mathbf{f}}$ is independent of the kinematic functions $\boldsymbol{\phi}_0$ and $\boldsymbol{\Lambda}$, it follows that

$$\Delta[\delta L_{ext}] = \int_{L_0} \Big[\left\{ \begin{matrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{matrix} \right\}^T \Delta \left\{ \begin{matrix} \hat{\mathbf{f}} \\ \hat{\mathbf{m}} \end{matrix} \right\} \Big] dl_0 + \Delta[\delta L_{boun}] = \mathbf{0}. \tag{3.5.17}$$

Linear part of the virtual work. Collecting our results, the linear part of the virtual work $\delta L(\boldsymbol{\phi}_0, \boldsymbol{\Lambda})$ at the configuration $(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})$

$$\mathcal{L}[\delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})] = \delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}}) + \Delta[\delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})]$$

becomes

$$\boxed{\mathcal{L}[\delta L(\check{\boldsymbol{\phi}}_0, \check{\boldsymbol{\Lambda}})] = \int_{L_0} \Big[\left\{ \begin{matrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{matrix} \right\}^T \check{\boldsymbol{\Xi}}\check{\boldsymbol{\Pi}}\mathbf{C} \left\{ \begin{matrix} \check{\boldsymbol{\gamma}}_r \\ \check{\boldsymbol{\omega}}_r \end{matrix} \right\} \Big] dl_0 - \delta L_{ext} + \int_{L_0} \left\{ \begin{matrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{matrix} \right\}^T \check{\mathbf{K}} \left\{ \begin{matrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{matrix} \right\} \mathrm{d}l_0.} \tag{3.5.18}$$

The linearization of the virtual work which uses the indirect form of rotation linearization is not computed in this work. We remarck that in such a case the direction of linearization would be $\Delta\boldsymbol{\theta}$, since the equation depends on the rotation vector field $\boldsymbol{\theta}$.

Chapter 4

Finite Element approach for the finite-deformation small strain model

The one-dimensional equilibrium equations of the finite-deformation small-strain model are solved by a Finite Element (FE) approach. Hence, as usual, we subdivide the beam in elements connected by nodes, i.e. we discretize the beam axis as $[0, L] = \bigcup_{e=1}^{\text{numel}} I_e^h$ where I_e^h denotes a typical element with length $h > 0$ and "numel" is the total number of elements. Then, approximating the independent model functions as the linear combination of interpolating shape functions $N(X_3)$ weighted by their nodal values, we transform the integral-differential problem into an algebraic problem. A Newton-Raphson's procedure is adopted to solve the algebraic system since the nonlinearity of the equations with respect to the beam configuration functions.

The point of interest for a finite-deformation theory is how to deal with finite rotation approximation and updating procedures. For our model, both of them depend on which form of equilibrium equations we work with (those which uses the direct variation of rotation or those which uses the indirect one). Next section is devoted to a brief discussion of this topic. In the other two sections, we specifically focus on our finite element, which results from the approximation of the equilibrium which uses the direct variation of rotation. We first provide the formulation of the element and then the results of four tests widely investigated in the literature (see [6, 11, 24, 27]).

4.1 Approximation and updating procedures for rotations

First of all it must be pointed out which are the rotational measures in the case of direct or indirect linearized equilibrium equations. Considering the first one (see equation (3.5.18)), the searched measures are

$$\mathbf{\Lambda}, \qquad \mathbf{w}_\delta, \qquad \mathbf{w}_\Delta,$$

where $\mathbf{\Lambda}$ is the actual rotation and $\mathbf{w}_\delta$ and $\mathbf{w}_\Delta$ are, respectively, the rotational virtual variation direction and the rotational incremental linearization direction.

Instead, considering the indirect form the searched measures are

$$\mathbf{\Lambda}(\boldsymbol{\theta}), \qquad \delta\boldsymbol{\theta}, \qquad \Delta\boldsymbol{\theta},$$

where $\mathbf{\Lambda}$ is the actual rotation evaluated through the rotation vector $\boldsymbol{\theta}$ and $\delta\boldsymbol{\theta}$ and $\Delta\boldsymbol{\theta}$ are, respectively, the virtual variation direction and the incremental linearization direction of rotation vector.

As usual, when we deal with linearization of weak form of equilibrium, the virtual variation directions and the incremental linearization directions are approximated. Consequently, considering first the case of indirect variation and linearization, $\delta\boldsymbol{\theta}$ and $\Delta\boldsymbol{\theta}$ are approximated. For this reason the nodal incremental values computed within a Newton-Raphson's iteration are the nodal values of the approximated function $\Delta\boldsymbol{\theta}$, i.e. $\Delta\boldsymbol{\theta}_I$. Observe now that the tangent space on which the direction of linearization $\Delta\boldsymbol{\theta}$ lays never changes with the point of linearization $\boldsymbol{\theta}$, because the tangent space and space of linearization point always coincide since both are linear spaces. For this reason, within Newton-Raphson's iterations we can directly superimpose the nodal value increments and consequently construct the vector of the new nodal point of linearization $\boldsymbol{\theta}_{i+1}$ as

$$\boldsymbol{\theta}^{i+1} = \boldsymbol{\theta}^i + \Delta\boldsymbol{\theta}^{i+1} \qquad \text{where} \quad \boldsymbol{\theta}^i = \boldsymbol{\theta}^{i-1} + \Delta\boldsymbol{\theta}^i...., \quad \boldsymbol{\theta}^1 = \boldsymbol{\theta}^0 + \Delta\boldsymbol{\theta}^1,$$

always remaining into the linear space of $\boldsymbol{\theta}$, see figure 4.1. The nodal values of the updated total rotation vector $\boldsymbol{\theta}_I^{i+1}$ are then interpolated within the element to obtain the elementary rotation vector $\boldsymbol{\theta}_e^{i+1}$ as

$$\boldsymbol{\theta}_e^{i+1} = \sum_{J=1}^{\text{nnodel}} N_J(X_3)\boldsymbol{\theta}_J^{i+1}.$$

The updated elementary rotation $\mathbf{\Lambda}_e^{i+1}$ is then computed by the exponential map of the skew tensor $\mathbf{\Theta}_e^{i+1} = [\boldsymbol{\theta}_e^{i+1}\times]$ as

$$\mathbf{\Lambda}_e^{i+1} = \exp[\mathbf{\Theta}_e^{i+1}],$$

which means that we pass from the linear space of $\boldsymbol{\theta}$ to the manifold $\mathcal{G}^{orth+}$ through the exponential map (see figure 4.1). The update is fully consistent with the indirect linearization procedure adopted to evaluate the virtual variations into the principle of virtual work and its second linearization.

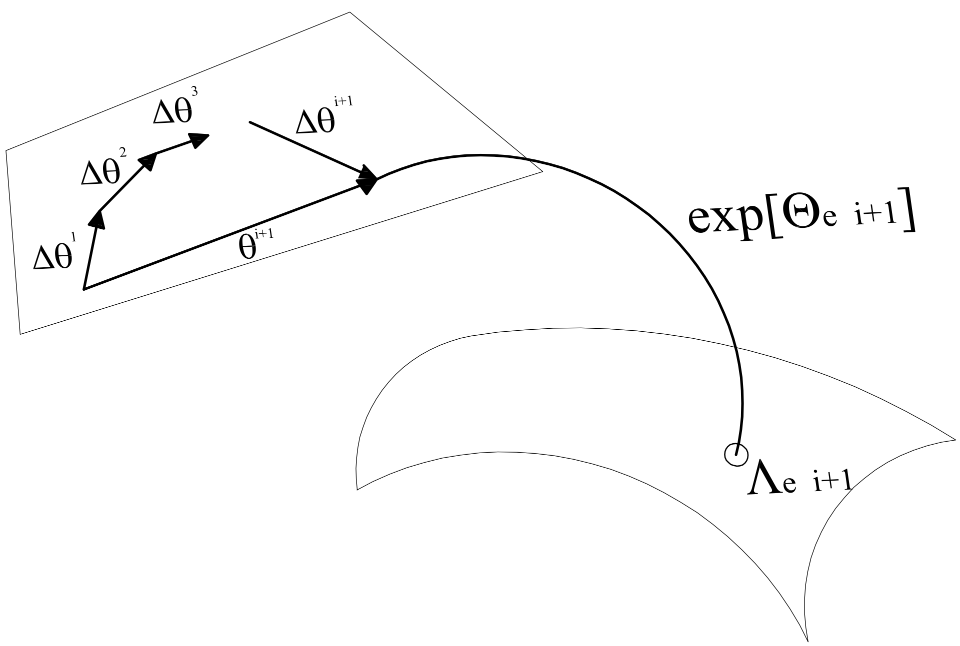


Figure 4.1: Updating indirect procedure

Let us consider now the case of direct linearization. The approximated functions are $\mathbf{w}_\delta$ and $\mathbf{w}_\Delta$ and consequently the nodal incremental values computed within a Newton-Raphson's iteration are the nodal values of $\mathbf{w}_\Delta$, i.e. $\mathbf{w}_{\Delta I}$. In this case the tangent space of $\mathbf{w}_\Delta$, $\mathcal{T}_{\mathbf{\Lambda}}\mathcal{G}^{orth+}$, changes every time that the point of linearization $\mathbf{\Lambda}$ changes, because the tangent space and the differentiable rotation manifold coincide only locally, see figure 4.2. For this reason, within Newton-Raphson's iterations we cannot directly superimpose the nodal values increments $\mathbf{w}_{\Delta I}$.

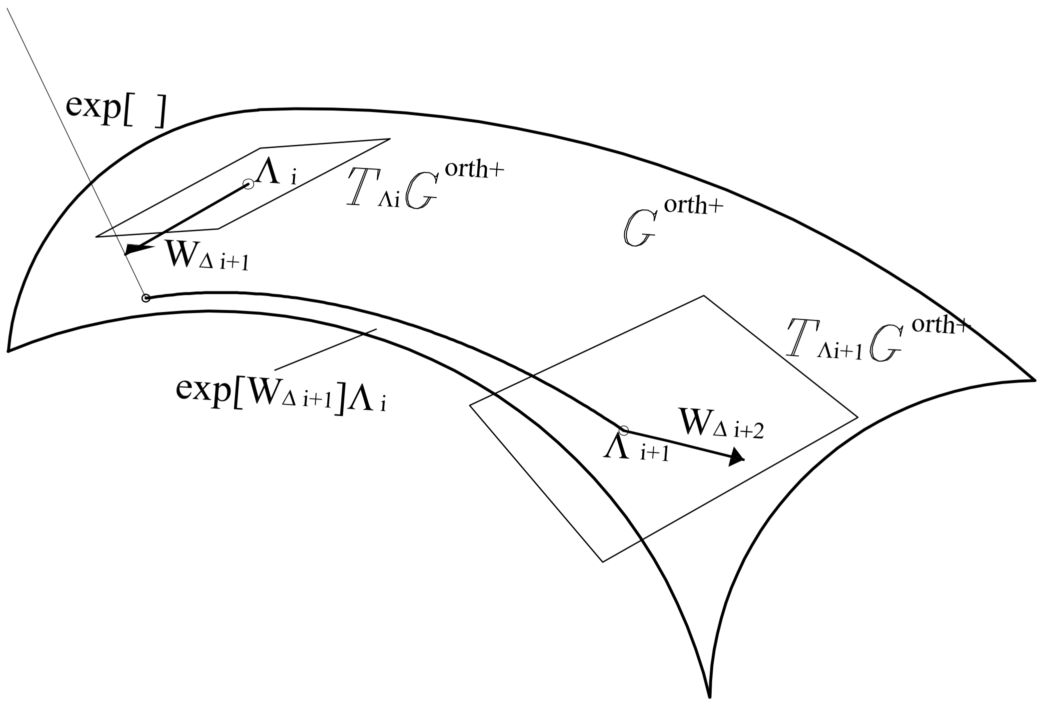


Figure 4.2: Updating direct procedure

Instead, we have to interpolate within the element the nodal increments ${\mathbf{w}_\Delta}_J^{i+1}$ located at the actual linearization point $\mathbf{\Lambda}^i$, obtaining the elementary increment ${\mathbf{w}_\Delta}_e^{i+1}$ as

$$ {\mathbf{w}_\Delta}_e^{i+1} = \sum_{J=1}^{\text{nnodel}} N_J(X_3) {\mathbf{w}_\Delta}_J^{i+1}. $$

Then, we have to pass from the local tangent space $\mathcal{T}_{\mathbf{\Lambda}}\mathcal{G}^{orth+}$ to the rotation manifold through the exponential map, $\exp[{\mathbf{W}_\Delta}_e^{i+1}]$ where ${\mathbf{W}_\Delta}_e^{i+1} = [{\mathbf{w}_\Delta}_e^{i+1}\times]$, and finally superimpose the new rotation on the elementary actual one

$$ \mathbf{\Lambda}_e^{i+1} = \exp[{\mathbf{w}_\Delta}_e^{i+1}]\mathbf{\Lambda}_e^i. $$

So, the updated elementary rotation $\mathbf{\Lambda}_e^{i+1}$ is computed (see figure 4.2). The update is fully consistent with the linearization procedure adopted to evaluate the virtual variations into the principle of virtual work and its linearization in the case of direct formulation.

The main difference between the updates is that in the indirect case the updating is done into a linear space while in the direct case it is done on the rotation manifold. For this reason in the first case we need to store only the nodal values of the last total rotation vector $\boldsymbol{\theta}_I^{i+1}$ while in the second case we must store for each element the last elementary rotation $\mathbf{\Lambda}_e^i$ in order to reconstruct the updated elementary rotation $\mathbf{\Lambda}_e^{i+1}$. In this sense the procedure adopted in case of indirect linearization of rotation is called *path-independent* with respect to the rotation, while the other procedure is called *path-dependent* with respect to the rotation.

At this point it would seem computationally convenient to use the direct procedure. Anyway, the direct form of equilibrium suffers from the singularity of the tensor $\mathbf{T}(\boldsymbol{\theta})$ for $\| \boldsymbol{\theta} \| = 2n\pi$ where n is an integer, as indicated in section (2.2.1). Moreover the tangent operator of the direct form result to be ill-conditioned well before the singularity value. For these reasons we choose to implement the indirect equilibrium form.

4.2 The finite element formulation

The linearized equilibrium equation we approximate is the direct equilibrium form (3.5.18)

$$ \mathcal{L}[\delta L(\check{\boldsymbol{\phi}}_0, \check{\mathbf{\Lambda}})] = \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^T \check{\mathbf{\Xi}} \begin{Bmatrix} \check{\mathbf{f}}^* \\ \check{\mathbf{m}}^* \end{Bmatrix} \mathrm{d}l_0 - \delta L_{ext} + \int_{L_0} \begin{Bmatrix} \delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\delta \end{Bmatrix}^t \check{\mathbf{K}} \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{Bmatrix} \mathrm{d}l_0. \quad (4.2.1) $$

Discretization and approximation. Let us consider the standard finite element discretization introduced above. According to the FE approach, the linear equilibrium $\mathcal{L}[\delta L(\check{\boldsymbol{\phi}}_0, \check{\mathbf{\Lambda}})]$ is discretized and approximated as

$$ \mathcal{L}[\delta L(\check{\boldsymbol{\phi}}_0, \check{\mathbf{\Lambda}})] = \sum_{e=1}^{\text{numel}} \mathcal{L}_e[\delta L(\check{\boldsymbol{\phi}}_{0e}, \check{\mathbf{\Lambda}}_e)] \quad (4.2.2) $$

where the subscript e indicates the approximated continuum function of a to a typical element I_e, numel is the total number of elements and $\mathfrak{L}_e[\delta L]$ is the elementary approximated linear work. We perform the calculation on an element basis and we approximate the virtual variations $\delta\boldsymbol{\phi}_0$ and $\mathbf{w}_\delta$ and the increments $\Delta\boldsymbol{\phi}_0$ and $\mathbf{w}_\Delta$ as follows

$$\begin{aligned} \delta\boldsymbol{\phi}_{0e} &= \sum_{I=1}^{\text{nnodel}} N_I(X_3)\delta\boldsymbol{\phi}_{0I}, \qquad \mathbf{w}_{\delta e} = \sum_{I=1}^{\text{nnodel}} N_I(X_3)\mathbf{w}_{\delta I}, \\ \Delta\boldsymbol{\phi}_{0e} &= \sum_{I=1}^{\text{nnodel}} N_I(X_3)\Delta\boldsymbol{\phi}_{0I}, \qquad \mathbf{w}_{\Delta e} = \sum_{I=1}^{\text{nnodel}} N_I(X_3)\mathbf{w}_{\Delta I}, \end{aligned} \tag{4.2.3}$$

where

- nnodel is the number of element nodes, i.e. 2 in this implementation;
- $N_I(X_3)$ represents the shape function associated with node I, which is *linear* in this implementation;
- $\delta\boldsymbol{\phi}_{0I}$, $\mathbf{w}_{\delta I}$, $\Delta\boldsymbol{\phi}_{0I}$ and $\mathbf{w}_{\Delta I}$ are the nodal virtual and incremental displacements and rotations of the element at node I.

Substituting the approximation into the linearized continuum equilibrium, we can take out of the integral the nodal values of the approximated quantities and hence obtain the approximated elementary work as

$$\mathfrak{L}_e[\delta L(\check{\boldsymbol{\phi}}_{0e}, \check{\boldsymbol{\Lambda}}_e)] = \begin{Bmatrix} \delta\boldsymbol{\phi}_{0I} \\ \mathbf{w}_{\delta I} \end{Bmatrix}^T \check{\mathbf{F}}_{intIe} - \delta L_{exte} + \begin{Bmatrix} \delta\boldsymbol{\phi}_{0I} \\ \mathbf{w}_{\delta I} \end{Bmatrix}^T \check{\mathbf{K}}_{IJe} \begin{Bmatrix} \Delta\boldsymbol{\phi}_{0J} \\ \mathbf{w}_{\Delta J} \end{Bmatrix}. \tag{4.2.4}$$

$\mathbf{F}_{intIe}$ and $\mathbf{K}_{IJe}$ represent the nodal elementary contribution respectively to the global internal force vector and to the global tangent matrix. We use the subscript e for them to indicate that they depend on continuum approximated functions of the element. Their dimensions are respectively $[6 \times 1]$ and $[6 \times 6]$. The internal force, $\mathbf{F}_{inte}$, and the tangent matrix, $\mathbf{K}_e$, for the whole element have dimensions respectively $[12 \times 1]$ and $[12 \times 12]$ in this implementation since we employ two node elements. The next paragraph is devoted to compute such quantities.
The external work is just indicated in (4.2.4) since in this implementation the external forces are not computed but directly assigned as external nodal values.

Elementary internal force and tangent matrix computation. Table 4.2 schematically summarizes all the relations needed to compute the elementary internal force and tangent matrix.

$$\check{\mathbf{F}}_{intIe} = \int_{L_{0e}} \left[\check{\mathbf{\Xi}}_{Ie} \begin{Bmatrix} \check{\mathbf{f}}^*{}_e \\ \check{\mathbf{m}}^*{}_e \end{Bmatrix}\right] \mathrm{d}l_{0e} \quad \text{and} \quad \check{\mathbf{K}}_{IJe} = \int_{L_{0e}} \check{\mathbf{S}}_{IJe} + \check{\mathbf{T}}_{IJe} \mathrm{d}l_{0e} \tag{4.2.5}$$

$$\check{\mathbf{S}}_{IJe} = \check{\mathbf{\Xi}}_{Ie} \check{\mathbf{\Pi}}_e \mathbf{C} \check{\mathbf{\Pi}}_e^T \check{\mathbf{\Xi}}_{Ie}^T \tag{4.2.6}$$

$$\check{\mathbf{T}}_{IJe} = \begin{bmatrix} \mathbf{0} & -N_{I,3}\mathbf{I}[\check{\mathbf{f}}_e^* \times] N_J \\ N_I \mathbf{I}[\check{\mathbf{f}}_e^* \times] N_{J,3} & N_I [\check{\mathbf{f}}_e^* \otimes \check{\boldsymbol{\phi}}_{0,3e} - (\check{\mathbf{f}}_e^* \cdot \check{\boldsymbol{\phi}}_{0,3e})\mathbf{I}] N_J - N_{I,3}\mathbf{I}[\check{\mathbf{m}}_e^* \times] N_J \end{bmatrix} \tag{4.2.7}$$

$$\check{\mathbf{\Pi}}_e = \begin{bmatrix} \check{\mathbf{\Lambda}}_e & \mathbf{0} \\ \mathbf{0} & \check{\mathbf{\Lambda}}_e \end{bmatrix} \quad \text{and} \quad \check{\mathbf{\Xi}}_{Ie} = \begin{bmatrix} N_{I,3}\mathbf{I} & \mathbf{0} \\ -N_I[\check{\boldsymbol{\phi}}_{0e,3} \times] & N_{I,3}\mathbf{I} \end{bmatrix} \tag{4.2.8}$$

$$\begin{Bmatrix} \check{\mathbf{f}}_e^* \\ \check{\mathbf{m}}_e^* \end{Bmatrix} = \check{\mathbf{\Pi}}_e \mathbf{C} \check{\mathbf{\Pi}}_e^T \begin{Bmatrix} \check{\boldsymbol{\gamma}}_e \\ \check{\boldsymbol{\omega}}_e \end{Bmatrix} \tag{4.2.9}$$

$$\check{\boldsymbol{\gamma}}_e = \check{\boldsymbol{\phi}}_{0,3e} - \check{\mathbf{\Lambda}}_e \mathbf{E}_3 \tag{4.2.10}$$

Table 4.2.1: Computation of elementary internal force and tangent matrix

We see that three approximated functions are needed to carry out the computation, i.e.

$$\check{\mathbf{\Lambda}}_e, \ \ \check{\boldsymbol{\phi}}_{0,3e}, \ \ \check{\boldsymbol{\omega}}_e.$$

The check states that the measures are evaluated in a point. In context of the Newton-Raphson's iterations, it means that they are the new values at iteration $i+1$. To evaluate them we use the elementary updating procedures of the direct approach. For the rotation we have

$$\boxed{\mathbf{\Lambda}_e^{i+1} = \exp[\mathbf{W}_{\Delta e}]\mathbf{\Lambda}_e^i}. \tag{4.2.11}$$

The computation of updated spin vector $\boldsymbol{\omega}_e^{i+1}$ is carried out consistently with its direct linearization form. We now compute $\mathbf{\Omega}_e^{i+1} = [\boldsymbol{\omega}_e^{i+1}\times]$ (in intermediate computational steps we omit the subscript e for simplicity) as

$$\begin{aligned}
\mathbf{\Omega}^{i+1} &= \mathbf{\Lambda}^{i+1}_{,3}(\mathbf{\Lambda}^{i+1})^T = \\
&= (\exp[\mathbf{W}_\Delta]\mathbf{\Lambda}^i)_{,3}(\exp[\mathbf{W}_\Delta]\mathbf{\Lambda}^i)^T = \\
&= (\exp_{,3}[\mathbf{W}_\Delta]\mathbf{\Lambda}^i + \exp[\mathbf{W}_\Delta]\mathbf{\Lambda}^i_{,3})(\mathbf{\Lambda}^i)^T\exp[-\mathbf{W}_\Delta] = \\
&= \exp_{,3}[\mathbf{W}_\Delta]\exp[-\mathbf{W}_\Delta] + \exp[\mathbf{W}_\Delta]\mathbf{\Omega}^i\exp[-\mathbf{W}_\Delta].
\end{aligned}$$

Simo proved in [24] that the axial vector $\boldsymbol{\beta}$ of the skew tensor $\exp_{,3}[\mathbf{W}_\Delta]\exp[-\mathbf{W}_\Delta]$ is

$$\boldsymbol{\beta} = \frac{\sin w_\Delta}{w_\Delta}\mathbf{w}_{\Delta,3} + \left(1 - \frac{\sin w_\Delta}{w_\Delta}\right)\left(\frac{\mathbf{w}_\Delta \cdot \mathbf{w}_{\Delta,3}}{w_\Delta}\right)\frac{\mathbf{w}_\Delta}{w_\Delta} + \frac{1}{2}\left(\frac{\sin\frac{1}{2}w_\Delta}{\frac{1}{2}w_\Delta}\right)^2 \mathbf{w}_\Delta \times \mathbf{w}_{\Delta,3}, \tag{4.2.12}$$

where $w_\Delta = \| \mathbf{w}_\Delta \|$. We proved that this axial vector can be equivalently given as

$$\boldsymbol{\beta} = \mathbf{T}(\mathbf{w}_\Delta)\mathbf{w}_{\Delta,3},$$

where $\mathbf{T}(\mathbf{w}_\Delta)$ is the tensor defined in (2.2.28), where $\mathbf{\Theta}$ is substituted with $\mathbf{W}_\Delta$. By substitution of $\boldsymbol{\beta}$ into the updating formula, $\mathbf{\Omega}_e^{i+1}$ becomes

$$\boxed{\mathbf{\Omega}_e^{i+1} = [\boldsymbol{\beta}_e\times] + \exp[\mathbf{W}_{\Delta e}]\mathbf{\Omega}_e^i\exp[-\mathbf{W}_{\Delta e}]}. \tag{4.2.13}$$

The derivative of axis displacement is then updated by

$$\boxed{\boldsymbol{\phi}_{0,3e}^{i+1} = \Delta\boldsymbol{\phi}_{0,3e} + \boldsymbol{\phi}_{0,3e}^i}. \tag{4.2.14}$$

The approximated elementary functions $\boldsymbol{\phi}_{0,3e}^i$, $\mathbf{\Omega}_e^i$ and $\mathbf{\Lambda}_e^i$ are stored from the previous iteration. Their initial values are evaluated from the initial values of $\boldsymbol{\phi}_0$ and $\mathbf{\Lambda}$

$$\boldsymbol{\phi}_0 = X_3\mathbf{E}_3 \qquad \mathbf{\Lambda} = \mathbf{I}$$

and consequently they are

$$\boldsymbol{\phi}_{0,3e} = \mathbf{E}_3, \quad \mathbf{\Lambda}_e = \mathbf{I}, \quad \boldsymbol{\omega}_e = \mathbf{0} \qquad \text{initial values} \tag{4.2.15}$$

The other quantities which constitute the updating formulae, $\Delta\boldsymbol{\phi}_{0,3e}$, $\mathbf{w}_\Delta$ and $\mathbf{w}_{\Delta,3}$ are interpolated as

$$\Delta\boldsymbol{\phi}_{0,3e} = \sum_{I=1}^{\text{nnodel}} N_{,3I}(X_3)\Delta\boldsymbol{\phi}_{0I} \tag{4.2.16}$$

$$\mathbf{w}_{\Delta e} = \sum_{I=1}^{\text{nnodel}} N_I(X_3)\mathbf{w}_{\Delta I}, \qquad \mathbf{w}_{\Delta,3e} = \sum_{I=1}^{\text{nnodel}} N_{,3I}(X_3)\mathbf{w}_{\Delta I}. \tag{4.2.17}$$

The nodal values come from the global nodal increment vector, solution of the $(i+1)^{th}$ Newton-Raphson's residual equation. Passing from latter vector to the local one of the element, as usual we need to rotate it from the global reference system to the elementary one. To compute the rotation matrix, the position of the local reference system with respect to the global one has to be defined. For this reason we have assigned to each element an **auxiliary node** such that the first axis of the local system is defined by the vector connecting the first node of the element to this auxiliary node. The third axis of the system is oriented as the reference beam axis from the first to the second node and the second axis is defined by the cross product of the other two. The local system is consistent with that adopted in the definition of kinematics.

Integrations are performed by *Gauss integrations*. To avoid locking problems we have used a global reduced integration, i.e. we have adopted one Gauss point to evaluate every integral.

It must be pointed out that we choose to update $\boldsymbol{\phi}_0$ with a step-by-step updating as done for the rotation, even though this procedure is not necessary because it is a vectorial space. We do so for code structure harmony.

Newton-Raphson's residual equation. The elementary internal force vector, $\mathbf{F}_{inte}$, and the tangent matrix, $\mathbf{K}_e$, are valued in the local element coordinate system. Since the linearized equilibrium ($\mathcal{L}[\delta L]=0$) is solved with respect to the global system of the whole structure, the basis change from local to global system is done on $\mathbf{F}_{inte}$ and $\mathbf{K}_e$ coordinates as

$$\mathbf{R}^T\mathbf{F}_{inte} = \mathbf{F}_{integ} \qquad \text{and} \qquad \mathbf{R}^T\mathbf{K}_e\mathbf{R} = \mathbf{K}_{eg}, \tag{4.2.18}$$

where the rotation matrix $\mathbf{R}$ controls the change of coordinate from the global system to the local one. Then the assembling procedure

$$\sum_{e=1}^{\text{numel}} \mathbf{F}_{integ} = \mathbf{F}_{int} \qquad \text{and} \qquad \sum_{e=1}^{\text{numel}} \mathbf{K}_{eg} = \mathbf{K} \tag{4.2.19}$$

gives us the nodal global internal force vector $\mathbf{F}_{int}$ and tangent matrix $\mathbf{K}$.

The linearized virtual work (4.2.1) can now be stated in the global form

$$\mathcal{L}[\delta L] = \begin{Bmatrix}\delta\boldsymbol{\phi}_0\\ \mathbf{w}_\delta\end{Bmatrix}_{\text{n}}^t \left(\mathbf{F}_{int} - \mathbf{F}_{ext} + \mathbf{K}\begin{Bmatrix}\Delta\boldsymbol{\phi}_0\\ \mathbf{w}_\Delta\end{Bmatrix}_{\text{n}}\right) = 0 \qquad \forall\ \{\delta\boldsymbol{\phi}_0, \mathbf{w}_\delta\}_{\text{n}}, \tag{4.2.20}$$

where "n" is the number of structural nodes, which leads to the incremental nodal solution

$$\boxed{-\mathbf{K}^{-1}\mathbf{Res} = \begin{Bmatrix} \Delta\boldsymbol{\phi}_0 \\ \mathbf{w}_\Delta \end{Bmatrix}_{\mathrm{n}} \qquad \text{with } \mathbf{Res} = \mathbf{F}_{int} - \mathbf{F}_{ext}.} \tag{4.2.21}$$

Above, **Res** is the global *residual vector* or out-of-balance force vector.

The new configuration $(\boldsymbol{\phi}_0^{i+1}, \boldsymbol{\Lambda}^{i+1}, \boldsymbol{\omega}^{i+1})$ is an equilibrium configuration if $\mathbf{Res} = \mathbf{0}$. The convergence condition in the actual implementation is

$$res_{rel} < \mathrm{tol}$$

where res_{rel} is the ratio between the norm of residual vector at the actual and at the first Newton-Raphson's iteration, while "tol" is a fixed tolerance.
Finally, box 4.2.2 summarizes the iteration and updating procedures.

Iteration i	Computation of the increment vector $-\mathbf{K}^{-1}\mathbf{Res}^i = \begin{Bmatrix} \Delta\boldsymbol{\phi}_{0Ig}^{i+1} \\ \delta\mathbf{w}_{Ig}^{i+1} \end{Bmatrix}_{I=\text{nnod}}$
Iteration i+1	<u>Within each element</u> Extraction of the increments relative to the element nodes from the structural increment vector $\begin{Bmatrix} \Delta\boldsymbol{\phi}_{0Ig}^{i+1} \\ \Delta\mathbf{w}_{Ig}^{i+1} \end{Bmatrix}_{I=\text{nnod}} \rightarrow \begin{Bmatrix} \Delta\boldsymbol{\phi}_{0Jg}^{i+1} \\ \Delta\mathbf{w}_{Jg}^{i+1} \end{Bmatrix}_{J=\text{nnodel}}$ Rotation from the global reference system to the local one Calculation of approximated continuum element increments and their derivative by interpolation on Gauss point X_h $\Delta\boldsymbol{\phi}_0^{i+1} = \sum_1^{\text{nnodel}} N_J \Delta\boldsymbol{\phi}_{0J}^{i+1}$, $\Delta\boldsymbol{\phi}_{0,3}^{i+1} = \sum_1^{\text{nnodel}} N_{J,3} \Delta\boldsymbol{\phi}_{0J}^{i+1}$ $\mathbf{w}_\Delta^{i+1} = \sum_1^{\text{nnodel}} N_J \Delta\mathbf{w}_J^{i+1}$, $\Delta\mathbf{w}_{0,3}^{i+1} = \sum_1^{\text{nnodel}} N_{J,3} \Delta\mathbf{w}_J^{i+1}$ Call of kinematic quantities stored for the element from previous iteration $\boldsymbol{\phi}_0^i$, $\boldsymbol{\Lambda}^i$ Updating of kinematical quantities $\boldsymbol{\phi}_{0,3}^{i+1} = \boldsymbol{\phi}_{0,3}^i + \Delta\boldsymbol{\phi}_{0,3}^{i+1}$ $\boldsymbol{\Lambda}^{i+1} = \exp[\mathbf{W}_\Delta^i]\boldsymbol{\Lambda}^i$ Calculation of approximated continuum element strain and stress resultants $\boldsymbol{\gamma}^{i+1}$, $\boldsymbol{\omega}^{i+1}$, $\mathbf{f}^{*i+1}$, $\mathbf{m}^{*i+1}$ Calculation of elementary internal force vector and tangent matrix $\mathbf{f}_{int,el}^{i+1}$, $\mathbf{K}_{el}^{i+1}$ Integration along reference axis length $\mathbf{f}_{int,el}^{i+1} = \sum_h wg_h L_e \mathbf{f}_{int,el}^{i+1}(X_h)$ $\mathbf{K}_{el}^{i+1} = \sum_h wg_h L_e \mathbf{K}_{el}^{i+1}(X_h)$ where wg_h is Gauss weight and L_e the reference element length Rotation of the internal force vector and tangent matrix from the local reference system to the global one <u>Outside the element</u> Assembling procedure Calculation of residual vector If no convergence, calculation of the next increment vector for iteration $i+2$
Iteration i+2	

Table 4.2.2: Small-strain finite-deformation beam model using the direct form of linearization: Newton-Raphson's and updating procedures

4.3 Four tests for the Finite-Deformation Small-Strain element

In order to check the reliability of our finite-deformation small-strain element (FD-SS), we develop four significative tests and we compare our element results with those obtained by the element *framef3d.f* of FEAP. FEAP is an open source finite element analysis program developed by R. Taylor at University of California, Berkeley. The element framef3d.f has been programmed by A. Ibrahimbegovic and M. Al Mikdad in 1996 and subsequently modified by E. Kasper and R. Taylor in 1998. It uses the rotation vector parametrization of rotation, and consequently the form of variation and linearization we indicated as indirect. Our element instead works with the equations which use the direct linearization of rotation. Both employ two node elements, with linear shape function.

In all tests we consider an elastic isotropic material for which we assign the Young's modulus, E, the Poisson's ratio, ν, and calculate the shear modulus G as $G = \frac{E}{2(1=\nu)}$. We consider a cross-section for which we assign the height of the section along first and second axes of the reference system, respectively $h1$ and $h2$; the area, A; the moments of inertia around first and second axes of reference system, respectively J_1 and J_2; the torsional moment of inertia, J_t. . The performed tests are

- a three-dimensional problems without buckling, indicated as Cantilever 45° bend
- a planar problem without buckling, indicated as Cantilever roll-up
- a three-dimensional problem with buckling, indicated as Clamped beam under lateral buckling load
- a planar problem with buckling, indicated as end loaded column, i.e. the Elastica

The planar problems are of interest since we found in literature analytical solutions. Moreover we perform tests with and without buckling to check the response of the program with and without passing through the critical buckling point.

4.3.1 Three-dimensional problem without buckling: Cantilever 45° bend

This test aims at studying the equilibrium configuration of a clamped curved beam loaded by a concentrated tip force, P (see figure 4.3).
The bend has a radius of 10 [m] and an angle of 45°, for a total beam length of 15.70796 [m]. The real curve of the axis is approximated by 8 linear elements of length 0.98135. Each element approximates a subtended arch of length 0.3125π [m]. Beam material and geometric section properties are summarized below.
The load P as a maximum value of 50 [KN] and is applied in 60 load steps.

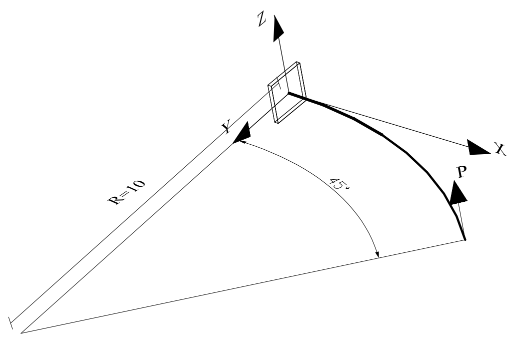


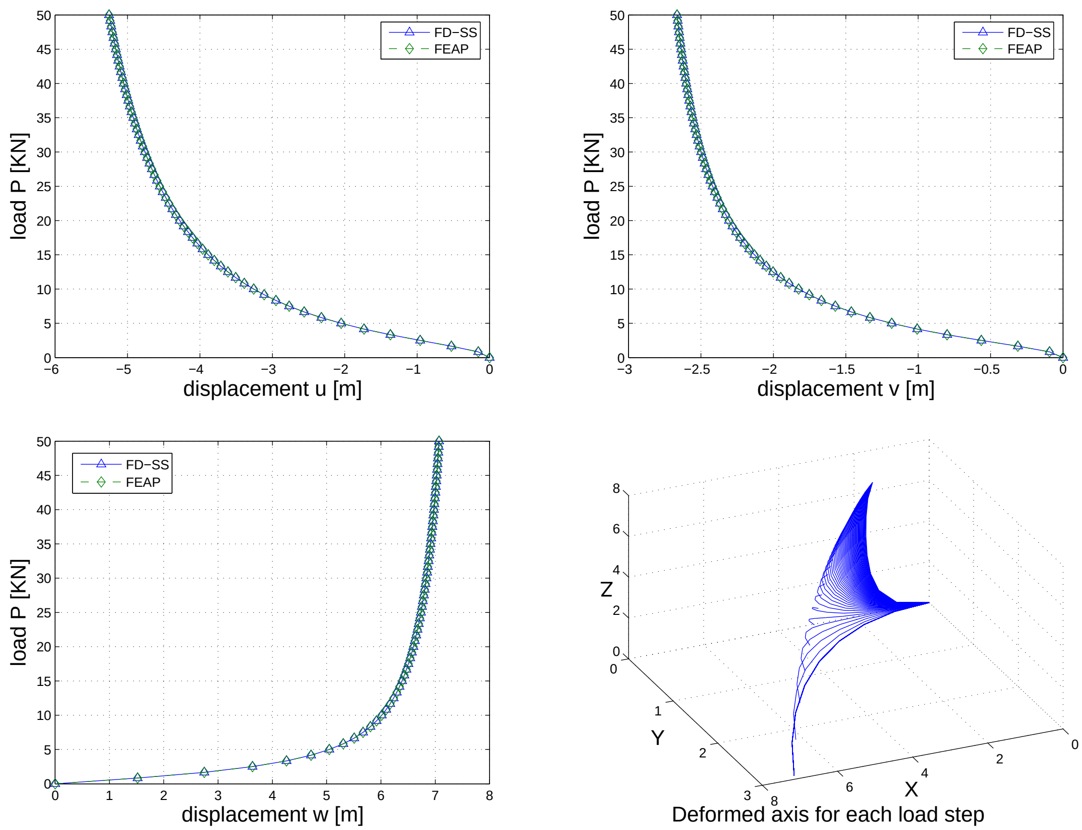
Figure 4.3: Cantilever 45° bend test: problem geometry, reference system and applied load.

Material properties
E=1e7[KN\ m^2],
G=0.5e7[KN\ m^2], ν=0.

Cross-section properties
h_1=0.1[m], h_2=0.1[m],
A=0.01[m^2],
J_1=0.1^4/12[m^4],
J_2=0.1^4/12[m^4],
$J_t = 0.1406 \cdot 0.1^4$ [m^4].

Analysis data
numel=8, L_e=0.98135[m],
tol =10^{-8},
load step number = 60.

The structure under investigation evidently experiences all modes of deformation: bending, shear, extension and torsion. For this reason a lot of References dealing with finite-rotations have performed this test, as for example [11, 24, 27]. In the following we compare FD-SS and FEAP analysis results, both graphically and numerically. The convergence of the two elements are compared in Table 4.3.1.



Displacements at final equilibrium configuration

	u[m]	v[m]	w[m]
FD-SS	-5.2543E+00	-2.6650E+00	7.0699E+00
FEAP	-5.2543E+00	-2.6650E+00	7.0699E+00

Figure 4.4: Cantilever 45° bend Test. FD-SS and FEAP results comparison: end node beam displacement (u) in x direction versus end applied load (top left); end node beam displacement (v) in y direction versus end applied load (top right); end node beam displacement (w) in z direction versus end applied load (bottom left). Triangles and squares represent the convergence point for each load step. Beam deformed configuration at each load step (bottom right). Table of displacement numerical values at equilibrium configuration.

	Load step	Total iterations	Residual final norm
FD-SS	1	10	1.341303E-011
	2	10	1.778115E-011
	3	9	1.418029E-011
	...	...	...
	15	5	3.559326E-009
	...	...	...
	60	5	1.443747E-011
FEAP	1	9	6.49E-10
	2	10	1.72E-10
	3	8	5.36E-07
	...	...	...
	15	6	2.11E-10
	...	...	...
	60	4	2.06E-08

Table 4.3.1: Cantilever 45° bend test. Table of convergence: comparison of FD-SS and FEAP.

The results are very good. Both displacements and convergence path of FD-SS agree with those of FEAP. Analyzing the evolution of the residual norm through the iterations of each step we noticed that, curiously, both FD-SS and FEAP increase the residual norm passing from the first iteration to the second. This increment is higher in the first steps, where in fact the elements need more iteration to converge. Unfortunately at the moment we do not have an explanation for this behaviour. In Table 4.3.2 the evolution of the residual norm through iterations is shown for the first step, for both FD-SS and FEAP.

Iteration	FD-SS residual norm	FEAP residual norm
1	8.333333E-01	8.3333333E-01
2	4.865321E+03	4.8652374E+03
3	3.277604E+01	3.2765924E+01
4	2.212159E+01	4.7668544E+00
5	4.143674E+00	7.5874085E-01
...	...	...
9	1.347692E-07	6.4944294E-10
10	1.341303E-11	...

Table 4.3.2: Cantilever 45° bend test. Residual norm evolution through the first step: FD-SS and FEAP comparison.

4.3.2 Planar problem without buckling: Cantilever roll-up

This test aims at studying the equilibrium configuration of a clamped beam loaded by a concentrated tip moment, m (see figure 4.5).
The beam has length L 10 [m] and is discretized by 10 elements of 1 [m]. Beam material and geometric section properties are summarized below.
The load has a maximum value of 41.664π [KN] and it is applied both in 6 load steps and in 20 load steps.

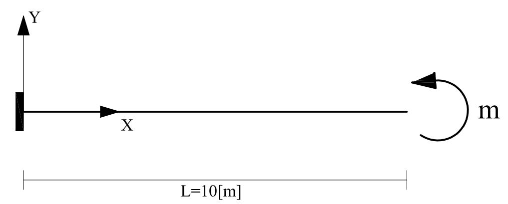


Figure 4.5: Cantilever roll-up test: problem geometry , reference system and applied load.

Material properties
E=2e5[KN\ m^2],
G=7.6923e4[KN\ m^2],
ν=0.3.

Cross-section properties
h_1=0.5[m], h_2=0.1[m],
A=0.05[m^2],
J_1=4.1667e-5[m^4],
J_2=1.041667e-3[m^4],
$J_t = 5e-4$ [m^4].

Analysis data
numel=10, L_e=1[m],
tol=10^{-8},
load step number : 6 and 20

This problem is a pure bending problem. According to the classical Euler formula for this case, the curvature of the beam deformed axis is constant along the beam and therefore the exact solution of the beam deformed shape is a part of a circle. The analytical expression of displacements, u and v, and rotation β at beam end ($X = L$) are[1]

$$\beta(L) = \frac{mL}{EJ}, \qquad u(L) = L - \frac{l}{\beta}\tan\frac{\beta}{2}(1+\cos\beta), \qquad v(L) = \frac{l}{\beta}\tan\frac{\beta}{2}\sin\beta.$$

For the chosen moment value m = 41.664π [KN] and the adopted material properties and geometry, the tip analytical displacements and rotation are summarized in Table 4.3.3.

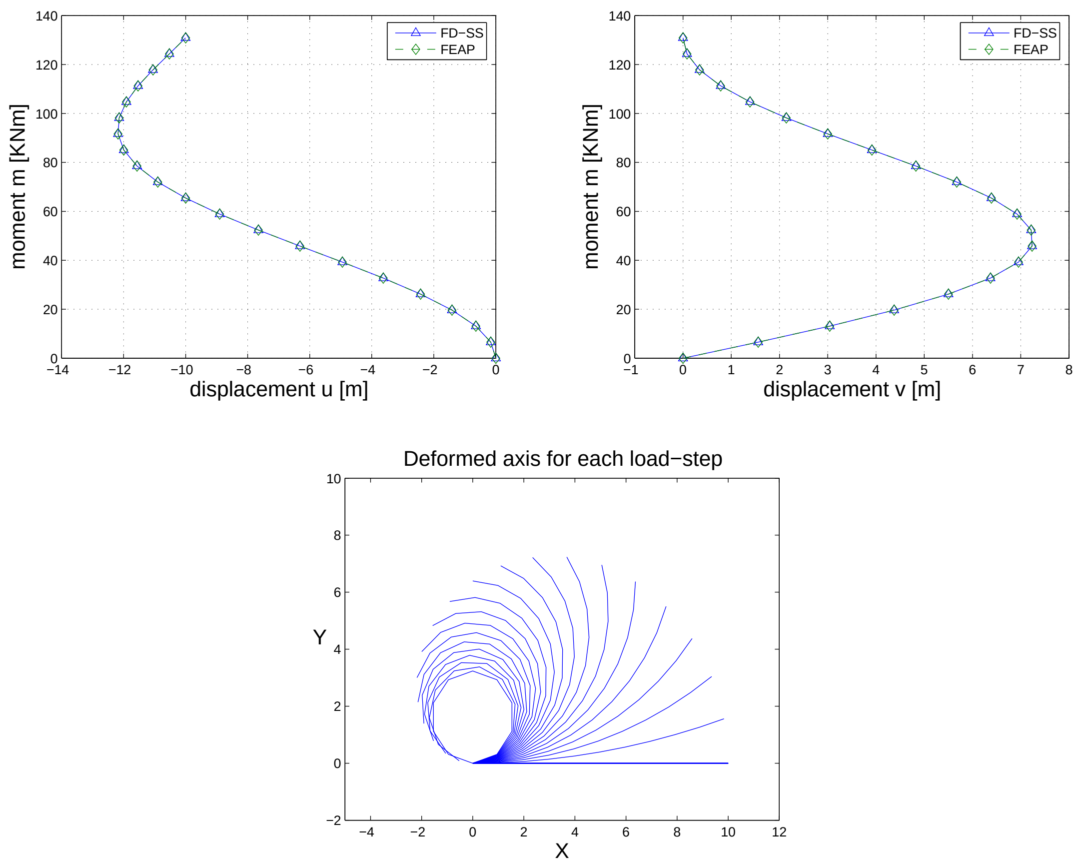
u[m]	v[m]	$\beta[rad]$
-10.0 E+00	0.0 E+00	2π E+00

Table 4.3.3: Cantilever roll-up test: analytical displacements in direction x (u) and y (v) and rotation β of beam tip for the assigned moment, geometry and material properties

[1]for further information on the analytical solution see [11]

A lot of authors who implemented a finite deformation model performed this test (see for example [11, 24, 27]).

In the following we compare FD-SS and FEAP analysis results, both graphically and numerically, using 20 load steps. The deformed axis of beam for each load step is plotted in Figure 4.6. We note that 6 is the minimum number of load steps needed by FD-SS and FEAP to converge. The convergence of the two elements are compared in Tables 4.3.4 and 4.3.5 using both 20 and 6 load steps.



Displacements and rotation of tip beam at final equilibrium configuration

	u[m]	v[m]	$\beta[rad]$
FD-SS	-10.00065069E+00	1.3082E-07	6.2832E+00
FEAP	-1.0000E+01	1.3806E-15	6.2832E+00

Figure 4.6: Cantilever roll-up test. FD-SS and FEAP results comparison using **20 load steps**: end node beam displacement (u) in x direction versus end applied moment m (top left); end node beam displacement (v) in y direction versus end applied moment m (top right); beam deformed configuration at each load step (center); displacements and rotation of beam tip at the equilibrium configuration (bottom).

	Load step	Total iterations	Residual final norm
FD-SS	1	6	5.548187E-009
	2	6	5.566500E-009
	...	...	...
	6	6	7.481233E-009
FEAP	1	6	5.57E-09
	2	6	5.58E-09
	...	...	...
	20	6	7.50E-09

Table 4.3.4: Cantilever roll-up test. Table of convergence: comparison of FD-SS and FEAP using 20 load steps

	Load step	Total iterations	Residual final norm
FD-SS	1	15	7.700860E-011
	2	15	7.973869E-011
	...	...	...
	6	15	1.094259E-010
FEAP	1	15	8.53E-11
	2	15	8.80E-11
	...	...	...
	6	15	1.08E-10

Table 4.3.5: Cantilever roll-up test. Table of convergence: comparison of FD-SS and FEAP using 6 load steps

4.3.3 Three-dimensional problem with buckling: Clamped beam under lateral buckling load

This test aims at studying the equilibrium configuration of a clamped beam loaded by two concentrated tip forces: a perturbation load $\mathrm{P_Z} = 0.01[KN]$ and a load $\mathrm{P_Y} = 10[KN]$, (see figure 4.10). The latter acts in direction of the beam height, while the perturbation load acts in the direction of the beam thickness. The beam height is 0.5 [m], the thickness is 0.1 [m] and the length is 10[m]. These load and geometry conditions are set up such that the beam experiences a lateral buckling or *bending-torsional* buckling. The beam axis is discretized by 10 elements of 1 [m]. Beam material and geometric section properties are summarized below.

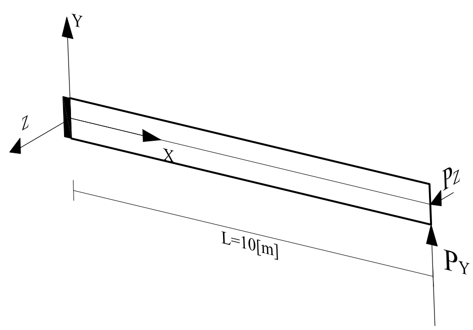


Figure 4.7: Clamped beam lateral buckling test: problem geometry, reference system and applied loads.

Material properties
E=2e5[KN\ m^2],
G=0.76923e5[KN\ m^2],
ν=0.3.

Cross-section properties
h_y=0.5[m], h_z=0.1[m],
A=0.05[m^2],
J_y=4.1667e-5[m^4],
J_z=1.0416e-3[m^4],
$J_t = 5.0e-4$ [m^4].

Analysis data
numel=10, L_e=1[m],
tol=10^{-8},
load step number: 100 and 400.

This problem has been investigated in [11, 24, 27]. In the following we compare FD-SS and FEAP analysis results, both graphically and numerically, for the cases of 100 and 400 load steps.

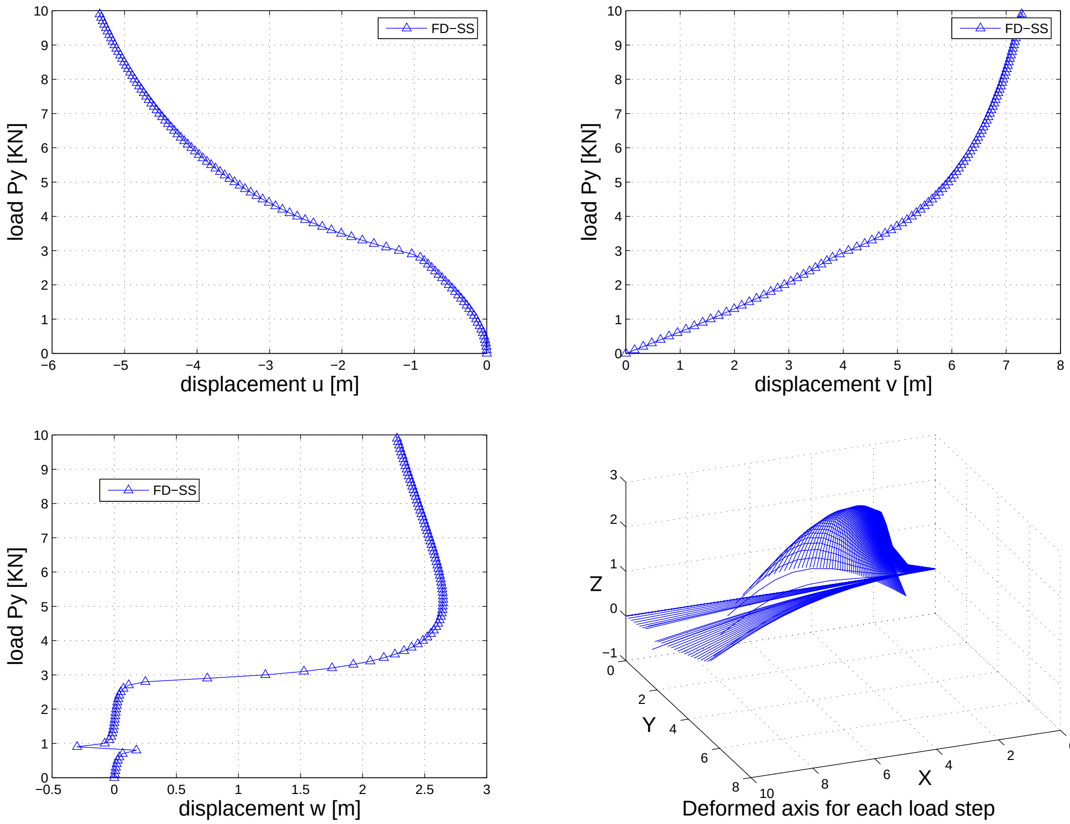


Figure 4.8: Clamped beam, lateral buckling test. FD-SS results for **100 load-steps**: end node beam displacement (u) in x direction versus end applied load Py (top left); end node beam displacement (v) in y direction versus end applied load Py (top right); end node beam displacement (w) in z direction versus end applied load Py (bottom left). Triangles represent the convergence point for each load step. Beam deformed configuration at each load step (bottom right). Note that in this case FEAP element *does not converge*. FD-SS passes through the first critical load, located at 0.9 [KN] and shows buckling at the *second critical load* of structure. The element uses 14 and 12 iterations respectively at steps 9 and 10, to pass through the first critical load.

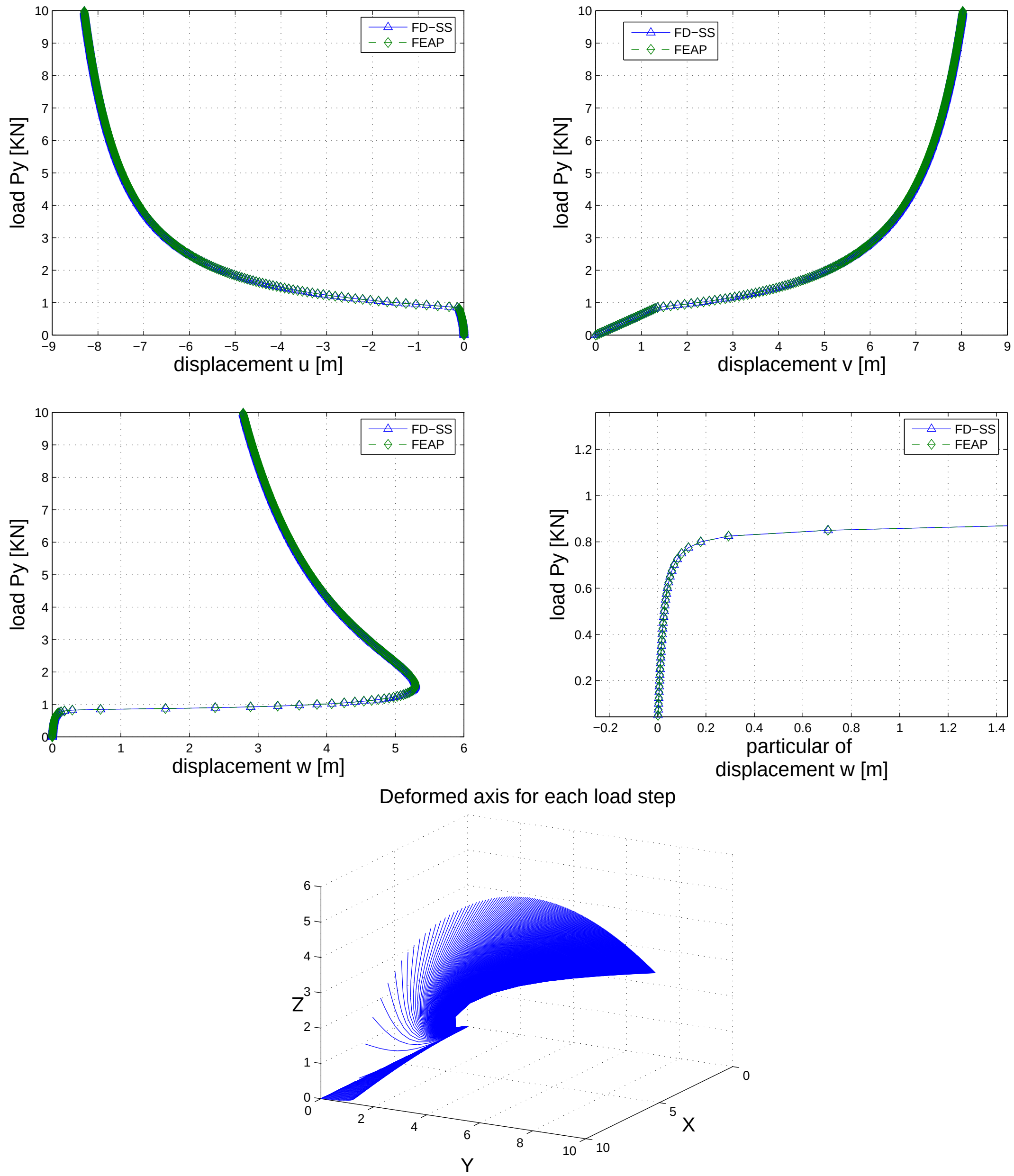


Figure 4.9: Clamped beam, lateral buckling test. FD-SS and FEAP results for **400 load-steps**: end node beam displacement (u) in x direction versus end applied load Py (top left); end node beam displacement (v) in y direction versus end applied load Py (top right); end node beam displacement (w) in z direction versus end applied load Py (bottom left); particular of displacement (w) at critical point(bottom right). Triangles represents the convergence point for each load step. Deformed beam configuration for each load step (bottom center). Results of two element fully agree.

	Load step	Total iterations	Residual final norm
FD-SS	1	4	1.092390E-011
	2	4	1.715679E-010
	...	...	...
	35	13	1.589177E-012
	36	12	2.252431E-012
	37	11	2.041948E-012
	...	...	...
	400	4	3.171766E-012
FEAP	1	4	2.31E-11
	2	4	1.76E-10
	...	...	...
	35	12	3.59E-11
	36	14	1.26E-11
	37	10	1.29E-08
	...	...	...
	400	4	1.88E-11

Table 4.3.6: Clamped beam lateral buckling test. Table of convergence: comparison FD-SS and FEAP using 400 loadsteps.

	u[m]	v[m]	w[m]
FD-SS	-8.3015E+00	-8.0286E+00	2.7802E+00
FEAP	-8.3015E+00	-8.0287E+00	2.7802E+00

Table 4.3.7: Displacement at final equilibrium configuration

It must be pointed out that both FD-SS and FEAP have some convergence problem if a small number of load step is employed. For 100 load steps FD-SS behaves better than FEAP since the latter does not converge while the former converge to the second critical load. For 200 loadsteps FD-SS does not converge in the step of the critical point in the maximum number of iterations (20) and therefore the solution is not accurate as in the case of 400 load steps. Using 300 load steps the analysis fails since the global tangent matrix result to be close to singular at critical point. FEAP instead from 300 load-step converges. We can conclude that FD-SS and FEAP do not suffer of convergence problem when the load-steps are sufficiently small when passing through the critical point.

4.3.4 Planar problem with buckling: end loaded column, i.e. the Elastica

This test aims at studying the equilibrium configuration of an end loaded column, (see figure 4.10). The end load is indicated as P_x. A perturbed load, P_y, acts orthogonally to the axis in order to numerically force the axial buckling of the column. The problem is known as Elastica and was first studied by Euler. In 1859 Kirchhoff obtained an analytical solution making use of simplified hypothesis and calculated it by elliptic integrals. Since we compare results of FD-SS with this solution, we present below the hypotheses and the final solution. In appendix it can be found the full development of the analytical solution.

For the FE analysis, the beam axis is discretized by 10 elements of 1 [m]. Beam material and geometrical section properties are summarized below.

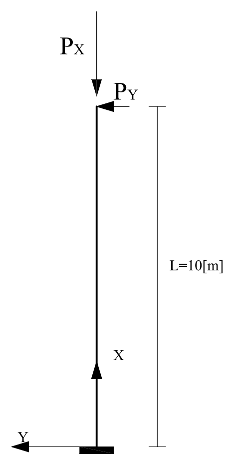


Figure 4.10: End loaded column test: problem geometry, reference system and applied loads.

Material properties
E=2e5[KN\ m^2],
G=0.76923e5[KN\ m^2],
ν=0.3.

Cross-section properties
h_y=0.1[m], h_z=0.1[m],
A=0.01[m^2],
J_1=0.1^4/12[m^4],
J_2=0.1^4/12[m^4],
$J_t = 0.1406 * 0.1^4$ [m^4].

Analysis data
numel=10, L_e=1[m],
tol=10^{-8},
load step number: 100 and 400

In the following we present two analysis: one carried out using 100 load-steps and P_y=-0.4[KN] and one carried out using 400 load-steps and P_y=-0.6[KN]. Both of them are compared with analytical solution. The latter is found by the equations in Table 4.3.8, for an assigned tip rotation θ_l which range from 0 [rad] to π [rad]. As shown in the graphics the result of FD-SS fully agree with the analytical ones.

Hypotheses

- the column is inextensible
- the curvature of the column axis is defined as the derivative of the angle of the tangent to the axis, θ, with respect to the length s along the column, i.e. there are not shear effects
- the bending moment is proportional to the local curvature

Top load

$$P(\theta_l) = \Big(\frac{2}{\pi}K(c)\Big)^2 P_0$$

Transverse displacement at top column

$$w_l = \frac{2c}{K(c)} l$$

Axial displacement at top column

$$u_l = 2\Big(1 - \frac{E(c)}{K(c)}\Big) l$$

where
θ_l is the tip column rotation
$c = \sin(\frac{1}{2}\theta_l)$ is a constant
$K(c)$ is the complete elliptic integral of the first kind
$E(c)$ is the complete integral of the second kind
$P_0 = \Big(\frac{\pi}{2l}\Big)^2 EJ$ is the Euler buckling load

Table 4.3.8: End loaded column: Kirchhoff hypotheses and analytical solution of tip displacement and load for assigned tip rotation

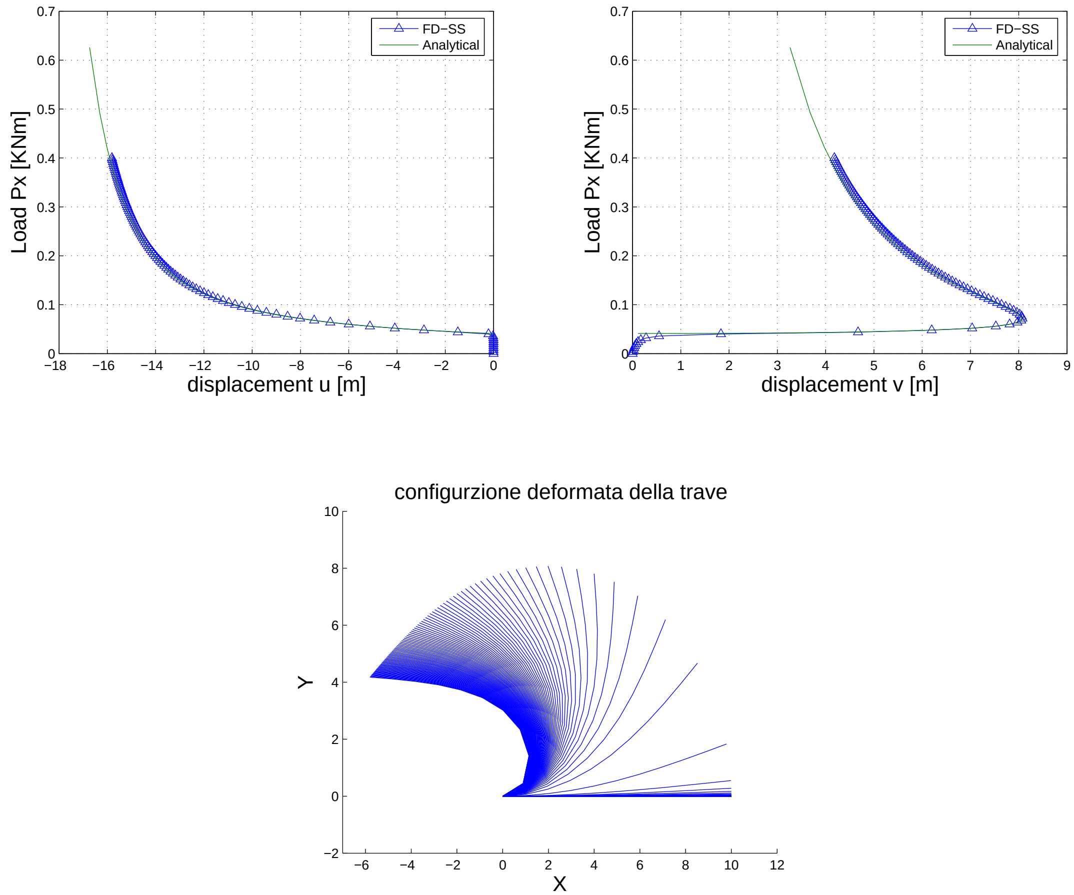


Figure 4.11: End loaded column test. FD-SS results for $\mathbf{P}_x$**=-0.4[KN]** and **100 load-steps** compared with analytical solution: end node beam displacement (u) in x direction versus end applied load Px (top left); end node beam displacement (v) in y direction versus end applied load Px (top right); beam deformed configuration at each load step (bottom).The results fully agree. The analytical solution have a longer path than FD-SS because it is obtained controlling θ.

	Load step	Total iterations	Residual final norm
FD-SS	1	5	5.989276E-013
	2	5	4.126241E-013
	...	...	...
	10	11	3.124480E-013
	11	12	4.021528E-013
	12	12	3.065757E-013
	...	...	...
	100	4	3.315613E-013

Table 4.3.9: End loaded column test: table of convergence for FD-SS.

Iteration	FD-SS residual norm
1	4.000200E-003
2	2.085982E+002
3	6.222225E+000
4	4.251998E-002
5	6.393637E-001
...	...
8	1.373952E-003
9	4.585684E-009
10	8.138703E-011
11	4.021528E-013

Table 4.3.10: End loaded column test. Residual norm evolution of FD-SS through the **11 step** (critical point).

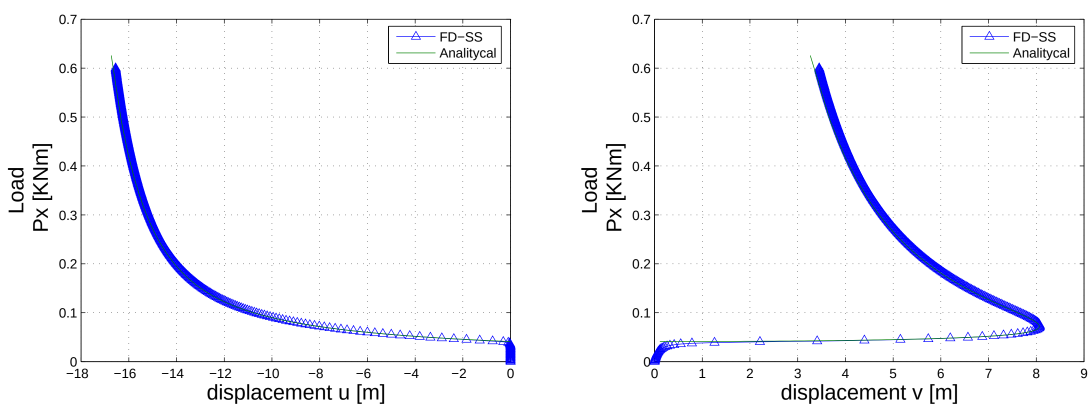


Figure 4.12: End loaded column test. FD-SS results for $\mathbf{P}_x$**=-0.6[KN]** and **400 load-steps** compared with analytical solution: end node beam displacement (u) in x direction versus end applied load Px (top left); end node beam displacement (v) in y direction versus end applied load Px (top right).The results fully agree.

Conclusions

In this work we presented a model for the description of deformation and static equilibrium of beams with no restriction on either displacements or rotations. We adopted the beam kinematic hypothesis first proposed by Simo in [23] and computed the associated three-dimensional deformation and strain beam measures. The introduction of a special polar decomposition for the deformation gradient allows to find the beam strain resultants and to express the beam Green-Lagrange strain tensor $\mathbf{E}$ in a compact and useful way.

The linerizations of the deformation gradient and of the Green-Lagrange strain tensor are then used to exploit the two forms of the three dimensional principle of virtual work defined on the reference beam configuration. For each form we recovered the associated one dimensional principle. The comparison between them showed that the one dimensional principle of virtual work obtained using the linearization of the Green-Lagrange strain tensor is the rotated-back expression of the one obtained using the linearization of the deformation gradient.

The obtained equilibrium equations describe a finite-deformation finite-strain regime. In order to develop a small strain theory we simplified the Green-Lagrange strain tensor by neglecting a quadratic term in the beam strain. The identification of this term is made clear thanks to the adopted special polar decomposition of the deformation gradient. We showed that postulating a linear elastic relation between the simplified Green-Lagrange strain and the second Piola-Kirchhoff stress tensors yields a linear elastic relation between beam strain and stress resultants. A similar proof is developed in [25] while in all other studied References the same constitutive relations on resultants are only postulated.
The simplification of equilibrium equations using the small strain hypothesis and the introduction of constitutive one-dimensional relations led to the final finite-deformation small-strain model equations.

These equations has been expressed for two possible linearizations of the rotation tensor which defines the cross section orientation. We referred to these linearizations procedures as direct or indirect. Developing the direct form we recovered exactly the equations proposed by Simo in [23, 24]. Instead, following the indirect form we recovered the equations given by Ibrahimbegovic in [11] and Ritto-Correa in [21].

For both formulations we schematically presented the finite element approach, pointing out that, with respect to rotations, the direct form is path-dependent while the indirect form is path-independent. The direct approach has been then described in detail and implemented. To conclude, numerical results of four significative tests have been presented which confirmed the good behaviour of the finite element formulation.

We think that the most important issues of this work are the clear computation of finite-deformation beam equilibrium equations in the framework of three-dimensional nonlinear principle of virtual work, the explanation of the small-strain hypothesis and of its role into the model, the comparison between the direct and indirect linearization produceres and their associated finite element approaches.

From a computational point of view our future task is the implementation of the indirect form of equations. Concerning with the theory of finite-deformation beams, the achieved knowledge open the doors to developments in the dynamic regime, very interesting for the analysis of dynamic multibody systems, and in the case of three-dimensional inelastic constitutive relations. Moreover could be of interest to develop the theory for a modified kinematics able to include description of warping phenomena and section striction.

We have done a lot of work, but as you see a lot can still be done.

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Appendix A

Proof for the pure stretch tensor A

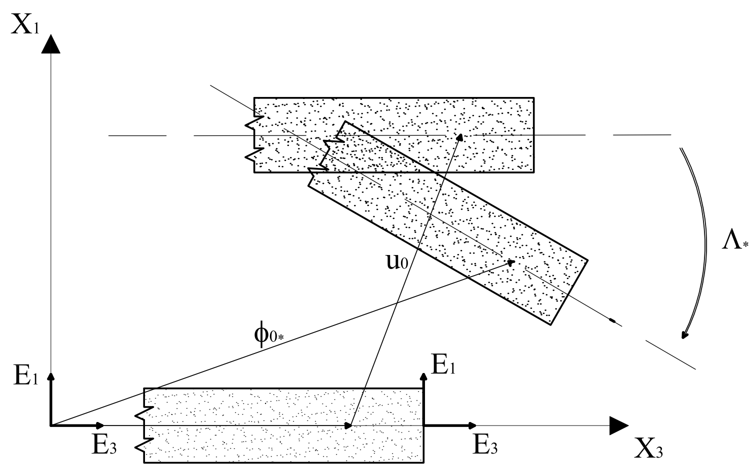

Figure A.1: Rigid motion of beam

We show that the tensor **A** of the left extended polar decomposition of the deformation gradient is a pure stretch tensor, i.e. $\mathbf{A} = \mathbf{I}$ in the case for a rigid translation and rotation around the origin. Looking at figure (A.1), the deformation map (3.1.2) takes the form

$$\boldsymbol{x}^* = (X_3\mathbf{E}_3 + \mathbf{u}_0^* + X_\alpha \mathbf{E}_\alpha)\mathbf{\Lambda}^*. \tag{A.0.1}$$

where $\mathbf{\Lambda}^*$ is the tensor $\mathbf{\Lambda}$ uniform in X_3 and $\mathbf{u}_0^*$ is the axis displacement vector uniform in X_3. We rewrite the equation (A.0.1) in the form

$$\boldsymbol{x}^* = \boldsymbol{\phi}_0^* + X_\alpha \mathbf{\Lambda}^* \mathbf{E}_\alpha \tag{A.0.2}$$
$$\boldsymbol{\phi}_0^* = (X_3\mathbf{E}_3 + \mathbf{u}_0^*)\mathbf{\Lambda}^* \tag{A.0.3}$$

where the vector field in (A.0.3) is the axis current position. From these positions it follows that

$$\boldsymbol{\Lambda}^*_{,3} = 0 \quad \text{(A.0.4)}$$
$$\boldsymbol{\Omega}^* = \boldsymbol{\Lambda}^*_{,3}\boldsymbol{\Lambda}^* = 0 \quad \text{(A.0.5)}$$
$$\boldsymbol{\phi}^*_{0,3} = \boldsymbol{\Lambda}^*\mathbf{E}_3 \quad \text{(A.0.6)}$$

From equations (A.0.5) and(A.0.6) and equations (3.1.14) and (3.1.15) we obtain

$$\boldsymbol{\gamma}^*_r = \boldsymbol{\Lambda}^{*T}\boldsymbol{\Lambda}^*\mathbf{E}_3 - \mathbf{E}_3 = 0 \quad \text{(A.0.7)}$$
$$\boldsymbol{\kappa}^*_r = 0 \quad \text{(A.0.8)}$$
$$\mathbf{A} = \mathbf{I} \qquad \text{for rigid body motion} \quad \text{(A.0.9)}$$

It confirms that **A** is a measure of pure stretch

Appendix B

Analytical solution for end loaded column: the Elastica

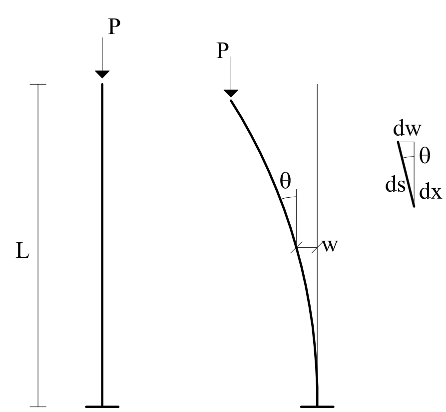

Figure B.1: End loaded column configuration

The hypothesis in which the analytical solution is calculated are the following

- the column is inextensible
- the curvature of the column axis is defined as the derivative of the angle of the tangent to the axis, θ, with respect to the length s along the column, i.e. there are not shear effects
- the bending moment is proportional to the local curvature

Thus, the kinematic and constitutive relations of the theory are

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \kappa, \qquad \frac{M}{EJ} = \kappa \tag{B.0.1}$$

where the bending stiffness EJ is assumed constant.
The moment $M(S)$ at any section is calculated from static as

$$M(S) = P[w_l - w(s)] \tag{B.0.2}$$

where w_l is the transverse displacement at the top of the beam. Differentiation with respect to the arc-length s gives

$$\frac{\mathrm{d}M}{\mathrm{d}s} = -P\frac{\mathrm{d}w}{\mathrm{d}s} \tag{B.0.3}$$

The derivative on the RHS is expressed in terms of the angle θ by use of B.0.1, and the derivative of the LHS follows from the detail in figure B.1 as $\mathrm{d}w/\mathrm{d}s = \sin\theta$. Thus, the differential equation for finite deflection of an homogeneous elastic column is obtained in the form

$$EJ\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = -P\sin\theta \tag{B.0.4}$$

It is convenient to introduce the parameter

$$k^2 = \frac{P}{EJ} \tag{B.0.5}$$

When introducing the Euler buckling load

$$P_0 = \left(\frac{\pi}{2l}\right)^2 EJ \tag{B.0.6}$$

the parameter k^2 can be expressed as

$$k^2 = \left(\frac{\pi}{2l}\right)^2 \frac{P}{P_0} \tag{B.0.7}$$

In term of the parameter k^2 the finite deflection column equation becomes

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} + k^2\sin\theta = 0 \tag{B.0.8}$$

This equation is similar to that governing oscillations of a pendolum under gravity, and in both cases the linearized equation is obtained by replacing $\sin\theta$ withθ

To integrate the equation B.0.8 it is necessary to manipulate it. Multiplicate B.0.8 by $\mathrm{d}\theta\mathrm{d}s$ and use the following differential relations from right to left into B.0.8

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}(\frac{\mathrm{d}\theta}{\mathrm{d}s})^2 = \frac{\mathrm{d}^2\theta}{\mathrm{d}s^2}\frac{\mathrm{d}\theta}{\mathrm{d}s} \tag{B.0.9}$$

$$2\frac{\mathrm{d}}{\mathrm{d}s}\left(\sin(\frac{1}{2}\theta)\right) = 2\sin(\frac{1}{2}\theta)\cos(\frac{1}{2}\theta)\frac{\mathrm{d}\theta}{\mathrm{d}s} = \sin\theta\frac{\mathrm{d}\theta}{\mathrm{d}s} \tag{B.0.10}$$

Then the following integrated form of B.0.8 is obtained

$$\frac{1}{4k^2}(\frac{\mathrm{d}\theta}{\mathrm{d}s})^2 = c^2 - \sin^2(\frac{1}{2}\theta) \tag{B.0.11}$$

where c is an arbitrary constant which depends by boundary condition. For the base clamped column in B.1, at the loaded end the moment vanishes, and then the curvature,$\mathrm{d}\theta\mathrm{d}s = 0$. Substituting the condition into B.0.11, c is found to be

$$c = \sin(\frac{1}{2}\theta_l) \tag{B.0.12}$$

Integration by elliptic integrals Taking the square root of the differential equation B.0.11 with a positive sign, and thereby considering displacement to the left as shown in B.1, the equation takes form

$$\frac{\mathrm{d}\theta}{\sqrt{c^2 - \sin^2(\frac{1}{2}\theta)}} = 2k\mathrm{d}s \tag{B.0.13}$$

This relation leads to elliptic integrals, when the following substitution is made,

$$\sin(\frac{1}{2}\theta) = c\sin\phi \tag{B.0.14}$$

Differentiation of this formula gives

$$\frac{1}{2}\cos(\frac{1}{2}\theta)\mathrm{d}\theta = c\cos\phi\mathrm{d}\phi \tag{B.0.15}$$

end substitution into B.0.13 gives

$$k\mathrm{d}s = \frac{\frac{1}{2}\mathrm{d}\theta}{c\cos\phi} = \frac{\mathrm{d}\phi}{c\cos(\frac{1}{2}\theta)} = \frac{\mathrm{d}\phi}{\sqrt{1 - c^2\sin^2\phi}} \tag{B.0.16}$$

From B.0.14 and the boundary condition, it follows that ϕ varies between 0 at s = 0 and $\frac{pi}{2}$ at $s = l$. Thus, the parameter kl can be obtained by integration of B.0.16

$$kl = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1 - c^2\sin^2\phi}} \tag{B.0.17}$$

The integral on the right is called the complete elliptic integral of the first kind, and it is denoted

$$K(c) = \int_0^{\pi/2} \frac{\mathrm{d}\phi}{\sqrt{1 - c^2\sin^2\phi}} \tag{B.0.18}$$

Using the relation eqn:bordo into B.0.17 and B.0.17 the load ratio P/P_0 from B.0.7 take the form

$$\sqrt{\frac{P}{P_0}} = \frac{2}{\pi}K[\sin(\frac{1}{2}\theta_l)] \tag{B.0.19}$$

in term of the end rotation θ_l.

The transverse displacement w_l of the top of the elastica can be determined from B.0.11 by using that $\theta_0 = 0$, whereby the parameter c becomes

$$c = \frac{1}{2k}\frac{\mathrm{d}\theta}{\mathrm{d}s} = \frac{1}{2k}\frac{M_0}{EJ} = \frac{w_l}{2k}\frac{P}{EJ} = \frac{1}{2}kw_l \tag{B.0.20}$$

Substitution from B.0.12 and B.0.17 leads to

$$\frac{w_l}{l} = \frac{2\sin(\frac{1}{2}\theta_l)}{K[\sin(\frac{1}{2}\theta_l)]} \tag{B.0.21}$$

giving the transverse displacement as function of the rotation θ_l
The vertical displacement of the top of the elastica is found by integration of the relation

$$\frac{\mathrm{d}x}{\mathrm{d}s} = \cos\theta \tag{B.0.22}$$

illustrated in the detail in **??**. This differential relation can be integrated in the following way. First $\cos\theta$ is reduced to half angle by a standard trigonometric relation,

$$\mathrm{d}x = \cos\theta \mathrm{d}s = [2\cos^2(\frac{1}{2}\theta) - 1]\mathrm{d}s \tag{B.0.23}$$

and then ds is substituted from B.0.16, giving

$$\mathrm{d}x + \mathrm{d}s = 2\cos^2(\frac{1}{2}\theta)\mathrm{d}s = \frac{2}{k}\cos(\frac{1}{2}\theta)\mathrm{d}\phi \tag{B.0.24}$$

Itegration of this relation over the full length of the elastica gives

$$(l - u_l) + l = \frac{2}{k}\int_0^{\pi/2} \sqrt{1 - c^2\sin^2\phi}\mathrm{d}\phi \tag{B.0.25}$$

where u_l denotes the downward displacement of the end of the elastica. The intergal is called the complete elliptic integral of the second kind, and is denoted

$$E(c) = \int_0^{\pi/2} \sqrt{1 - c^2\sin^2\phi}\mathrm{d}\phi \tag{B.0.26}$$

using the expression B.0.17 for kl, the final form of the axial displacement of the top of the elastica becomes

$$\frac{u_l}{l} = 2\Big(1 - \frac{E[\sin(\frac{1}{2}\theta_l)]}{K[\sin(\frac{1}{2}\theta_l)]}\Big) \tag{B.0.27}$$

which gives the axial displacement of the elastica as a function to the top rotation θ_l
For the analytical expression of the deformed shape of the elastica, see [17] cap 3.