Some Advances on Isogeometric Analysis at KAUST

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Joint work with:
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The Cost of Continuity

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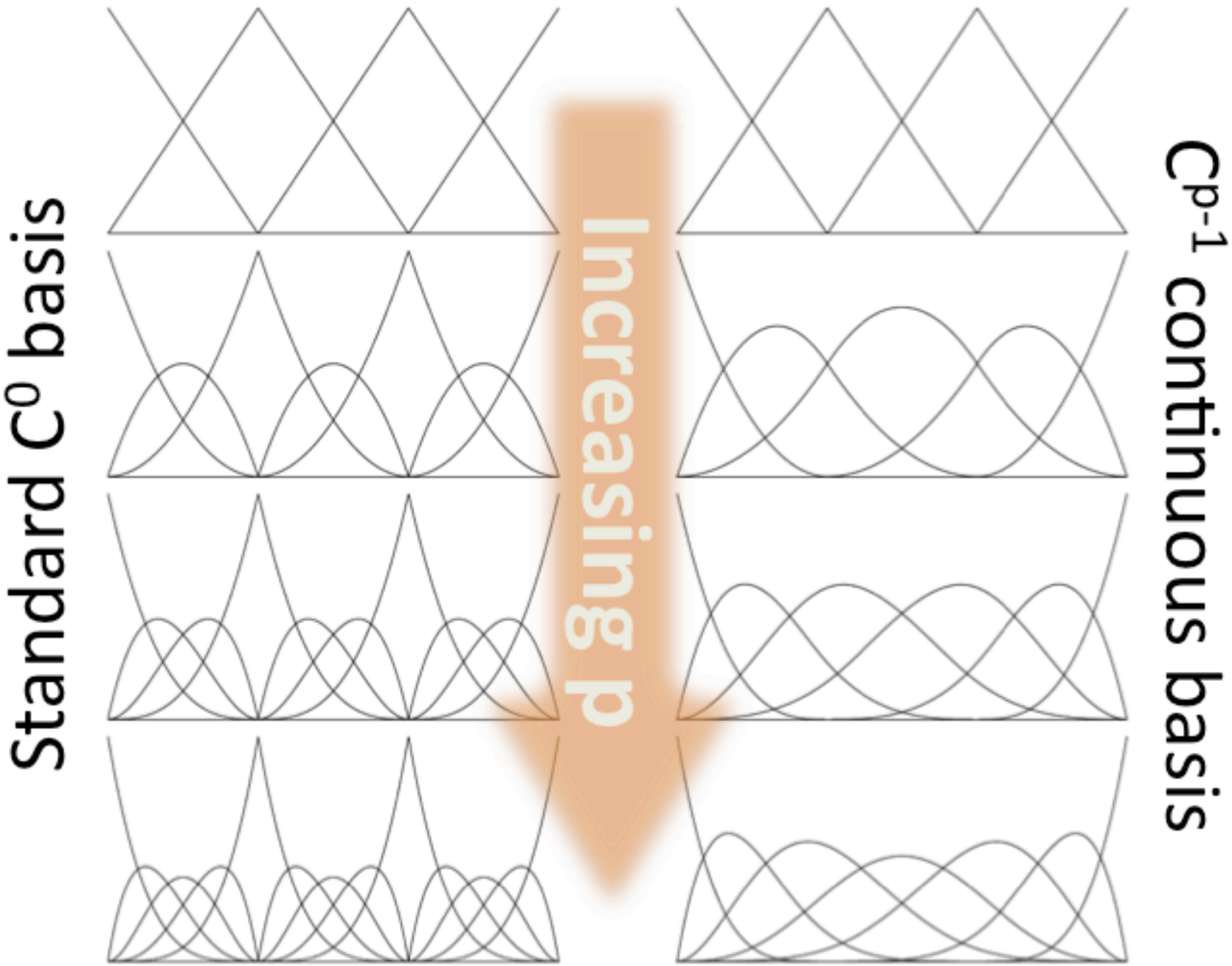
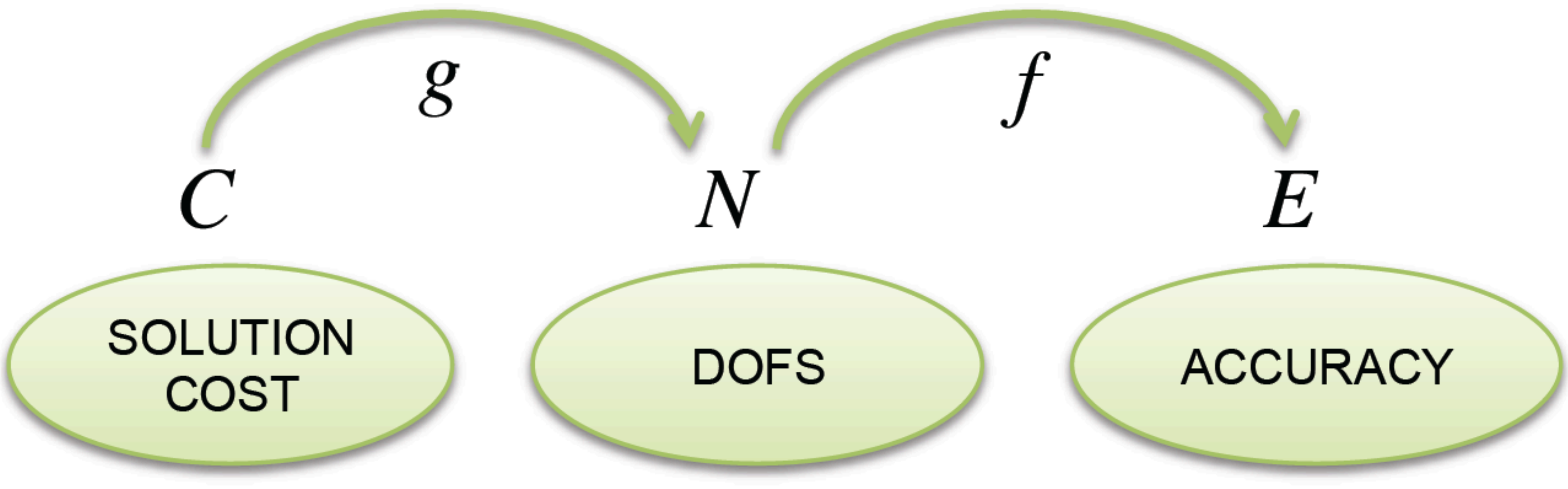
Claim: $C^{p\text{-}1}$ space economical *p*-refinement



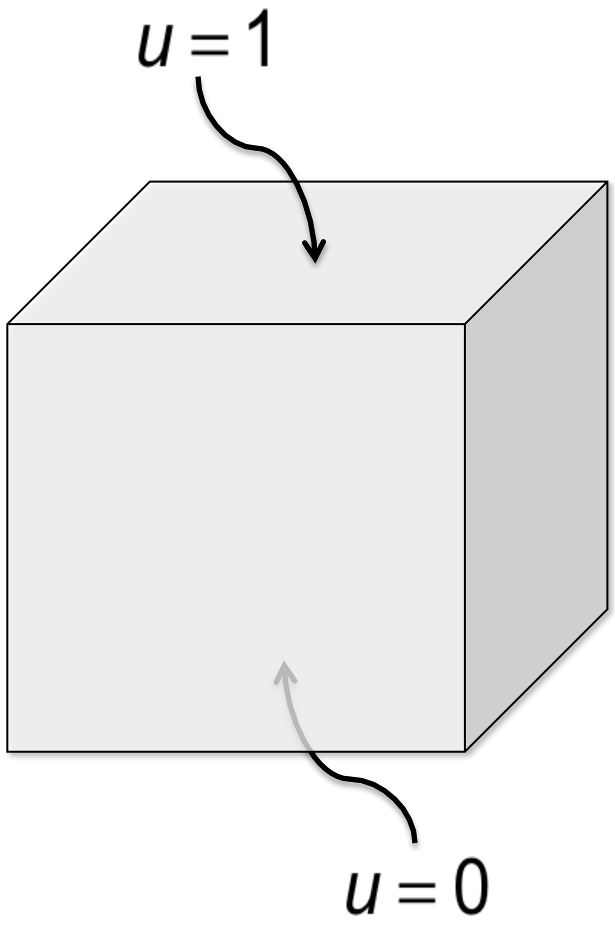
Problem:

True w.r.t. DOFs, but ignores solution cost



Goal: Understand solution cost

Canonical Laplace problem on unit cube



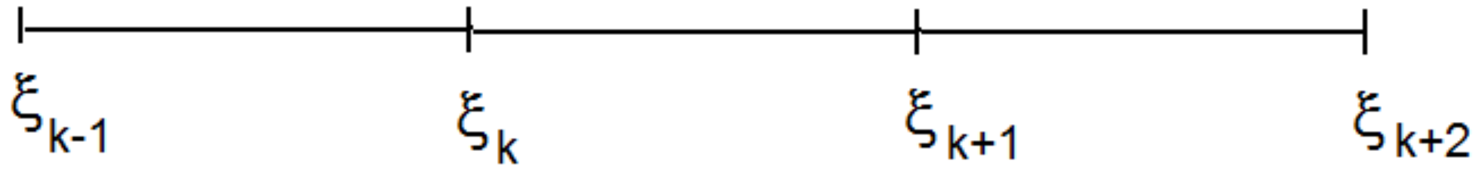
$\hat{n} \cdot \nabla u = 0$ elsewhere

Record for direct solver (MUMPS):

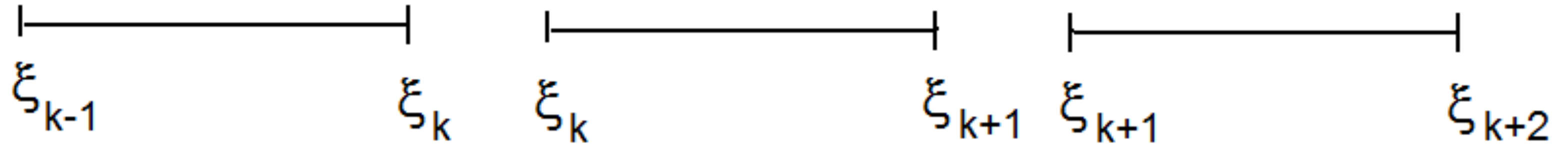
- Solve time
- Required memory

while varying polynomial order and continuity of discretization

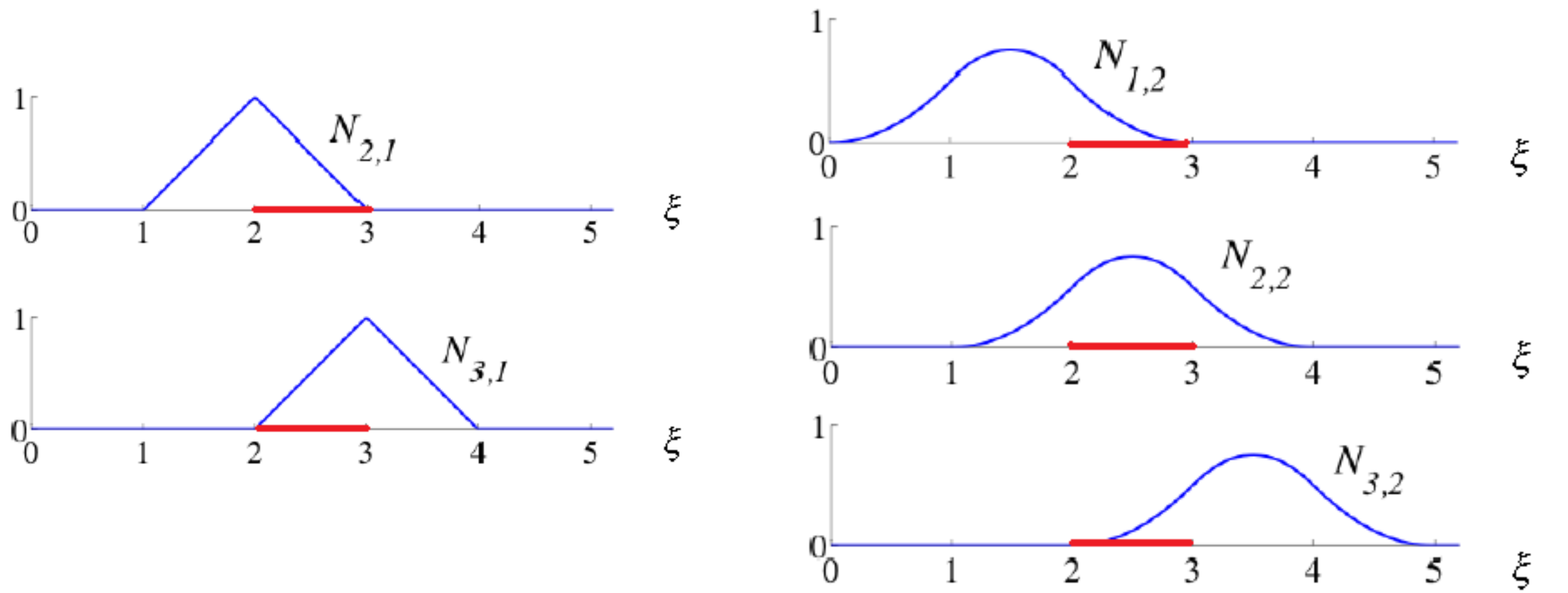
Isogeometric finite elements



Partition mesh into elements



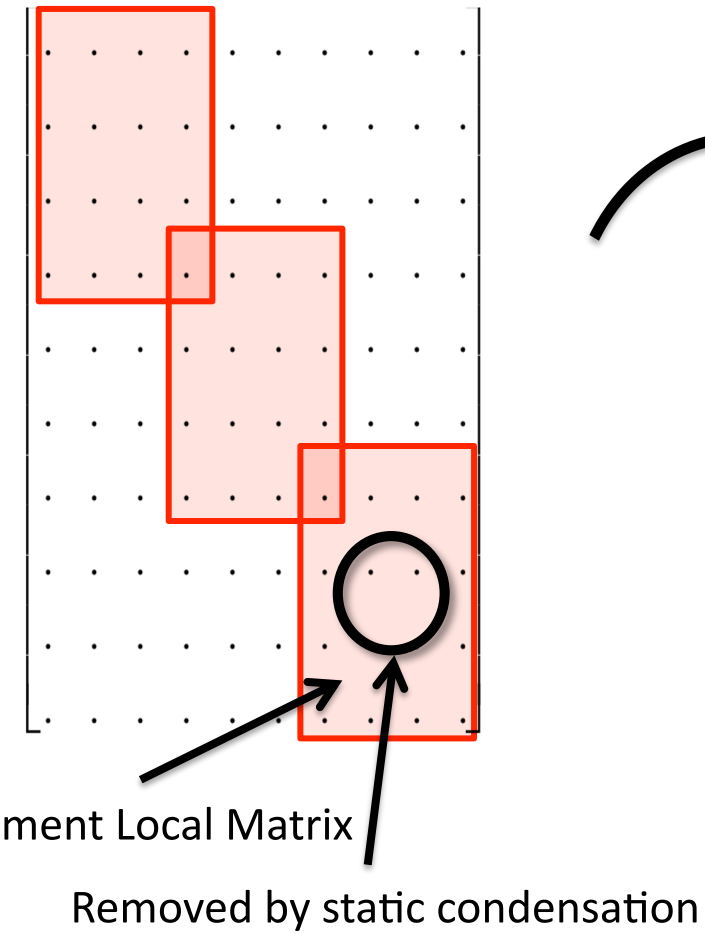
There are p+1 functions of order p assigned to an element $K = [\ \xi_k, \xi_{k+1}\ ]$

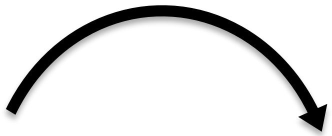


MUMPS: Multi-frontal direct solver

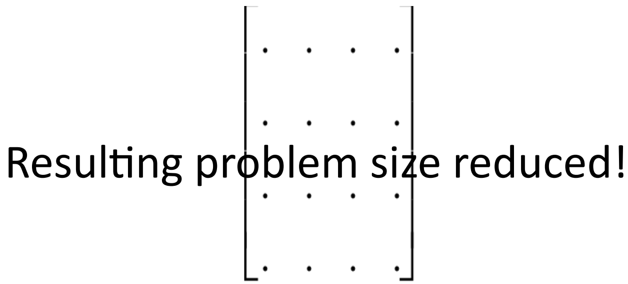
Solution strategy split into 2 parts:

1. Static condensation of fully assembled dofs
2. LU-factorization of remaining problem (skeleton problem)

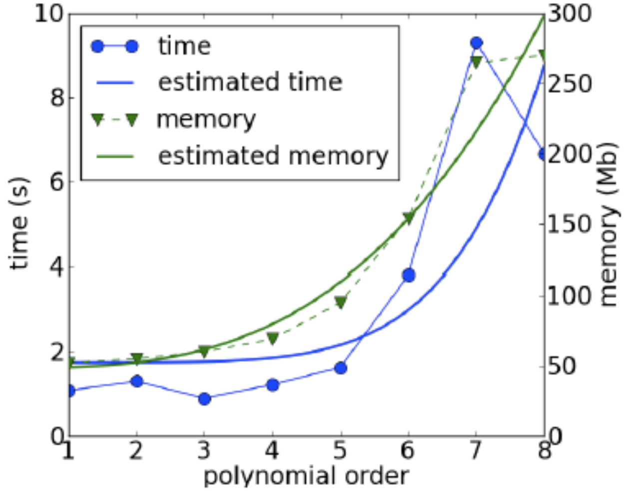




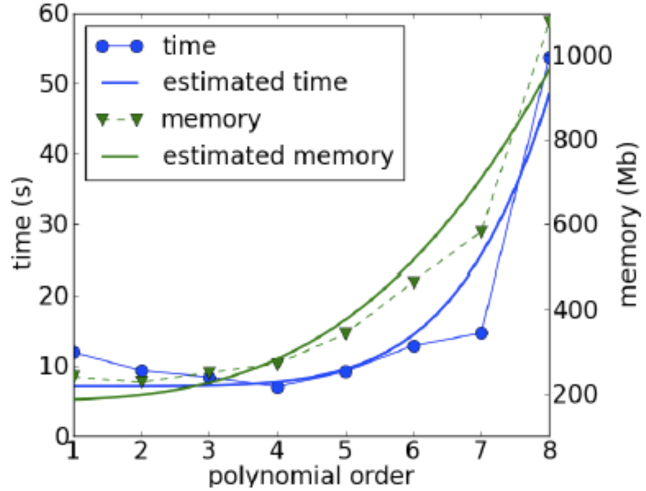
C^0 1D finite element matrices



C^0 B-spline spaces

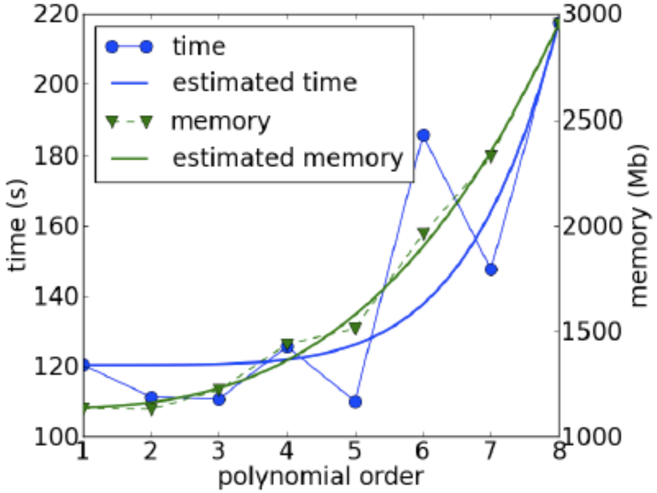


(a) 10k degrees of freedom



(b) 30k degrees of freedom

p=7 commensurate to p=1



(c) 100k degrees of freedom

Increasing polynomial order increases static condensation work, but skeleton problem is limiting cost and remains fixed!

Computational estimates of cost for C^0 spaces

- Bubbles in ea. element: $(p-1)^3 \Rightarrow$ Elimination time $O\left(\left((p-1)^3\right)^3\right) \approx O\left(p^9\right)$
  - N_e (# elements) $\approx O\left(N_{dof} / p^3\right) \Rightarrow$ Static condensation time $O\left(N_e p^9\right)$
- Squeleton problem:
  - N_e # elements $\Rightarrow$ N_f (# faces) $= 3\,N_e$ $\Rightarrow$ Size: $N_f p^2 = 3N_e p^2$
  - Average bandwidth $O\left((p-1)^2\right)$

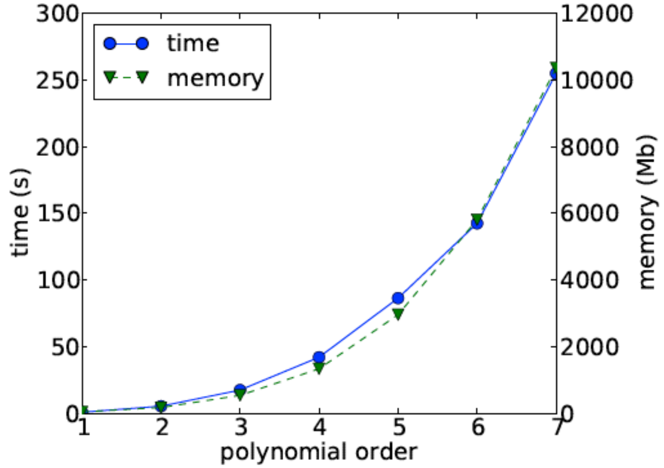
    $\Rightarrow$ Elimination time: $O\left(\left(3N_e p^2\right)^2 (p-1)^2\right) \approx O\left(N_e^2 p^6\right)$
- Total problem complexity estimate:

  $time = O\left(N_e p^9\right) + O\left(N_e^2 p^6\right) + \text{lower order terms}$

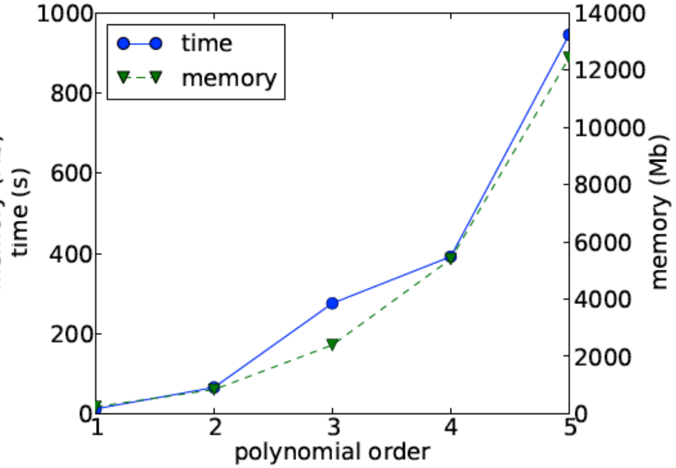
  $\Rightarrow$ for N_{dof} squeleton problem cost is $O\left(N_{dof}^2\right)$

  $\Rightarrow$ for $N_{dof} \gg O\left(p^6\right)$ then static condensation time $\ll$ squeleton problem
- Total problem memory usage estimate: $O\left(Np^3\right) + O\left(N^{4/3}\right) + \text{lower order terms}$

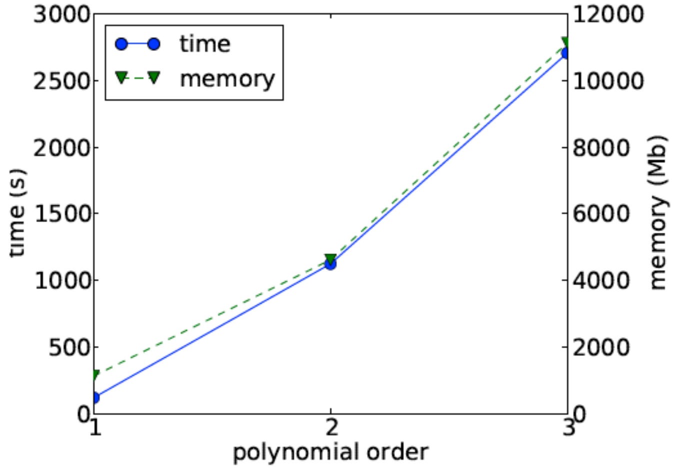
The high price of continuity



(a) 10k degrees of freedom



(b) 30k degrees of freedom



(c) 100k degrees of freedom

Computational estimates of cost for C^{p-1} spaces

- Elimination problem:
  - N_e # elements $\Rightarrow$ Size: $N_e\left(p+1\right)^3$
  - Average bandwidth $O\left(\left(p+1\right)^3\right)$

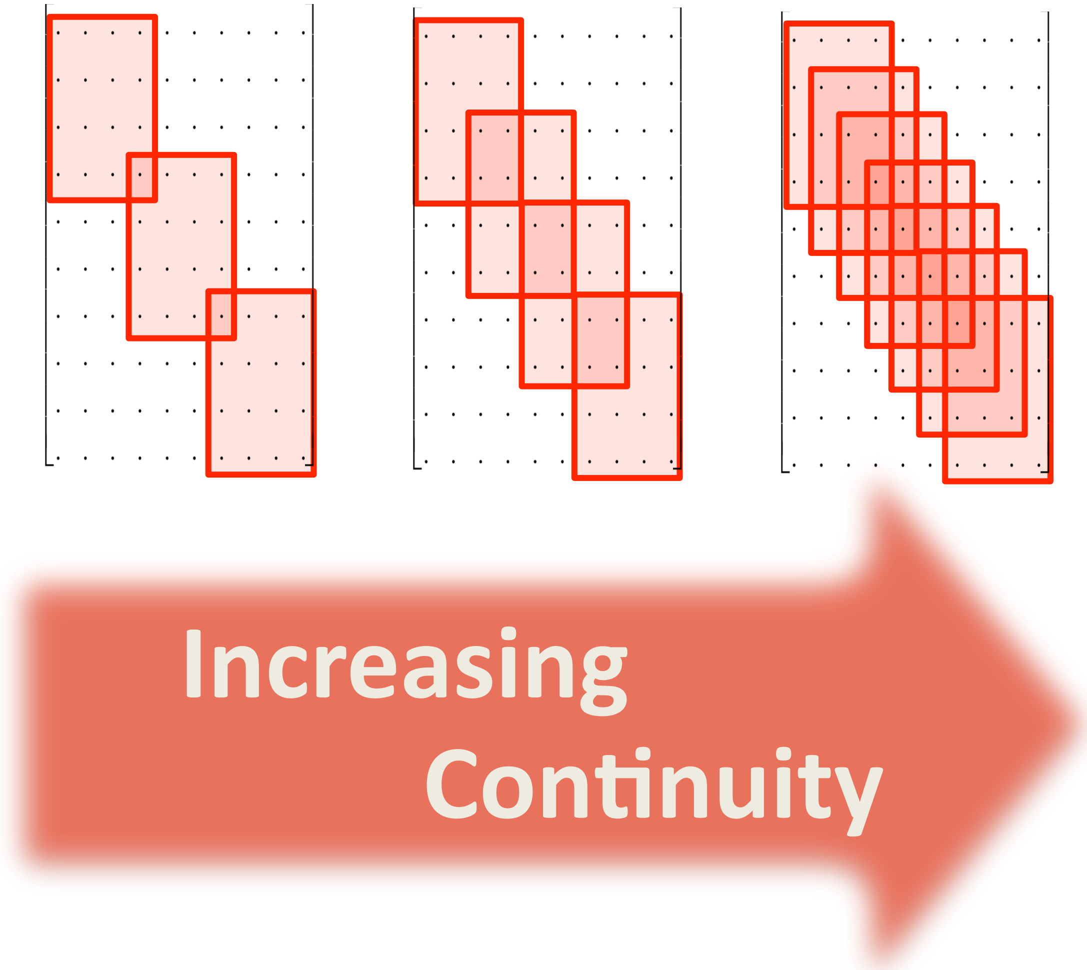
    $\Rightarrow$ Elimination time: $O\left(\left(N_e\left(p+1\right)^3\right)^2\left(p+1\right)^3\right) \approx O\left(N_e^2 p^9\right)$

- Total problem complexity estimate:

  $$time = O\left(N_e^2 p^9\right) + \text{lower order terms} \approx O\left(N^2 p^6\right)$$

- Total problem memory usage estimate: $O\left(N^{4/3} p^2\right) + \text{lower order terms}$

The high price of continuity



Higher continuous basis result in element stiffness matrix blocks overlapping, causes performance loss of multi-frontal algorithm

Goal: Understand value of continuity

Canonical Laplace problem on unit cube

$$-\nabla \cdot (\nabla u) = f \quad \text{on } \Omega$$
$$u = 0 \quad \text{on } \Gamma_D$$
$$(\nabla u) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_N$$

$$f(x,y,z) = \frac{3C^2\pi^2}{4}\left[\sin\left(\frac{C\pi}{2}x\right)\sin\left(\frac{C\pi}{2}y\right)\sin\left(\frac{C\pi}{2}z\right)\right]$$

$$u(x,y,z) = \sin\left(\frac{C\pi}{2}x\right)\sin\left(\frac{C\pi}{2}y\right)\sin\left(\frac{C\pi}{2}z\right)$$

$$\Omega = [0,1]^3,\ \Gamma_D = (0,:,:) \cup (:,0,:) \cup (:,:,0),$$
$$\Gamma_N = (1,:,:) \cup (:,1,:) \cup (:,:,1)$$

The value of continuity?

Table I. Results for Problem 1, $C = 5$, $\|E\|_{H_1} \approx 10^{-2}$

N_x	N_y	N_z	p	k	N_{dof}	$\|E\|_{L_2}$	$\|E\|_{H_1}$	t_{ass}	t_{solve}	t_{total}	$\frac{t_{solve}}{N_{dof}}$
62	62	61	1	0	246078	4.78e-04	1.77e-01	230.26	722.82	953.08	2.94e-03
33	33	34	2	0	309741	3.85e-05	9.96e-03	516.67	994.32	1510.99	3.21e-03
33	33	34	2	1	44100	3.94e-05	1.00e-02	146.42	183.72	330.14	4.17e-03
9	9	9	3	0	21952	9.15e-05	1.00e-02	13.23	3.84	17.07	1.75e-04
11	10	11	3	1	12672	1.34e-04	1.01e-02	17.45	9.45	26.9	7.46e-04
11	11	12	3	2	2940	1.53e-04	1.03e-02	13.36	2.02	15.38	6.87e-04
5	4	5	4	0	7497	1.31e-04	9.28e-03	6.52	0.63	7.15	8.40e-05
5	4	5	4	1	4046	2.14e-04	1.14e-02	5.76	1.07	6.83	2.64e-04
6	6	5	4	2	2925	2.40e-04	1.05e-02	9.84	1.78	11.62	6.09e-04
7	7	7	4	3	1331	2.79e-04	1.06e-02	17.35	1.14	18.49	8.56e-04

The value of continuity?

Table II. Results for Problem 1, $C = 3$, constant elements

N_x	N_y	N_z	p	k	N_{dof}	$\|error\|_{L_2}$	t_{ass}	t_{solve}	t_{total}	$\|error\|_{L_2} t_{total}$
20	20	20	1	0	9261	7.964055e-05	0.83	0.8	1.63	1.30e-04
20	20	20	2	0	68921	2.331762e-07	47.57	44.43	92.0	2.15e-05
20	20	20	2	1	10648	2.95832e-07	13.89	6.6	20.49	6.06e-06
20	20	20	3	0	226981	2.159312e-10	493.72	637.11	1130.83	2.44e-07
20	20	20	3	1	74088	1.942879e-09	250.91	462.83	713.74	1.39e-06
20	20	20	3	2	12167	3.086729e-09	108.78	26.12	134.9	4.16e-07
20	20	20	4	0	531441	1.456711e-13	3471.63	3878.85	7350.48	1.07e-09
20	20	20	4	1	238328	1.663401e-13	2029.75	6429.29	8459.04	1.41e-09
20	20	20	4	2	79507	1.320435e-11	1152.44	1744.75	2897.19	3.83e-08
20	20	20	4	3	13824	2.805228e-11	592.4	74.61	667.01	1.87e-08

Isogeometric-Specific Multi-Frontal Solver

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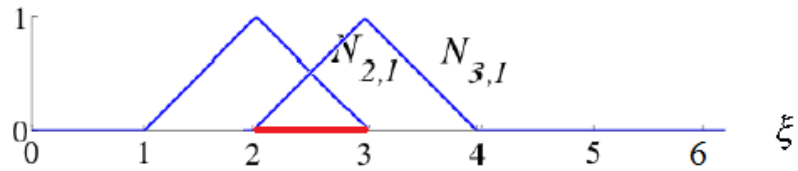
Multi-frontal algorithm

$$
\begin{array}{c}
 \\ N0,1 \\ N1,1 \\ N2,1 \\ N3,1 \\ N4,1 \\ N5,1 \\ N6,1
\end{array}
\begin{array}{c}
\begin{array}{ccccccc} N0,1 & N1,1 & N2,1 & N3,1 & N4,1 & N5,1 & N6,1 \end{array} \\
\begin{bmatrix}
1 & 0 & \dots & 0 & \dots & 0 & 0 \\
\frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 & \dots & 0 & 0 \\
0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 & \dots & 0 \\
0 & \dots & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & \dots & 0 \\
0 & \dots & 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} & 0 \\
0 & 0 & \dots & 0 & \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\
0 & 0 & \dots & 0 & \dots & 0 & 1
\end{bmatrix}
\end{array}
\begin{Bmatrix} x_1 \\ x_2 \\ \dots \\ x_i \\ \dots \\ x_6 \\ x_7 \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ \dots \\ 0 \\ \dots \\ 0 \\ 20 \end{Bmatrix}
$$



E.g., contribution of

b(N2,1;N3,1)

Construct multiple frontal matrices s.t. they sum up to the full matrix

Variables must be split into parts

$$x_i = x_i^{I} + x_i^{II}$$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Multi-frontal algorithm

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

First all frontal matrices are constructed

Multi-frontal algorithm

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

- Assemble frontal matrices 1 and 2 into new 3x3 frontal matrix
- Rows 1 and 2 are fully assembled

Multi-frontal algorithm

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Column 1 eliminated using first row

=> 2nd row = 2nd row – [1/h²] * 1st row

Multi-frontal algorithm

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2/h^2 & 1/h^2 \\ 0 & 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+

$$\begin{bmatrix} 1 & 0 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -1/h^2 & 1/h^2 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -1/h^2 & 1/h^2 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -1/h^2 & 1/h^2 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -1/h^2 & 1/h^2 \\ 1/h^2 & -1/h^2 \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -1/h^2 & 1/h^2 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

2nd row = 2nd row – [1/h^2] * 1st row

Multi-frontal algorithm

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4 \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Assemble frontal matrices 3 and 4 into new frontal matrix

Only row 2 is fully assembled

Multi-frontal algorithm

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ \frac{1}{h^2} & 0 & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+ +

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Change of the ordering

Multi-frontal algorithm

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} & 0 \\ \frac{1}{h^2} & 0 & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

+ +

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Eliminate entries in column 1 below row 1

2nd row = 2nd row – [1/h²] / [-2/h2] * 1st row

3rd row = 3rd row – [1/h2] / [-2/h2] * 1st row

Multi-frontal algorithm

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} & \frac{1}{h^2} \\ 0 & -\frac{1}{h^2} & \frac{1}{2h^2} \\ 0 & \frac{1}{2h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4 \\ x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Eliminate entries below the diagonal

2nd row = 2nd row – [1/h2] / [-2/h2] * 1st row

3rd row = 3rd row – [1/h2] / [-2/h2] * 1st row

Multi-frontal algorithm

$$\begin{bmatrix} -\frac{2}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{2h^2} \\ \frac{1}{2h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} & 0 \\ \frac{1}{h^2} & -\frac{2}{h^2} & \frac{1}{h^2} \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6 \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 20 \end{Bmatrix}$$

$$+ \qquad\qquad + \qquad\qquad +$$

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Recursevely eliminate remaining frontal matrices

…

Multi-frontal algorithm

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 1

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_2^{II} \\ x_3^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 2

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_3^{II} \\ x_4^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 3

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_4^{II} \\ x_5^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Procesor 4

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ \frac{1}{h^2} & -\frac{1}{h^2} \end{bmatrix} \begin{Bmatrix} x_5^{II} \\ x_6^{I} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

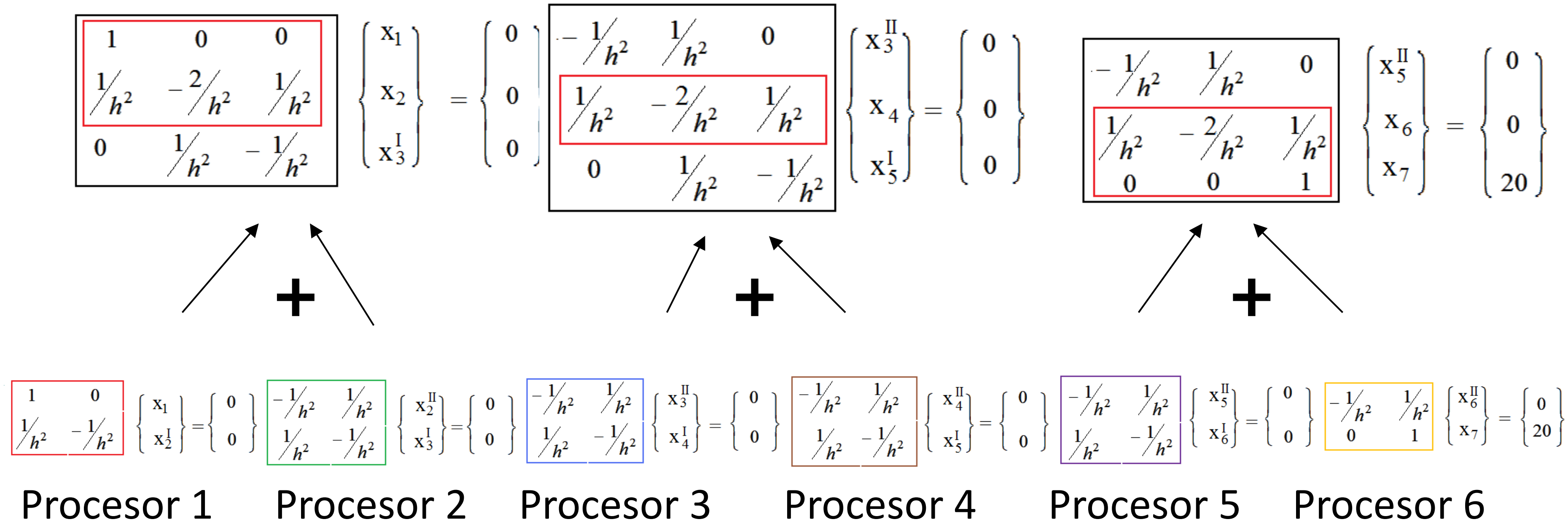
Procesor 5

$$\begin{bmatrix} -\frac{1}{h^2} & \frac{1}{h^2} \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} x_6^{II} \\ x_7 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 20 \end{Bmatrix}$$

Procesor 6

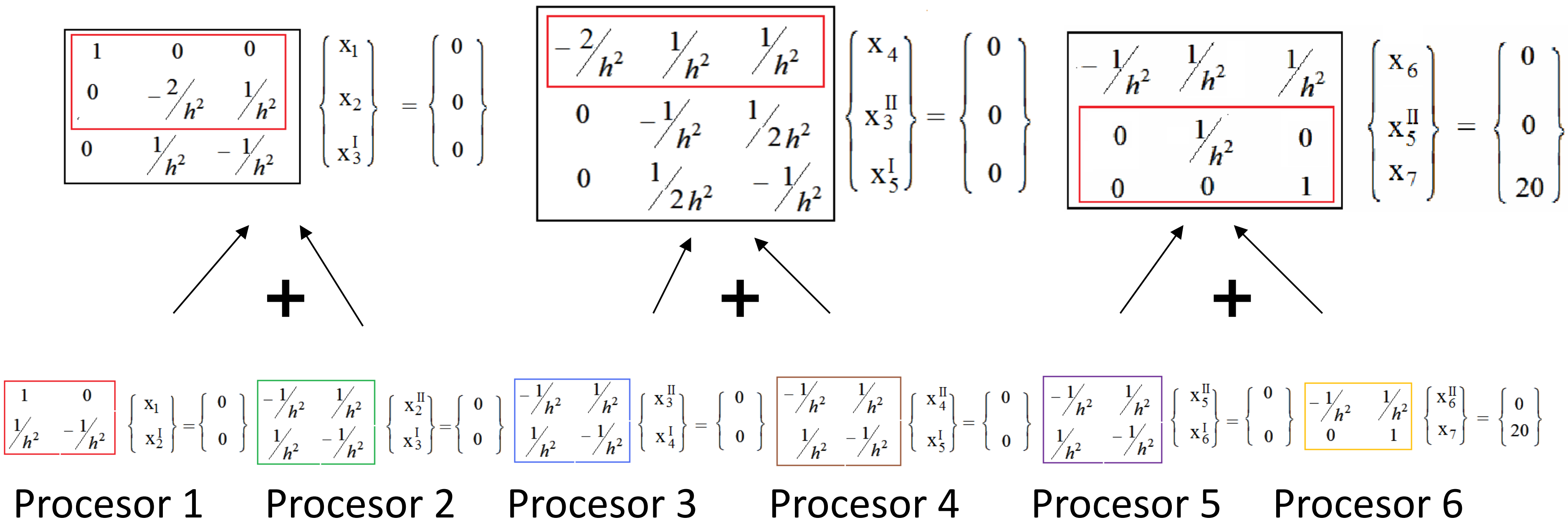
All frontal matrices are generated at the same time

Multi-frontal algorithm



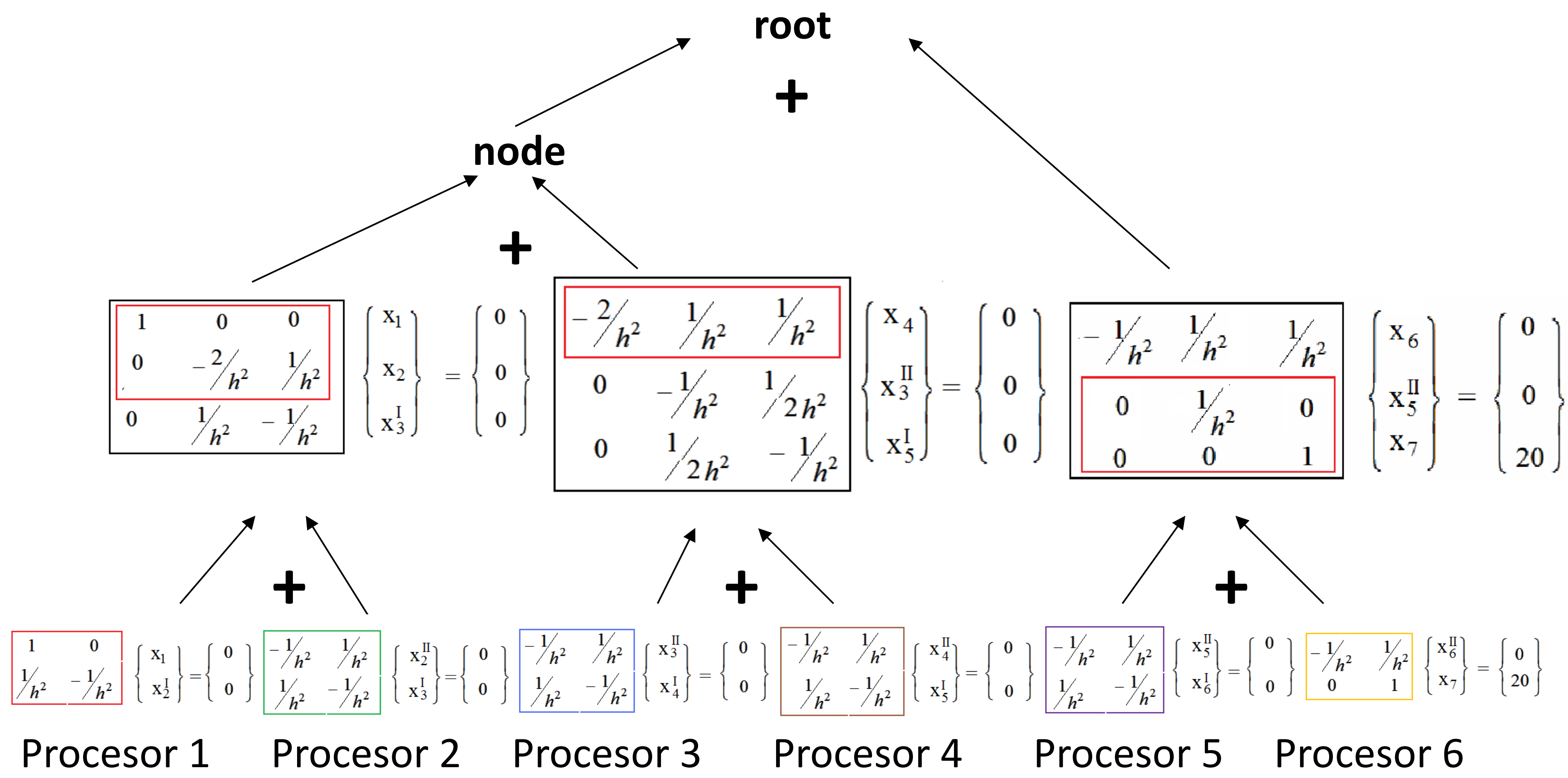
Assembly and elimination are executed concurrently over pairs of frontal matrices

Multi-frontal algorithm



Concurrent assembly and elimination executed over different pairs of frontal matrices

Multi-frontal algorithm



Algorithm recursively repeated until root of the tree is reached

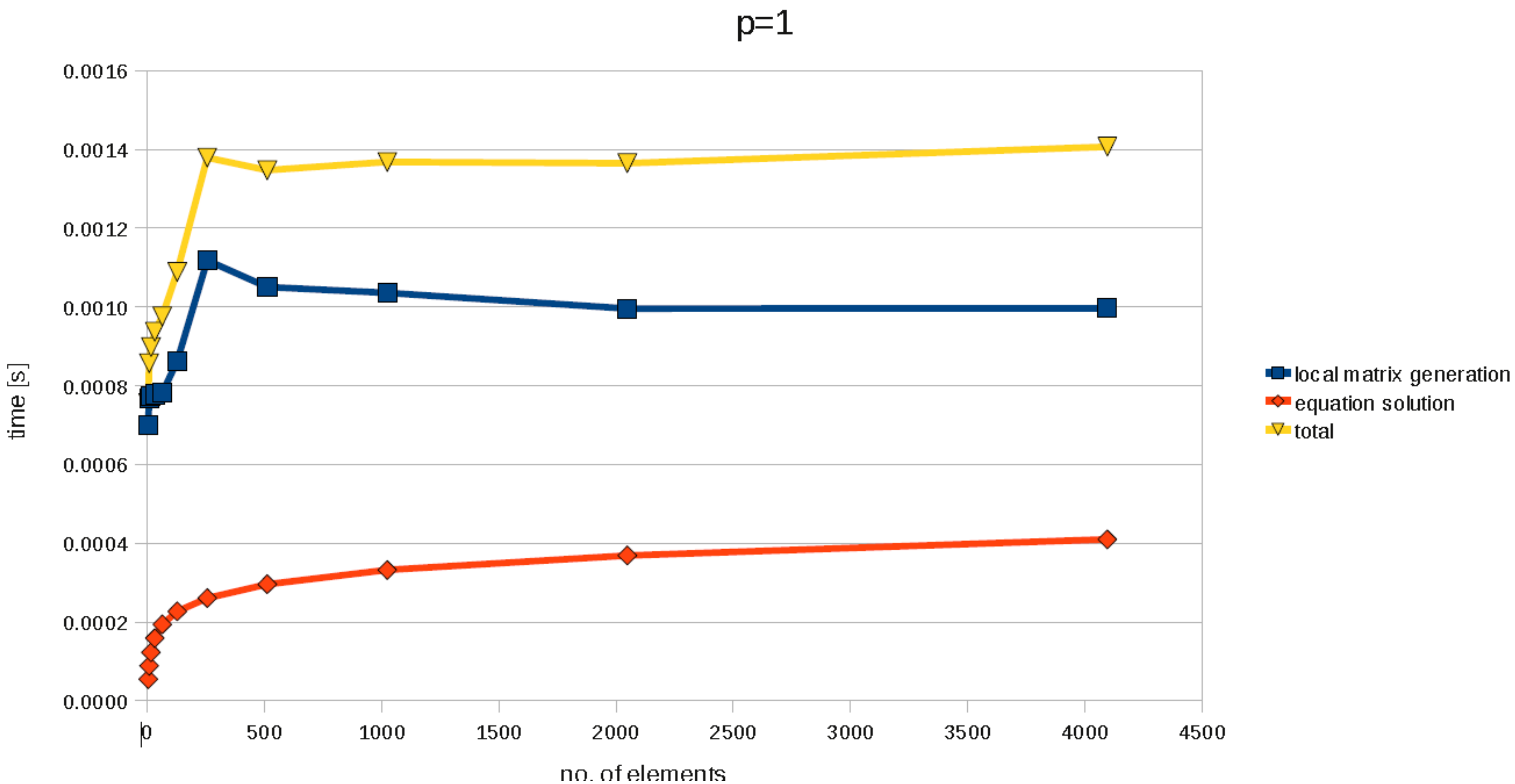
It results in upper trianguler matrix stored in distributed manner

Computational complexity = height of the tree = $\log(N_{dof})$

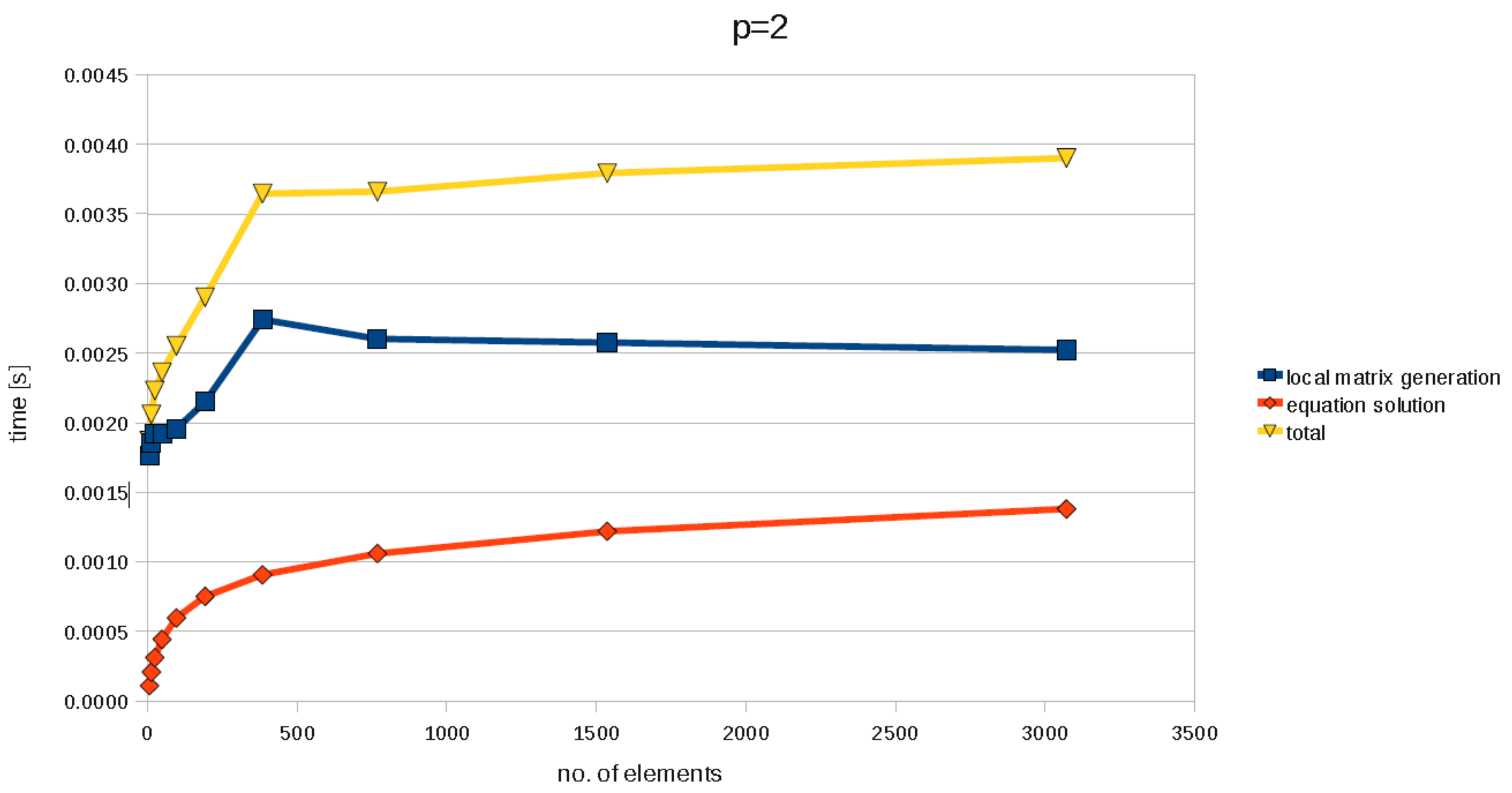
Numerical results

- NVIDIA CUDA
- GeForce GTX 260
- Multiprocessors x Cores/MP =
- Cores: 24 (MP) x 8 (Cores/MP) = 192 (Cores)
- Memory: 896MB

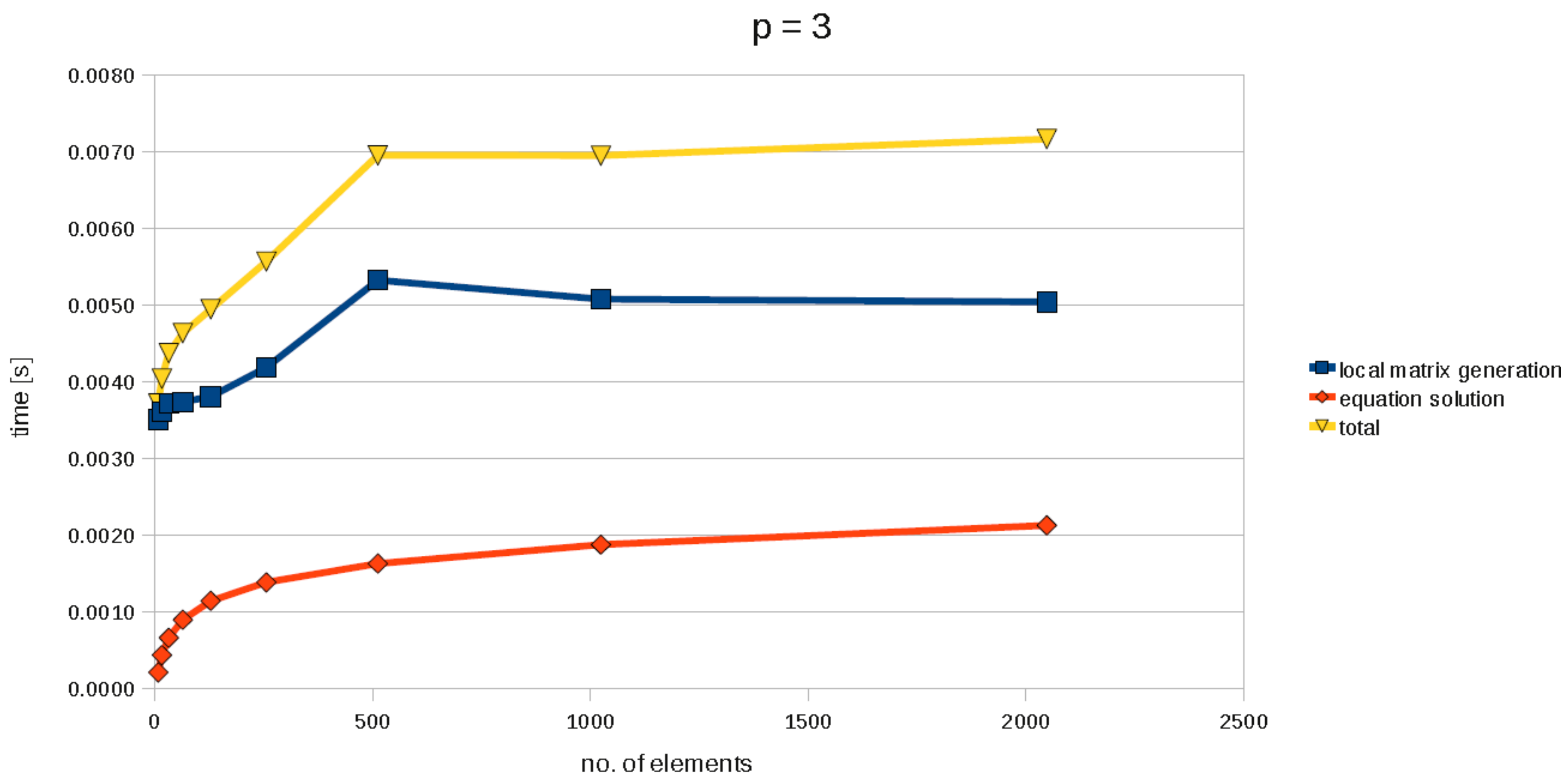
Numerical results



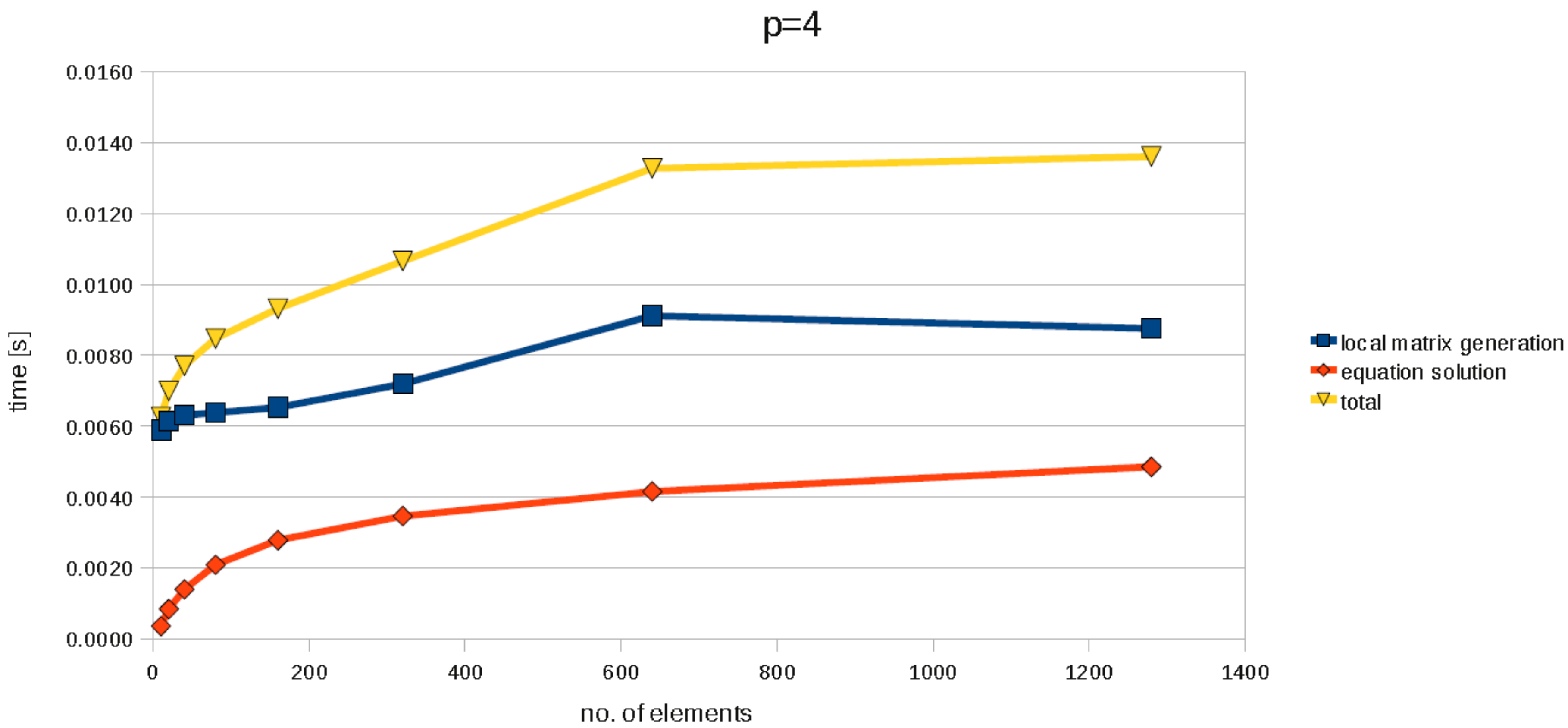
Numerical results



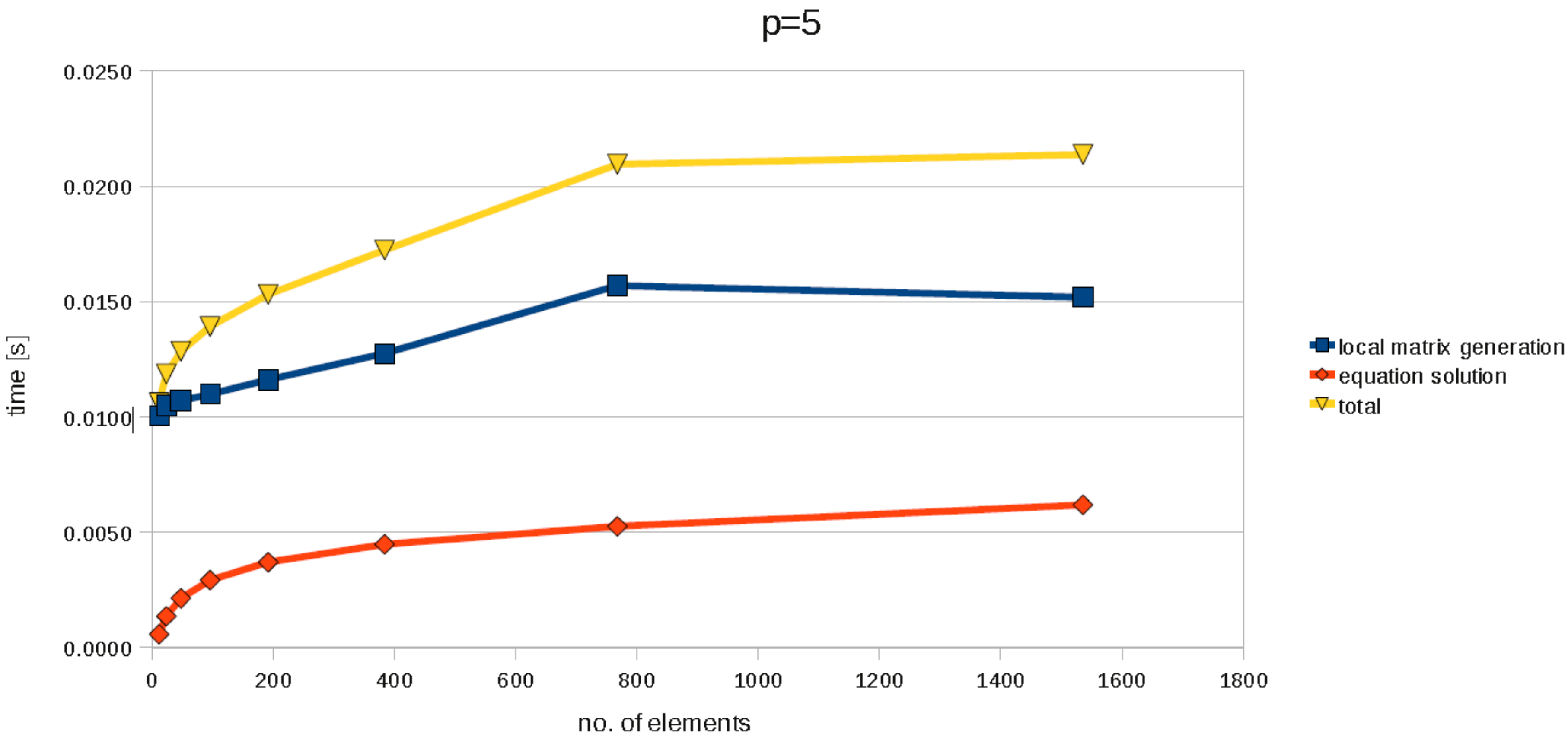
Numerical results



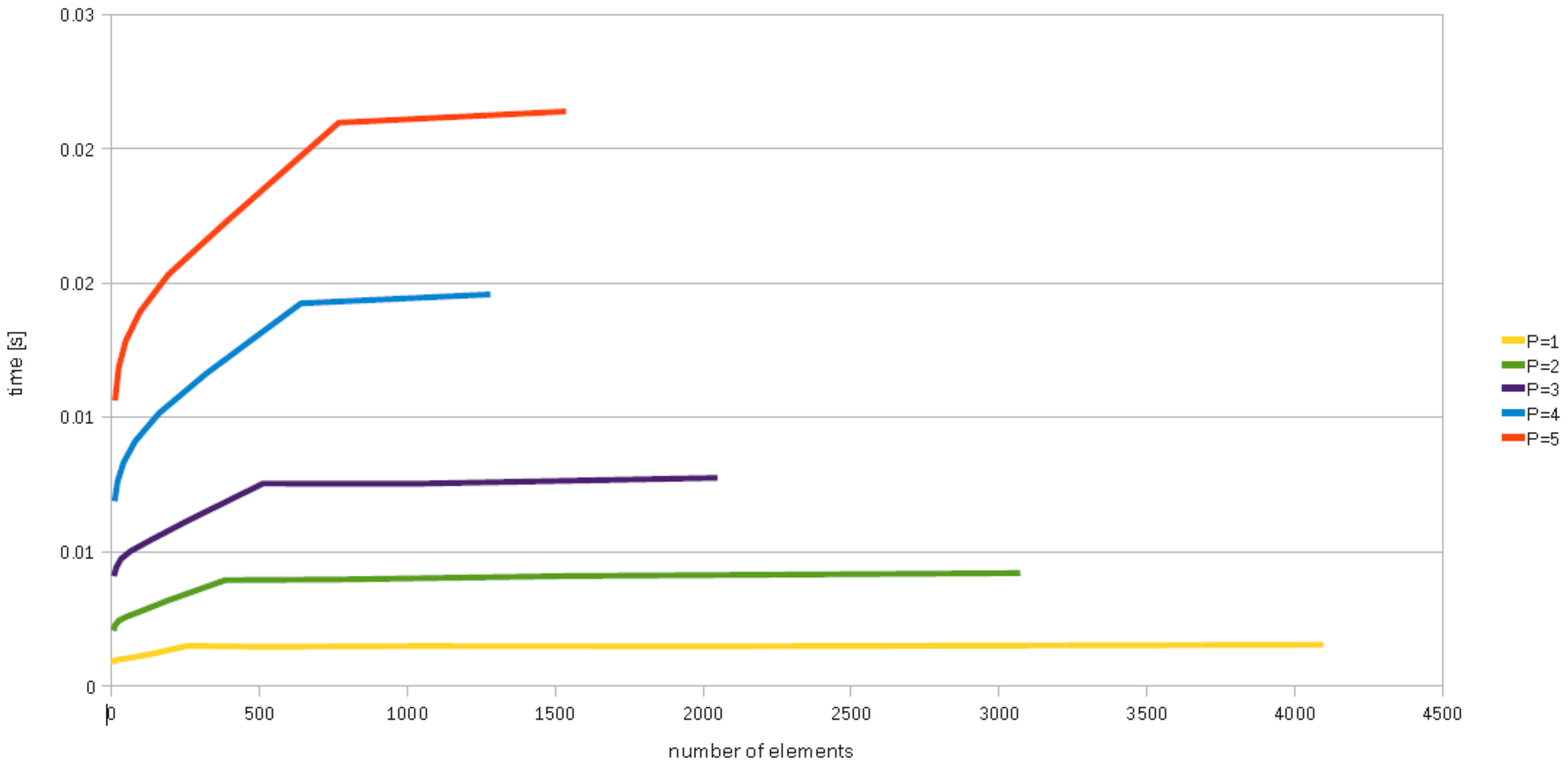
Numerical results



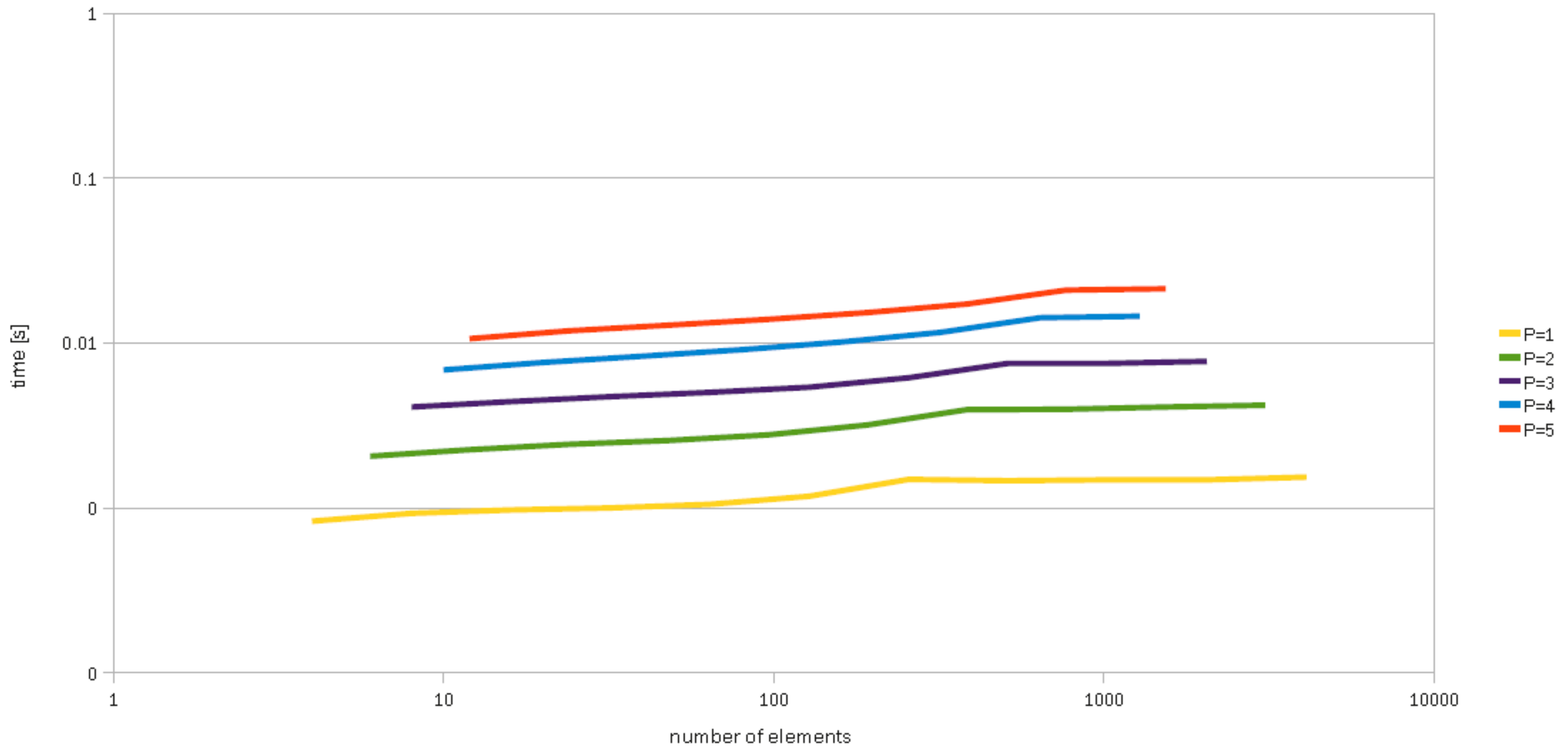
Numerical results



Numerical results



Numerical results



Flood Modeling

Joint work with:

N.O. Collier, H. Radwan, G. Stenchikov, and W. Tao

Overview

- Motivation:
  - KAUST and Jeddah flood
  - Model Problem: Manning's model
- Application to KAUST topographic data
  - Numerical results

Diffusive-Wave Approximation

- Strong Form

$$\begin{cases} \dot{u} - \nabla \cdot (\kappa(u, \nabla u)\nabla u) = f & \text{on } \Omega \times (0, T] \\ u = u_0 & \text{on } \Omega \times \{t = 0\} \\ (\kappa(u, \nabla u)\nabla u) \cdot n = B_N & \text{on } \Gamma_N \times (0, T] \\ u = B_D & \text{on } \Gamma_D \times (0, T] \end{cases}$$

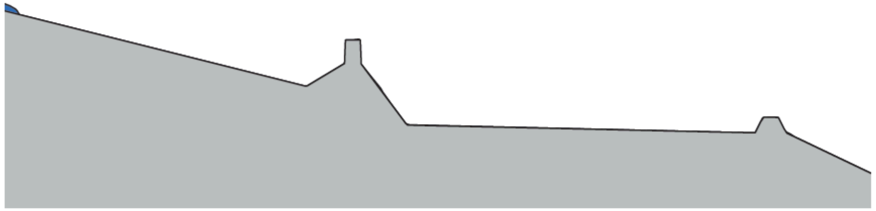
where

$$\kappa(u, \nabla u) = \frac{(u - z)^{\alpha_M}}{C_f |\nabla u|^{1-\gamma_M}}$$

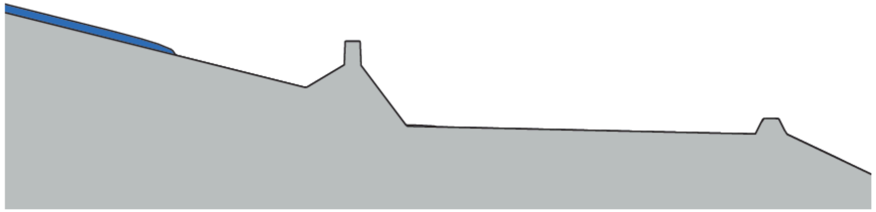
- Weak Form

$$B(w, u) = \left(w, \frac{\partial u}{\partial t}\right)_\Omega + (\nabla w, \kappa(u, \nabla u)\nabla u)_\Omega + (w, f)_\Omega = 0$$

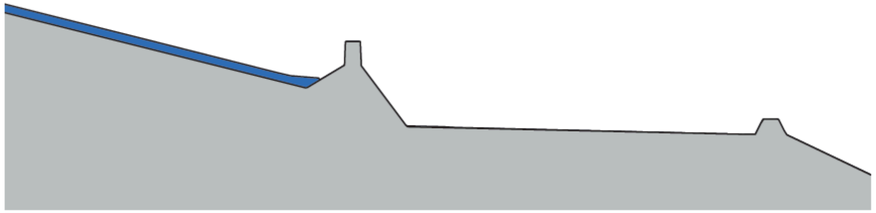
1D Results



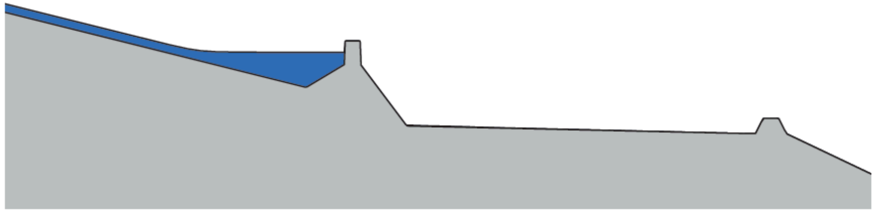
(a) $t = 0.4$, $step = 25$



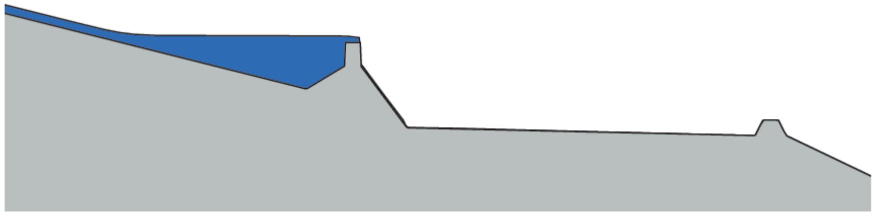
(b) $t = 3.9$, $step = 90$



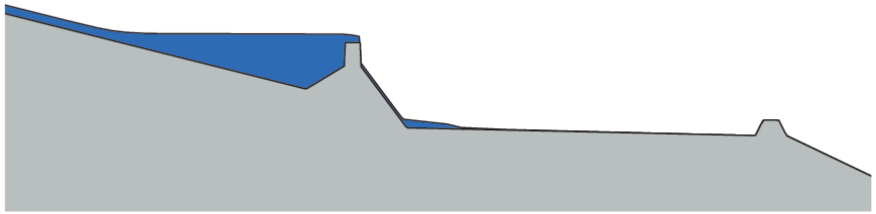
(c) $t = 7.5$, $step = 145$



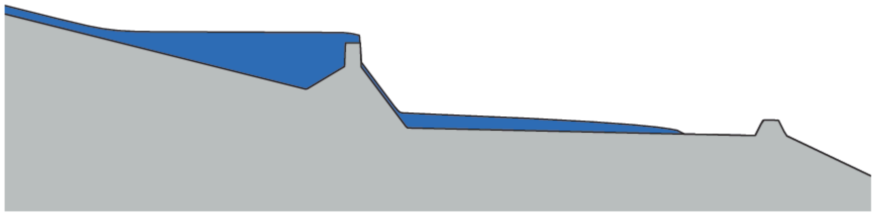
(d) $t = 13.8$, $step = 163$



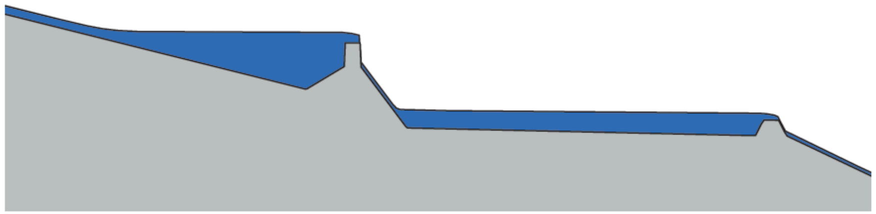
(e) $t = 23.8$, $step = 184$



(f) $t = 26.0$, $step = 195$



(g) $t = 35.5$, $step = 235$

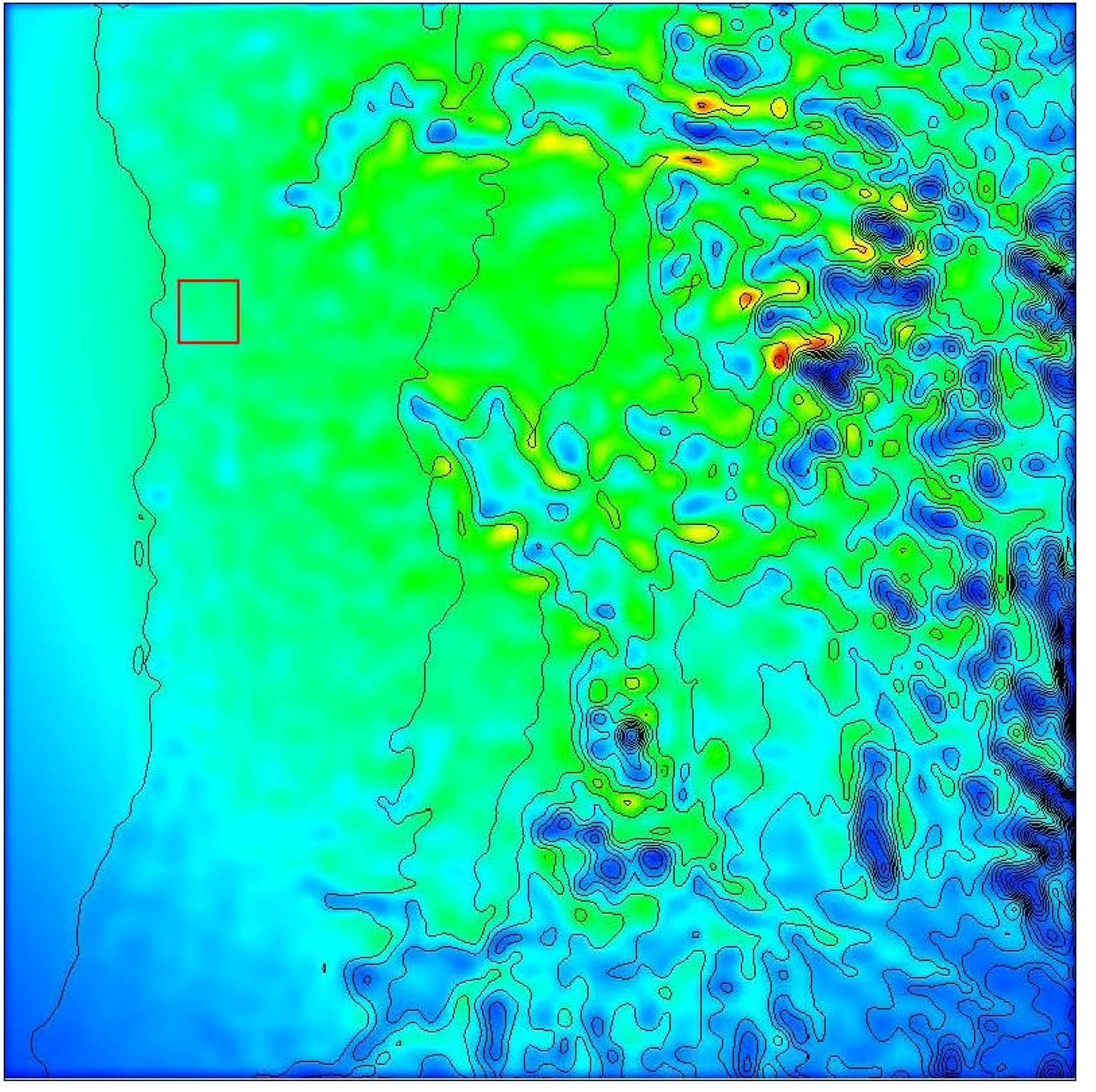


(h) $t = 48.0$, $step = 275$

Relevant Topography

Relevant Topography (Zoom in)

2D Results



Discontinuous Petrov-Galerkin Method

Joint work with:
N.O. Collier, L. Demkowicz, J. Gopalakrishnan,
I. Muga, A.H. Niemi, D. Pardo, and J. Zitelli

Towards Discretization without Numerical Dispersion

- Motivation
  - Model Problem
  - Dispersion/Phase error - Pollution effect
- DPG framework
  - Features and Characteristics
- Application to the Helmholtz Equation
  - Numerical results
- DPG formulation details

Wave Propagation

- Equation governing wave propagation (at speed c): $\Delta p - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$
- Assuming time-harmonic dependance: $p(\mathbf{x},t) = \exp(i\omega t)\, p(\mathbf{x})$
- ω is the angular frequency
- Helmholtz equation (second order formalism):

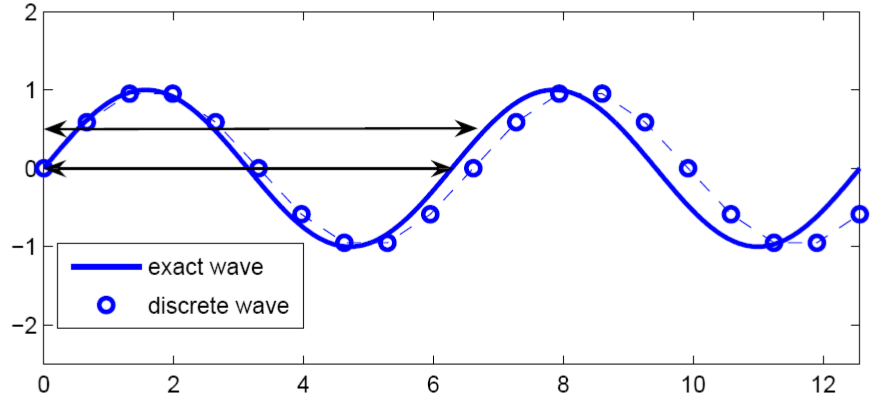
$$\begin{cases} \Delta p + k^2 p = -ikf & \text{in } \Omega \\ \partial_n p + ikp = 0 & \text{on } \partial\Omega \end{cases} \qquad (\blacktriangle)$$

- $k = \frac{\omega}{c}$ is the wave number
- $p(\mathbf{x}) = \exp(ik\mathbf{v}\cdot\mathbf{x})$ are particular solutions to $(\blacktriangle)$

(time-harmonic wave trains)

Dispersion Analysis

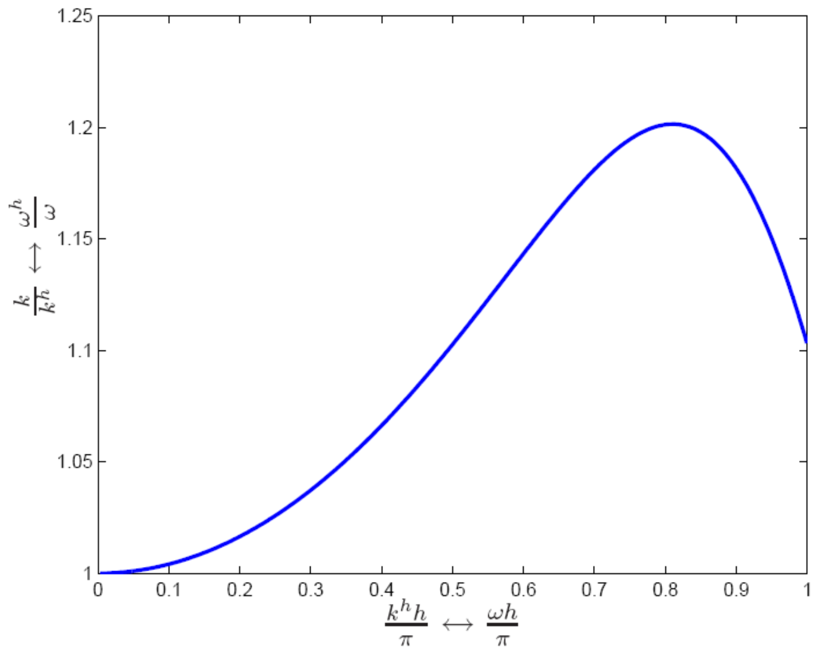
After discretization and numerical solution of (▲), wave trains with discrete wave-number $k^h(\neq k)$ are obtained



Dispersion analysis: analysis of wave-number error

Spectrum analysis is equivalent to dispersion analysis in the regime where k^h is real

Duality principle



Linear approximation case

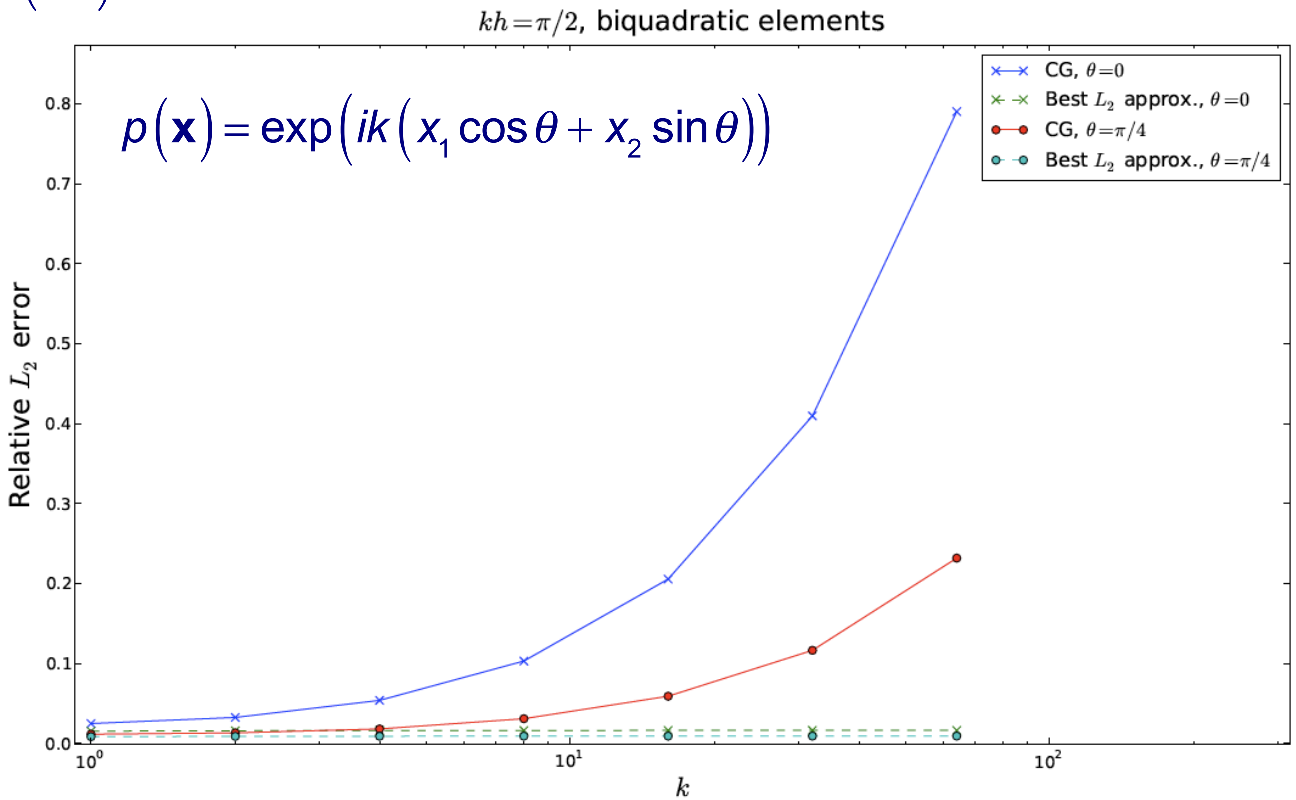
Pollution Effect

- Mathematical characterization of dispersion
- Given exact solution $p \in U$ and its discrete approximation $p^h \in U^h \subset U$

$$\frac{\left\| p - p^h \right\|}{\left\| p \right\|} \le C(k) \inf_{w^h \in U^h} \frac{\left\| p - w^h \right\|}{\left\| p \right\|}$$

where $C(k) = C_1 + C_2 k^{\beta} (kh)^{\gamma}$

2D numerical evidence



Discontinuous Petrov-Galerkin Method

Objective

- Eliminate pollution error in multi-dimensions

Features

- Hermitian positive definite algebraic systems
- Unconditional stability

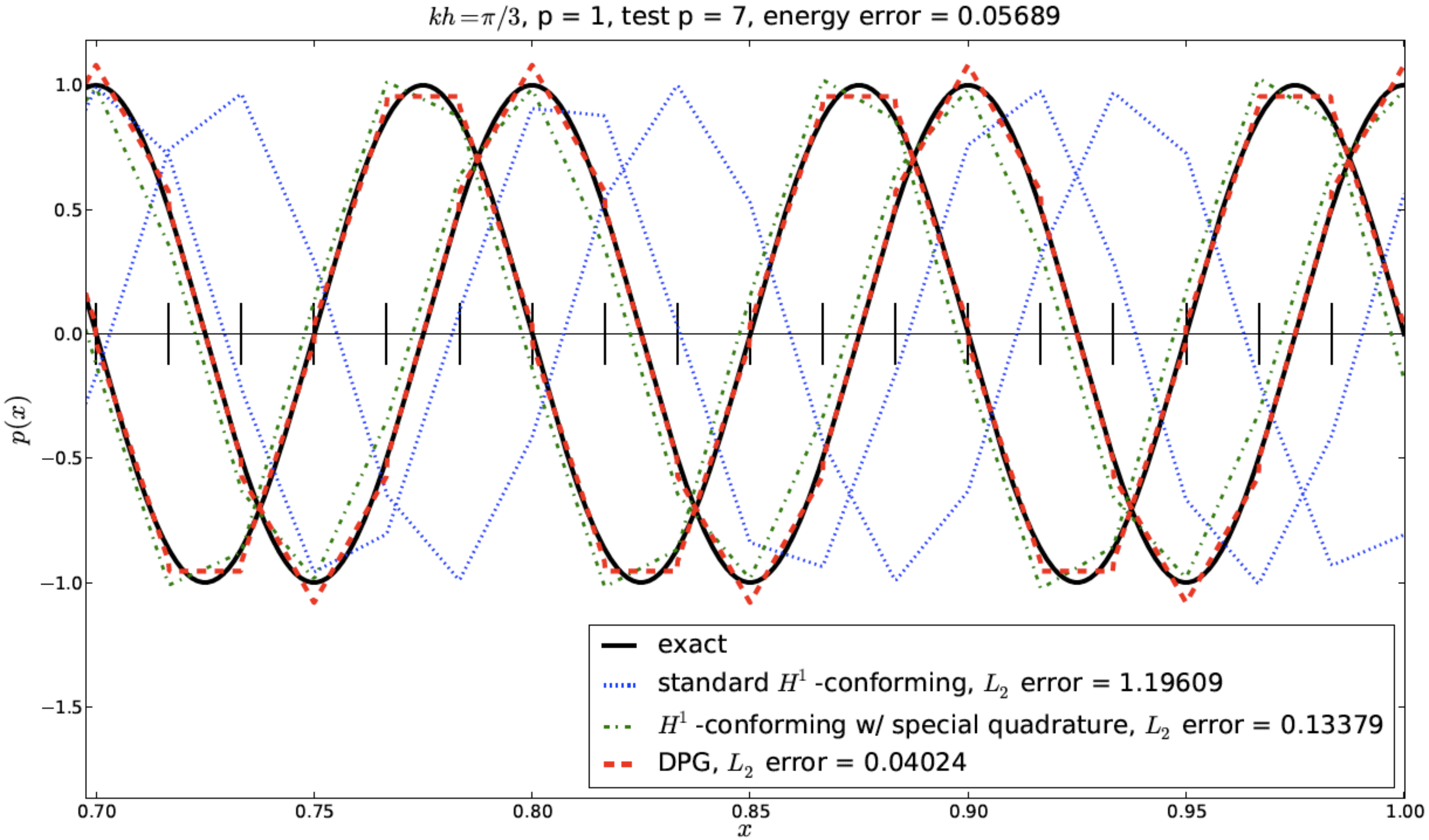
Characteristics

- Discontinuous Galerkin (DG) method
- Petrov-Galerkin method
- Least-squares-type Galerkin method

Discontinuous Petrov-Galerkin Method

- Optimal convergence rate in “energy” norm of problem irrespective of physical parameters
  - Constructed discrete weighting space guarantees optimal convergence in energy norm
    - Uses weighting test function space fully discontinuous
    - Discrete variational problem is local to each element
    - Local problems are symmetric and are solved approximately using standard Galerkin method
  - Attractive for problem with strong dependence on physical parameters
    - Parametric dependence complicates computation of optimal basis functions
- Discrete approximation of field variables, traces, and fluxes

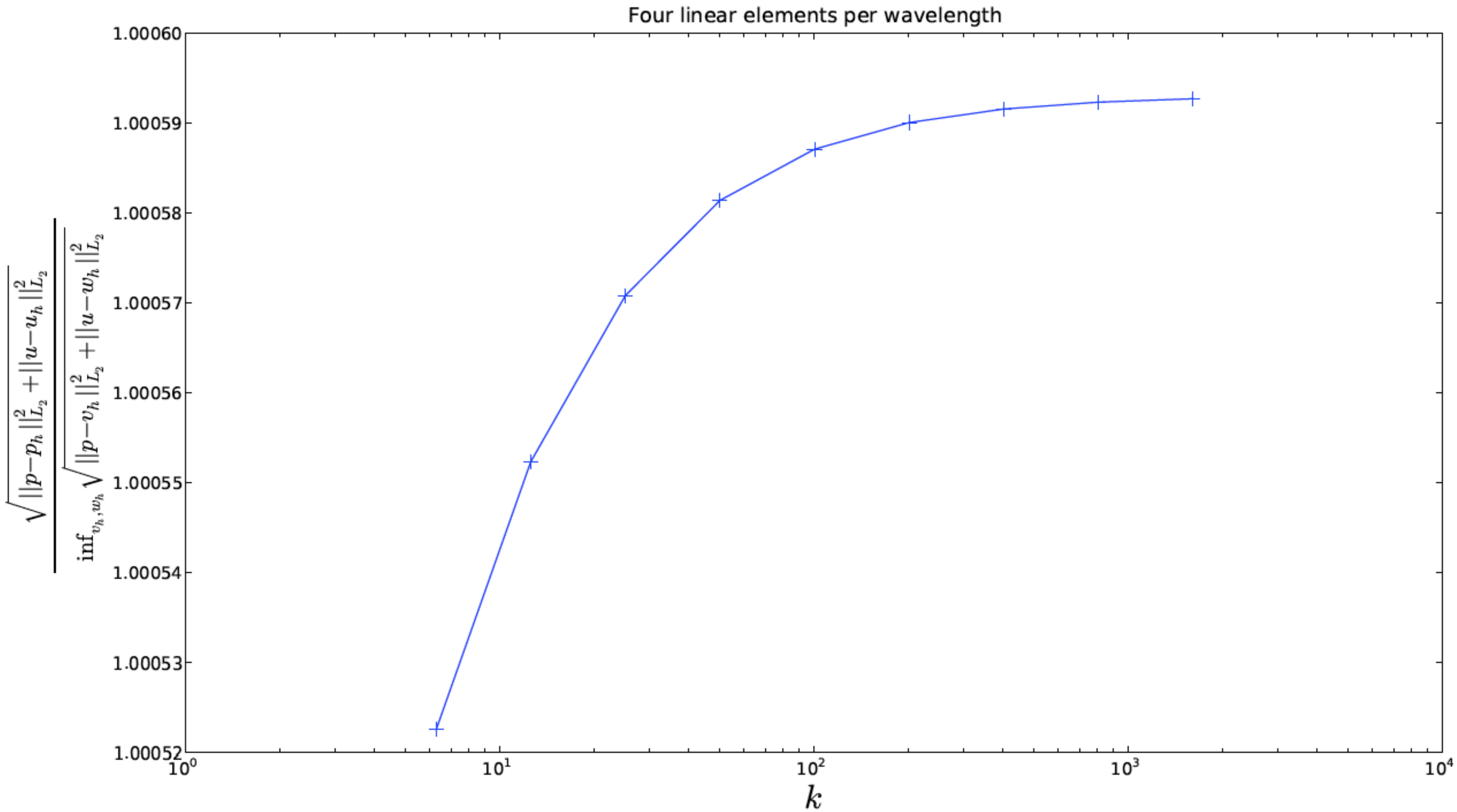
1D Result



Six linear elements per wavelength

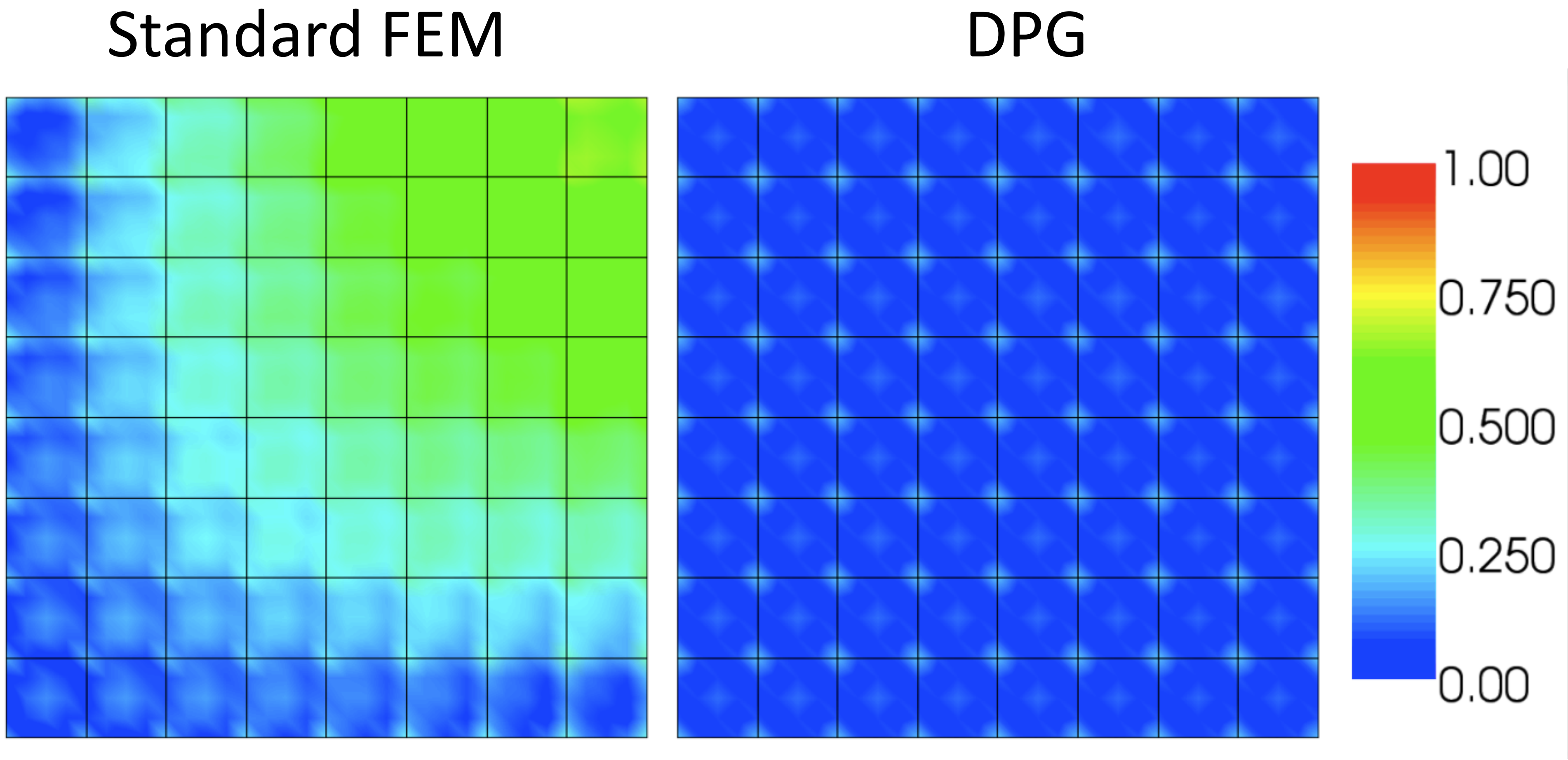
$$p(x) = \exp(ikx)$$

1D Result



Ratio between error and best approximation error as a function of k

2D Result

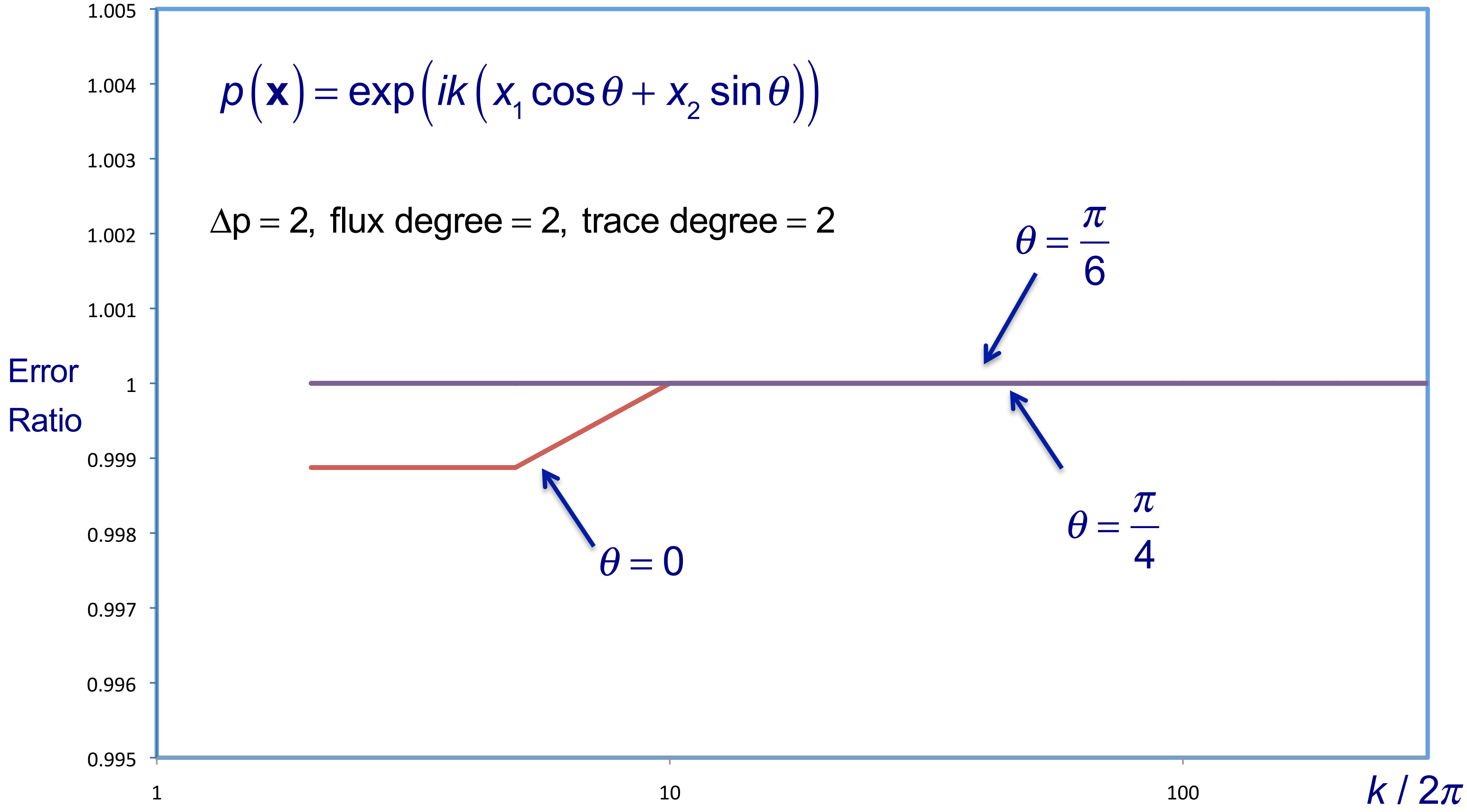


Error magnitude

$$p(\mathbf{x}) = \exp\left(ik\left(x_1 \cos\frac{\pi}{4} + x_2 \sin\frac{\pi}{4}\right)\right)$$

2D Result

4 bilinear elements per wavelength, pure impedance boundary cond.



Ratio between error and best approximation error as a function of $\frac{k}{2\pi}$

Steady Transport Problem

- Motivation for DPG framework
  - Design discretization such that
    - Trial space grants approximability
    - Weighting space grants stability
    - Parameter-free convergence rate
- Application to the Transport Equations
  - Numerical results

Advection-Diffusion-Reaction in 1D

- Conservative second order form:

$$\begin{cases} s(x)u(x)+\Big(a(x)u(x)-\kappa u(x)'\Big)' = f(x) & \text{in }]a,b[\\ u(a)=g_a, \quad u(a)=g_a \end{cases}$$

where

- $\kappa > 0$ is the constant diffusion coefficient
- $a(x), s(x)$ smoothly varying coefficients (advection velocity, reaction term)
- $f(x)$ source term

- Equivalent first order form:

$$\begin{cases} \sigma(x)-\Big(a(x)u(x)-\kappa u(x)'\Big)=0 & \text{in }]a,b[\\ s(x)u(x)+\sigma(x)'=f(x) & \text{in }]a,b[\end{cases}$$

Ultra Weak Formulation

- Discontinous Petrov-Galerkin formalism (ultra weak form):
  - Integration by parts over $K=\left(x_k, x_{k-1}\right)$,

    where $a=x_0<x_1<\ldots<x_N=b$

$$0=\int_{x_{k-1}}^{x_k} \tau\left(\sigma-a u\right) dx-\int_{x_{k-1}}^{x_k} \tau' \kappa u\, dx+\kappa \tau u\Big|_{x_{k-1}}^{x_k}$$

$$+\int_{x_{k-1}}^{x_k} v\left(s u-f\right) dx-\int_{x_{k-1}}^{x_k} v' \sigma\, dx+v \sigma\Big|_{x_{k-1}}^{x_k}$$

    for all test functions $v, \tau \in H^1\left(x_k, x_{k-1}\right)$

  - Let $\hat{\sigma}, \hat{u}$ denote traces of σ, u, which are independent variables

Ultra Weak Formulation

- The resulting abstract ultra weak form is:

Find $\mathbf{u}=\{\sigma,u,\hat{\sigma},\hat{u}\}\in U \ s.t.\ b(\mathbf{v},\mathbf{u})=l(\mathbf{v}) \quad \forall\mathbf{v}\in V$

where

$$b(\mathbf{v},\mathbf{u})=\sum_{k=1}^{N}\left\{\int_{x_{k-1}}^{x_k}(\tau-v')\,\sigma\,dx+\int_{x_{k-1}}^{x_k}(-\tau'\kappa-a\,\tau+s\,v)\,u\,dx+(v\,\hat{\sigma}-\kappa\,\tau\,\hat{u})\Big|_{x_{k-1}}^{x_k}\right\}$$

$$l(\mathbf{v})=\sum_{k=1}^{N}\int_{x_{k-1}}^{x_k}v\,f\,dx$$

and

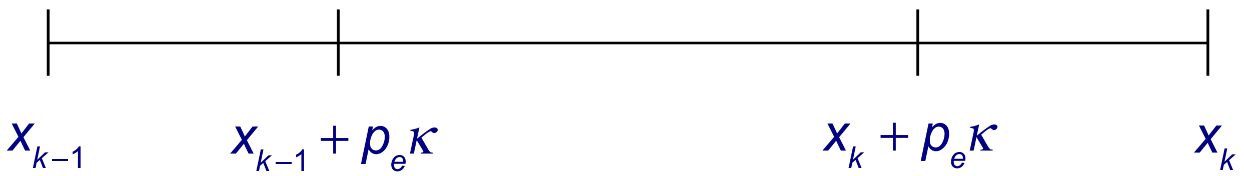
$$U=L_2(a,b)\times L_2(a,b)\times R^{N+1}\times R^{N+1}$$

$$V=W_N\times W_N$$

$$W_N=\left\{w:w|_{(x_k,x_{k-1})}\in H^1(x_k,x_{k-1}),\ k=1,2,\ldots,N\right\}$$

Computational Considerations

- Fields $\{\sigma, u\}$ approximated using Bernstein polynomials of order p
  - Optimal weighting space computed on an enriched polynomial space
    - Enrichment: uniform Δp degree elevation to $p_e = p + \Delta p$
  - Discrete variational problem develops exponential boundary layers
    - Using p-FEM recipes, boundary layer mesh of size $p_e \kappa$ is added
    - C^0 and $C^{p_e - 1}$ B-splines are used

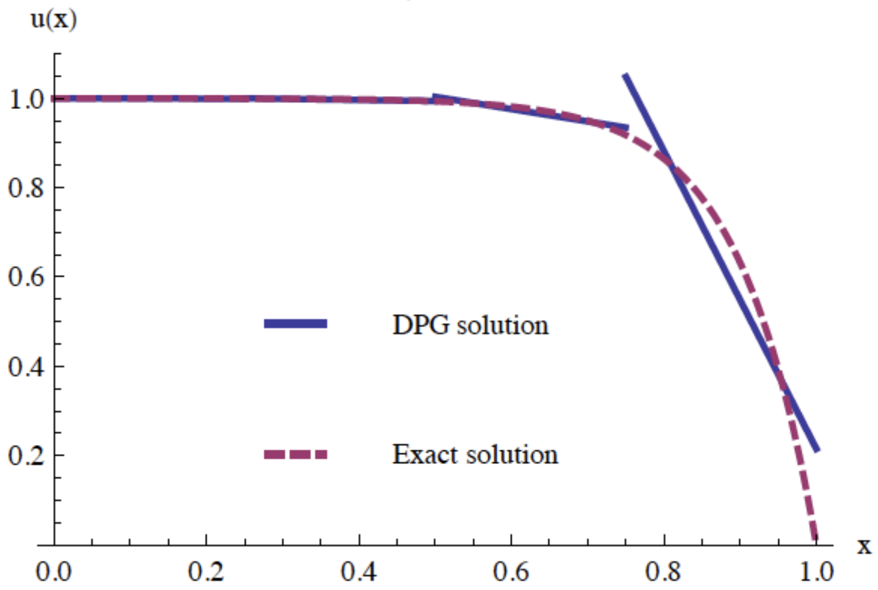


Subgrid within each element (boundary layer mesh)

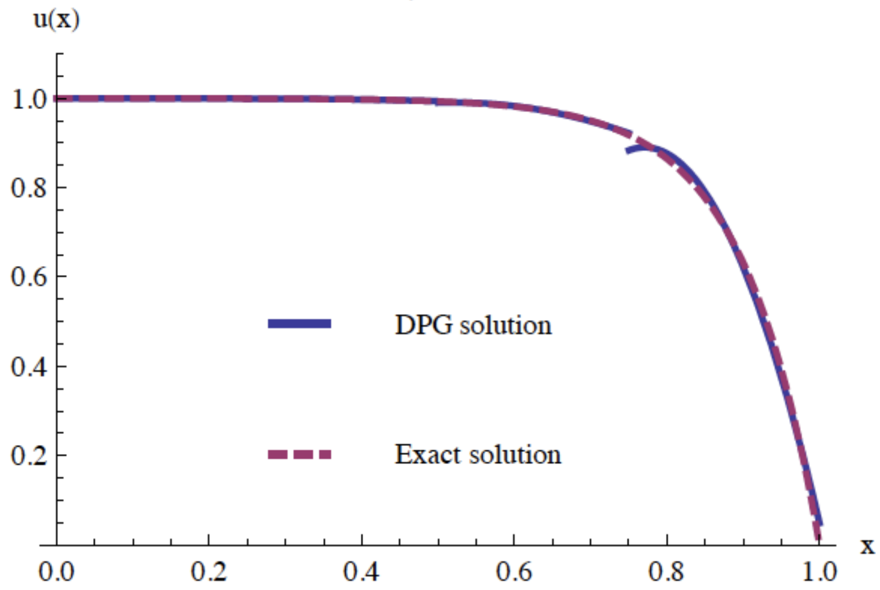
Model Transport Problem

$$\begin{cases} \left(u(x) - \kappa u(x)'\right)' = 0 \quad \text{in }]0,1[\\ u(0) = 1, \quad u(1) = 0 \end{cases} \Rightarrow u(x) = \frac{1 - \exp\left(\frac{x-1}{\kappa}\right)}{1 - \exp\left(\frac{-1}{\kappa}\right)}$$

$p = 1$

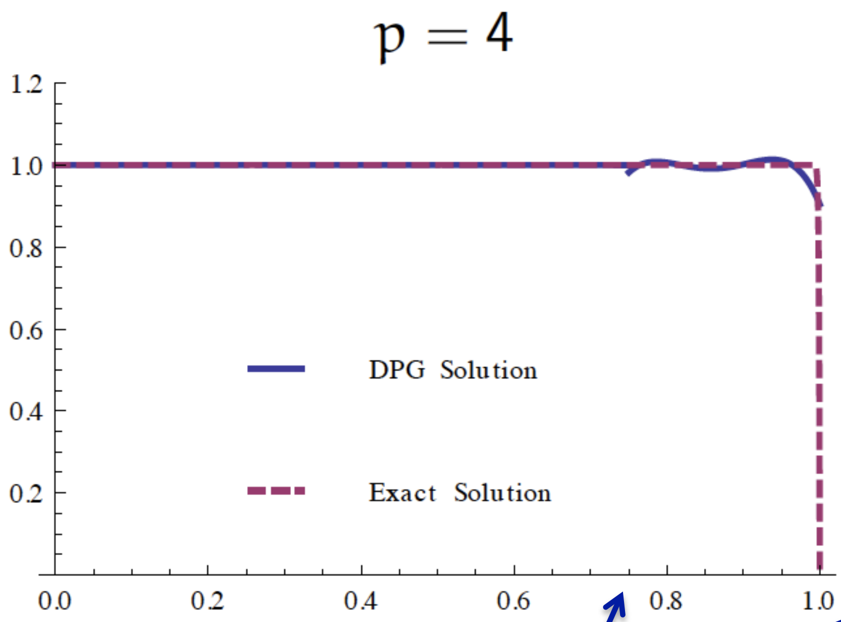


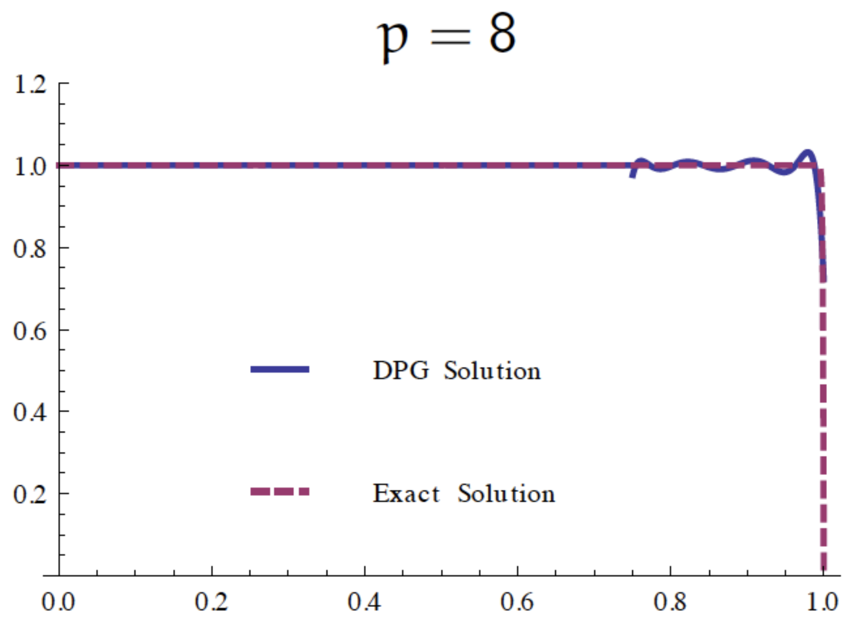
$p = 2$



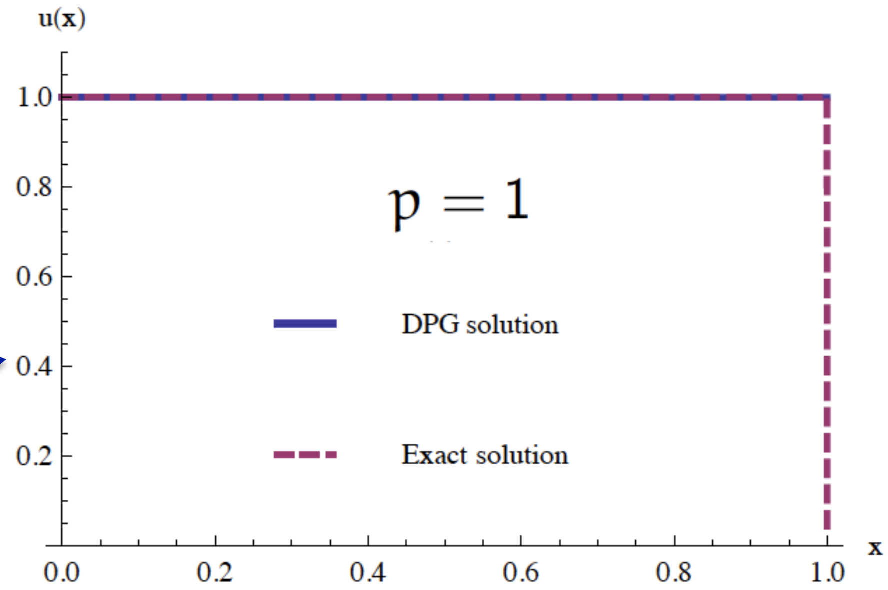
Four elements, $\kappa = 10^{-1}$, $\Delta p = 2$

Model Transport Problem





Four elements, $\kappa = 10^{-3}$, $\Delta p = 2$



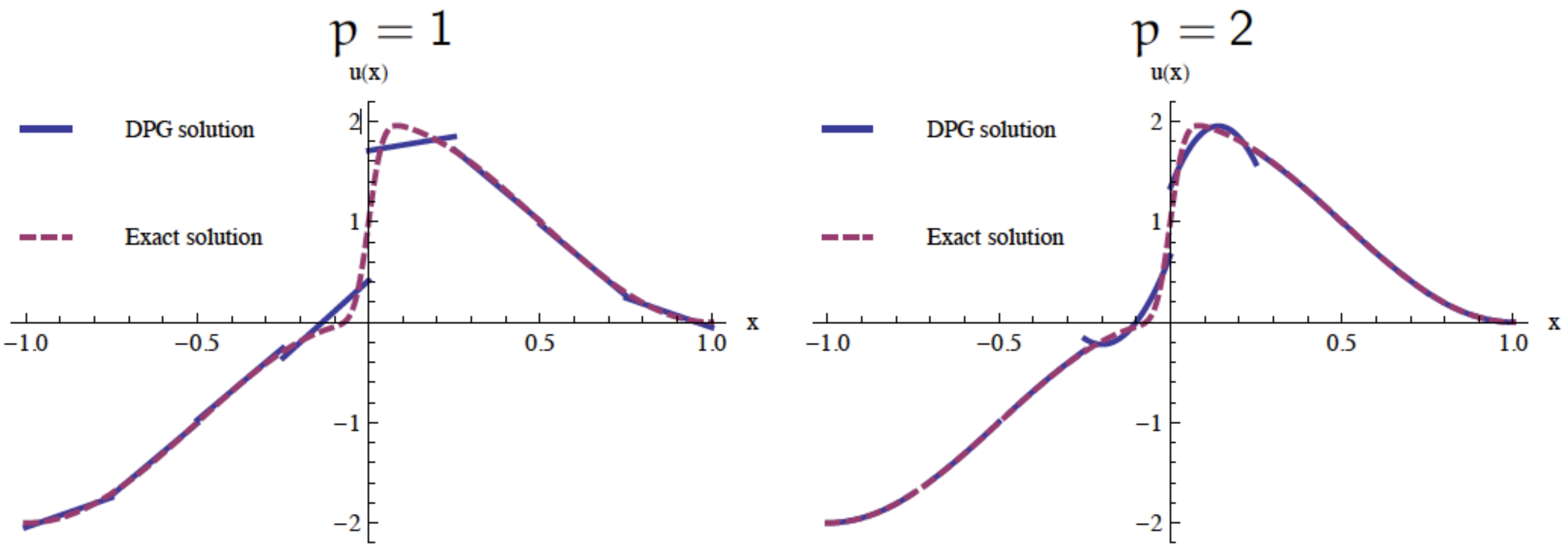
Four elements,

$\kappa = 10^{-6}$, $\Delta p = 2$

Hemker's Transport Problem

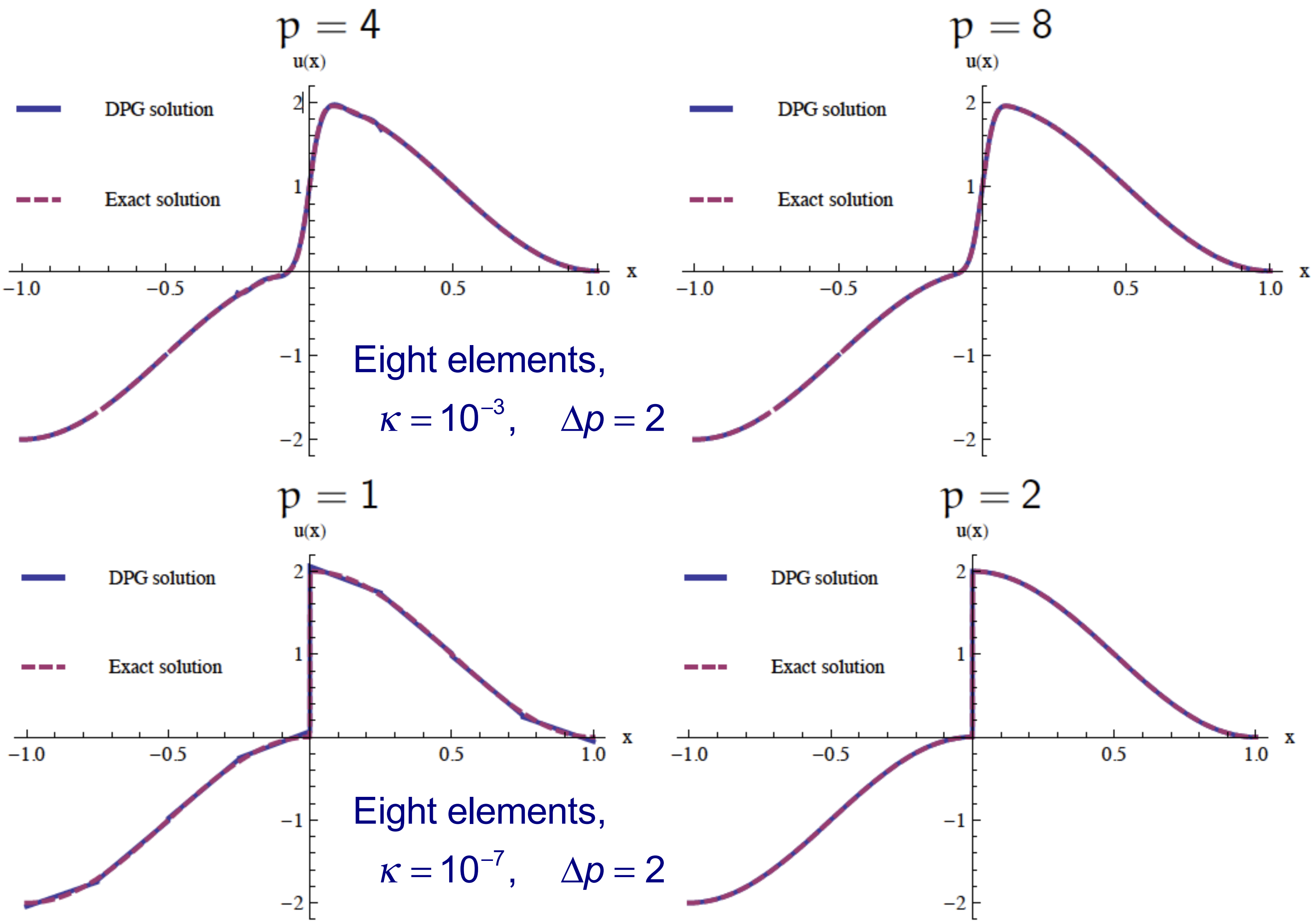
$$\begin{cases} \left(x u(x) - \kappa u(x)' \right)' = \kappa \pi^2 \cos(\pi x) + \pi x \sin(\pi x) & \text{in }]0,1[\\ u(-1) = -2, \quad u(1) = 0 \end{cases}$$

$$\Rightarrow \quad u(x) = \cos(\pi x) + \frac{\operatorname{erf}\left(\frac{x}{\sqrt{2\kappa}} \right)}{\operatorname{erf}\left(\frac{1}{\sqrt{2\kappa}} \right)}$$



Eight elements, $\kappa = 10^{-3}$, $\Delta p = 2$

Hemker's Transport Problem



Advection-Diffusion-Reaction in 2D

- Conservative second order form:

$$\begin{cases} s\,u + \nabla \cdot \left(\mathbf{a}\, u - \kappa \nabla u \right) = f, & \text{in} \nabla \\ u = g, & \text{on } \partial\Omega \end{cases}$$

where

- ◦ $\kappa > 0$ is the constant diffusion coefficient
- ◦ $\mathbf{a}(\mathbf{x}), s(\mathbf{x})$ smoothly varying coefficients (adv. velocity, reaction term)
- ◦ $f(\mathbf{x})$ source term

- Equivalent first order formalism:

$$\begin{cases} \boldsymbol{\sigma} - \left(\mathbf{a}\, u - \kappa \nabla u \right) = 0 & \text{in } \Omega \\ s\,u + \nabla \cdot \boldsymbol{\sigma} = f & \text{in } \Omega \end{cases}$$

Ultra Weak Formulation

- Discontinous Petrov-Galerkin formalism:
  - Integration by parts over a single element K in a mesh Ω_h

$$0 = \int_K \boldsymbol{\tau} \cdot (\boldsymbol{\sigma} - \mathbf{a}\, u)\, dx - \int_K \nabla \cdot \boldsymbol{\tau}\, \kappa\, u\, dx + \int_{\partial K} \kappa\, \mathbf{n} \cdot \boldsymbol{\tau}\, u$$

$$+ \int_K v (s\, u - f)\, dx - \int_K \nabla v \cdot \boldsymbol{\sigma}\, dx + \int_{\partial K} v\, \mathbf{n} \cdot \boldsymbol{\sigma}$$

for all test functions $v \in H^1(K)$, $\boldsymbol{\tau} \in \mathbf{H}(\text{div}, K)$

  - Let $\hat{\sigma}_n, \hat{u}$ denote traces of $\mathbf{n} \cdot \boldsymbol{\sigma}$, u, which are independent variables
    - The functional setting is not trivial

$$\hat{u} \in H_0^{1/2}(\partial \Omega^h), \qquad \hat{\sigma}_n \in H_0^{-1/2}(\partial \Omega^h)$$

Variational Crimes

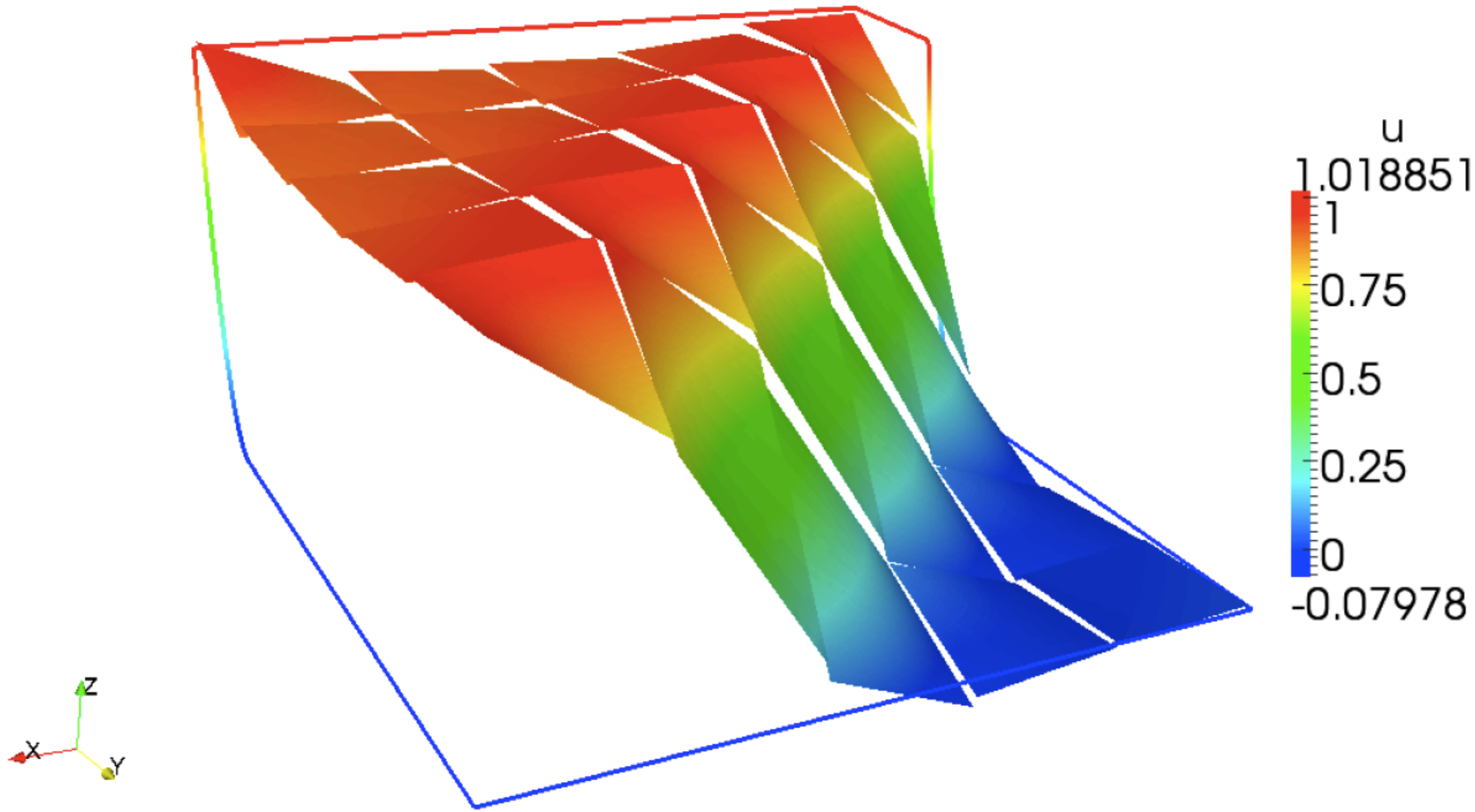
- Discrete Trial-to-Test operator:

$$\begin{cases} \text{Find } v_A^h \in V^{h+} \text{ s.t.} \\ \left(v_A^h, w^h\right)_V = b\left(N_A, w^h\right) \quad \forall w^h \in V^{h+} \end{cases}$$

where:

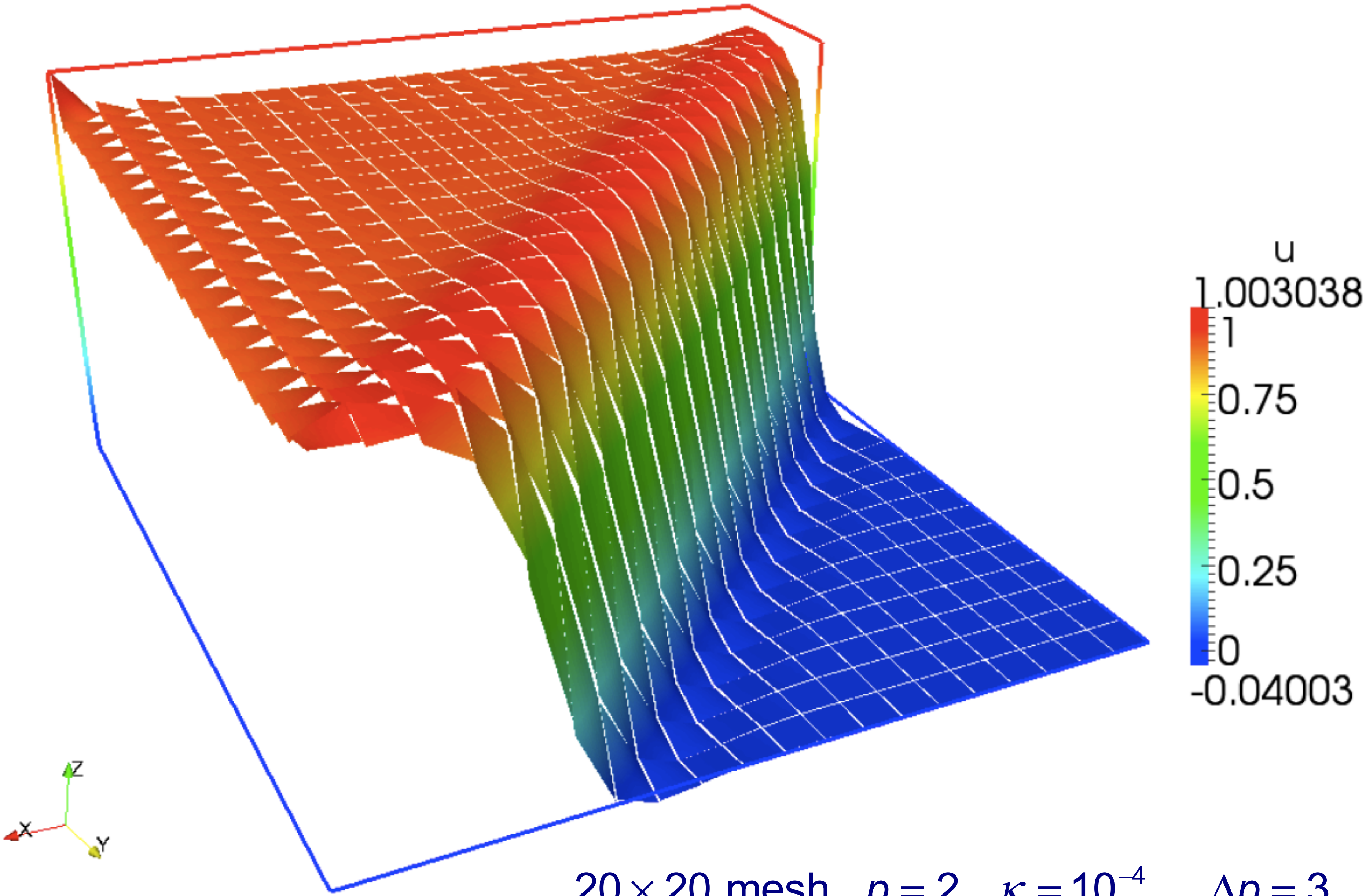
- V^{h+} is an enriched (h and p) finite element space, not in infinite dimensional (continuous) V
  - 3×3-spline grid constructed analogously to the 1D case
- N_A is each trial basis function
  - $\boldsymbol{\sigma}, u$ are approximated using L_2-conforming piecewise Bernstein polynomials of degree p
  - $\hat{\sigma}_n, \hat{u}$ are approximated using $H^{1/2}$- and $H^{-1/2}$-conforming piecewise Bernstein polynomials of degree $p+1$ on wireframe (continuous/discontinuous)

Advection Skew to Mesh



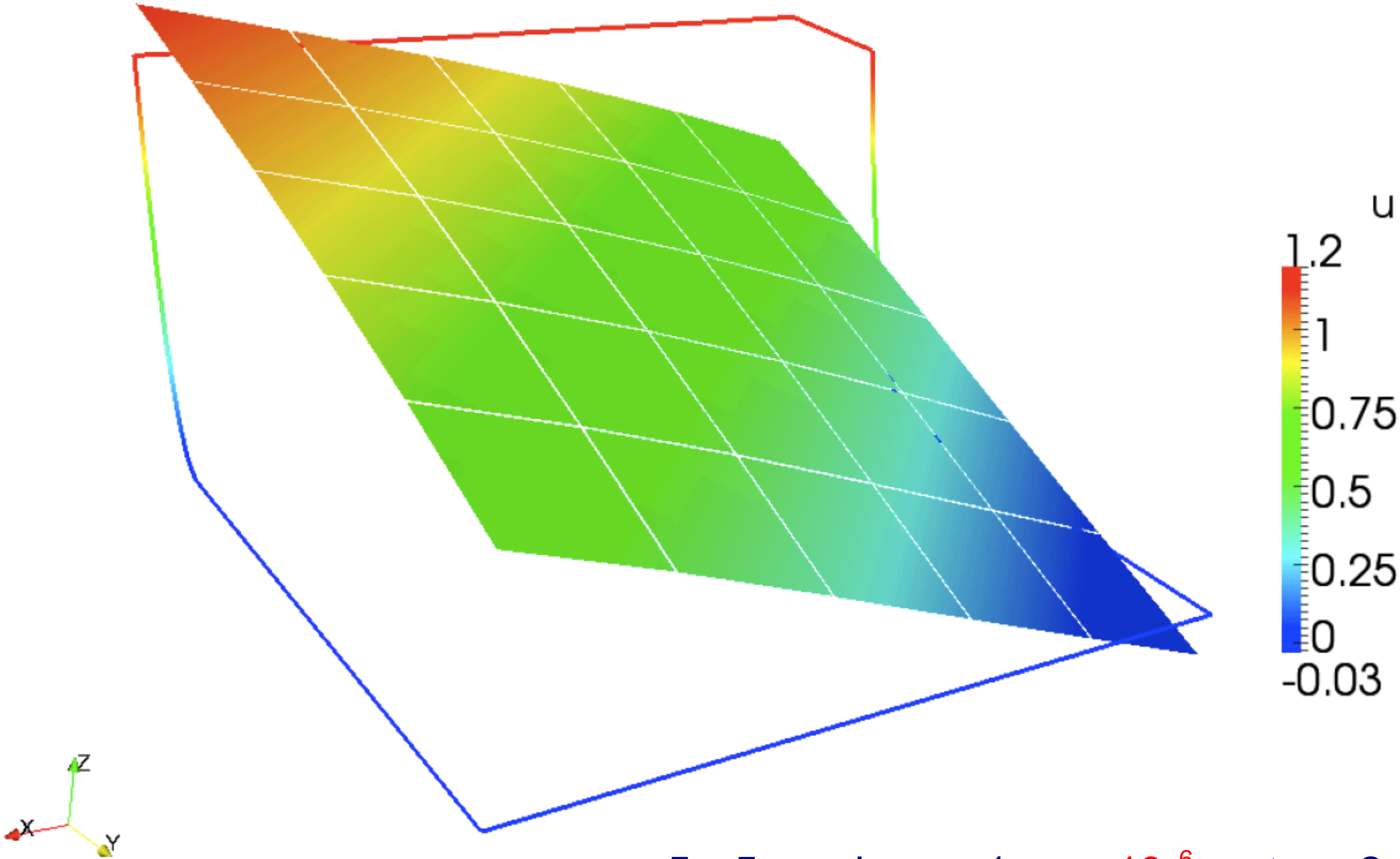
5×5 mesh, $p = 1$, $\kappa = 10^{-4}$, $\Delta p = 2$

Advection Skew to Mesh



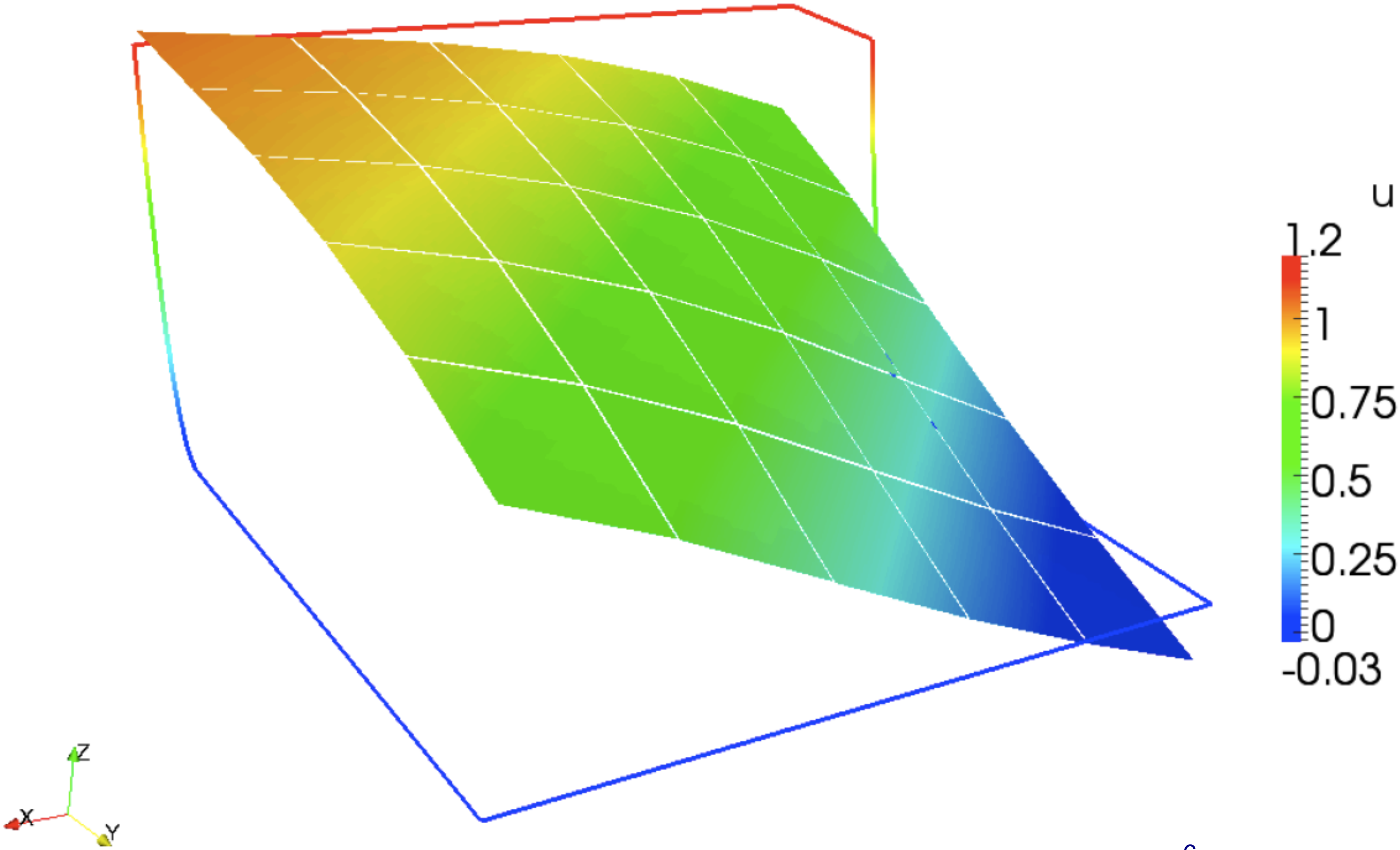
20×20 mesh, $p = 2$, $\kappa = 10^{-4}$, $\Delta p = 3$

Advection Skew to Mesh



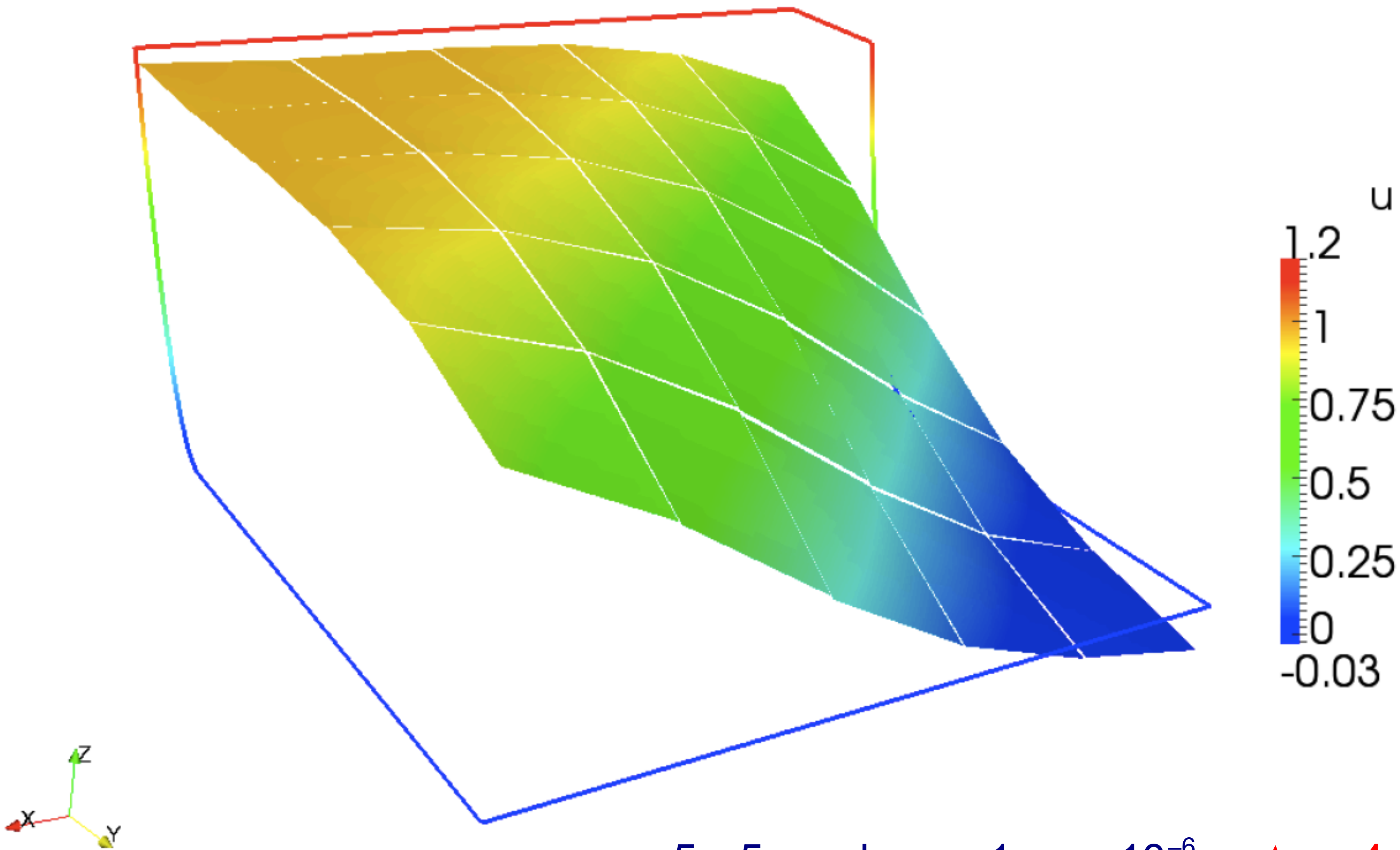
5×5 mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 2$

Advection Skew to Mesh



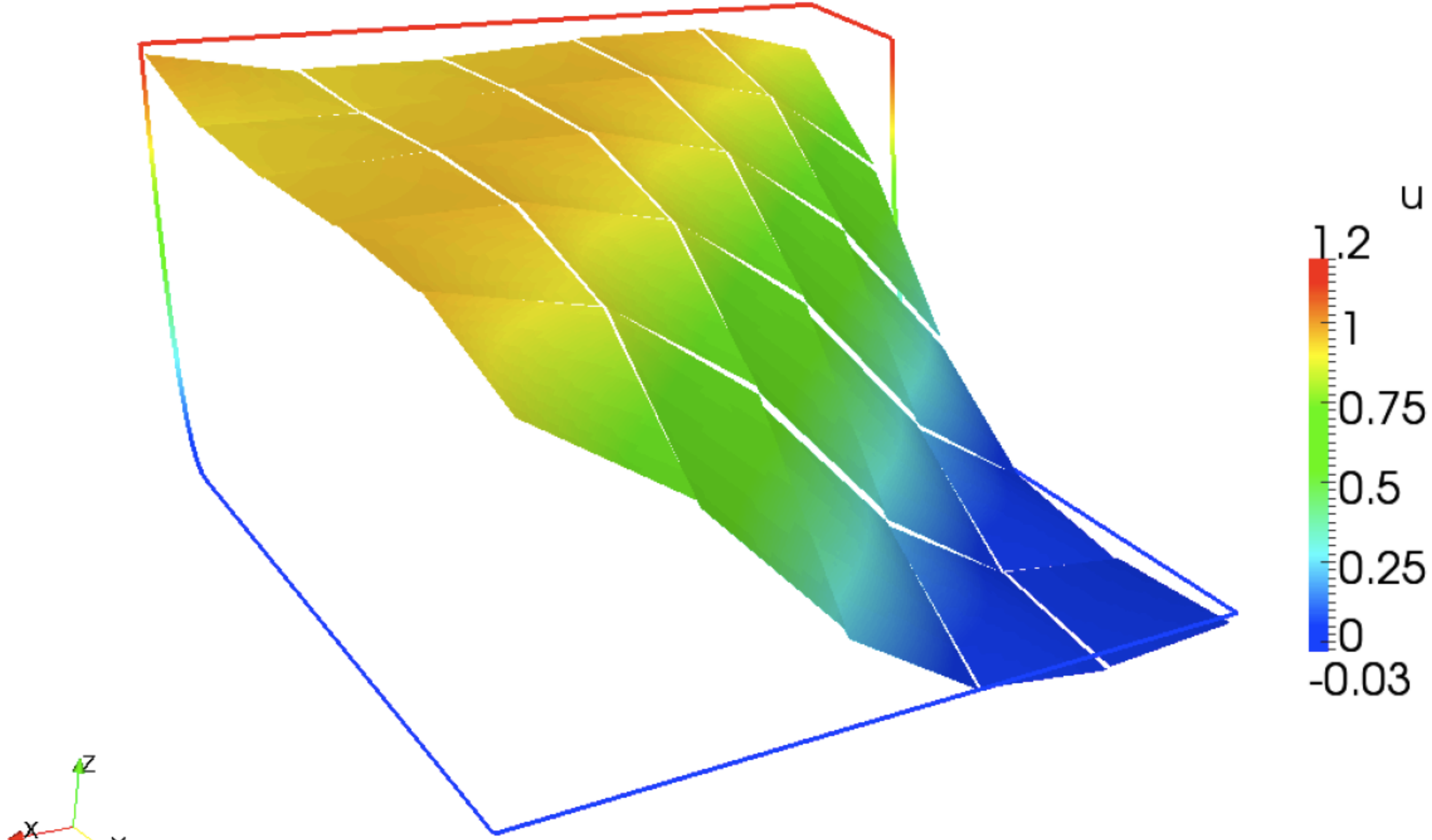
5×5 mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 3$

Advection Skew to Mesh



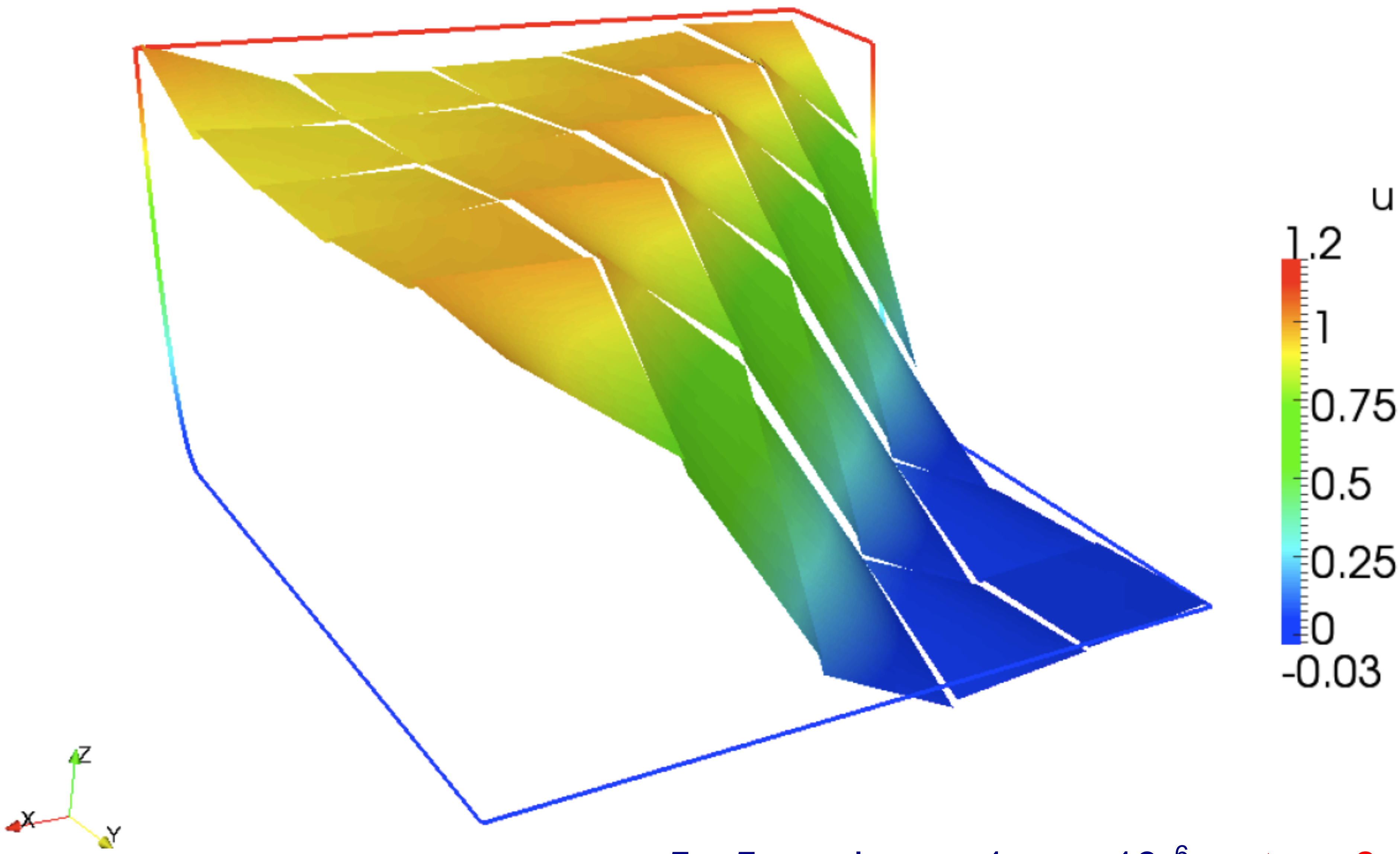
5×5 mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 4$

Advection Skew to Mesh



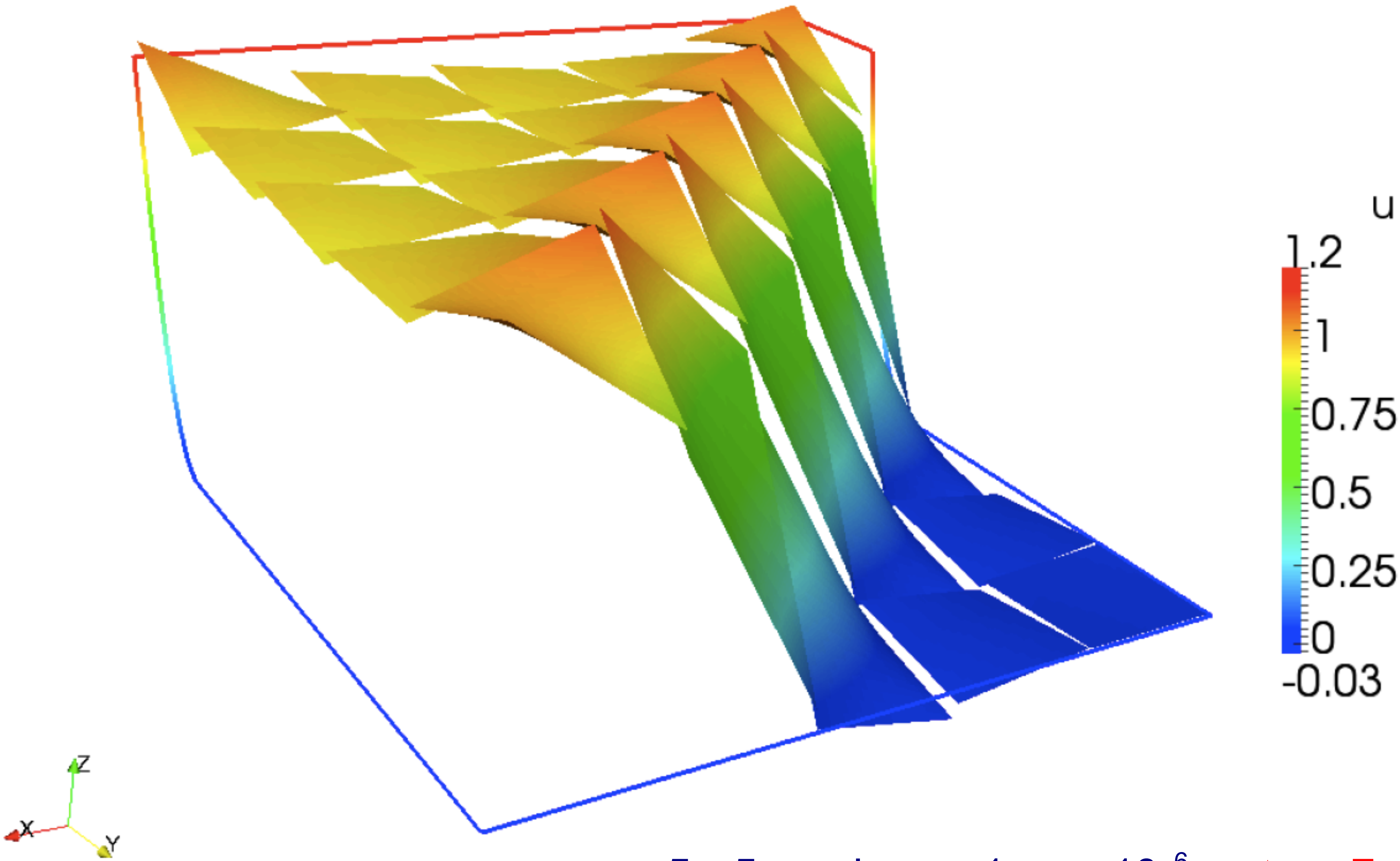
5×5 mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 5$

Advection Skew to Mesh



5×5 mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 6$

Advection Skew to Mesh



5×5 mesh, $p = 1$, $\kappa = 10^{-6}$, $\Delta p = 7$

Conclusions

Helmholtz

- DPG for multiD Helmholtz has best approximation property
  - Pollution/Dispersion effects are significantly reduced for discrete approximation (numerical evidence)
- Robust methodology for small perturbation problems
  - Advection-diffusion, Helmholtz, thin structures, etc.

Steady Transport

- Studied numerically the approximation of advection-diffusion-reaction problems using the discontinuous Petrov-Galerkin method with Bernstein polynomials and B-splines
- Constructed robust algorithm based on optimal test space norm and element subgrid enrichment for resolving weighting functions
- Smooth B-splines yield some computational saving when computing weighting function space