Isogeometric mortar methods with applications in contact mechanics

PhD dissertation defense

IUSS Scuola Universitaria Superiore Pavia

Brivadis Ericka

Supervisor: Annalisa Buffa

Doctor of Philosophy in Computational Mechanics and Advanced Materials, XXVIII Cycle (2012-2015)

Pavia, January 19th, 2017

Context of the work

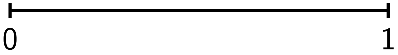
This research work is part of a project between the imati of Pavia and the company MICHELIN which aims to treat complex contact problems with isogeometric analysis.

It has been done in collaboration with A. Buffa (imati Pavia), G. Elber (TECHNION Israel Institute of Technology), M. Martinelli (imati Pavia), F. Massarwi (TECHNION Israel Institute of Technology), L. Wunderlich (TUM) and B. Wohlmuth (TUM).

Outline

1. IGA basics
2. Isogeometric mortar methods
3. Applications in contact mechanics

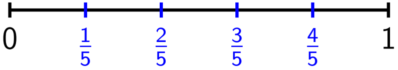
- The **parametric** unit interval



IGA basics

Univariate B-Splines

- The **parametric** unit interval is split by the **breakpoints** $\zeta_j \, (j = 1, \ldots, N)$ leading to a **parametric mesh**.

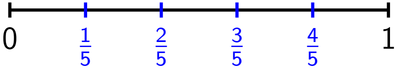


$$U = \{0, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, 1\}$$

IGA basics

Univariate B-Splines

- The **parametric** unit interval is split by the **breakpoints** $\zeta_j \,(j = 1, \ldots, N)$ leading to a **parametric mesh**.



$$U = \{0, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, 1\}$$

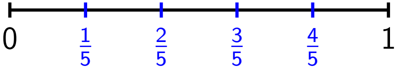
- Each breakpoint ζ_j has a multiplicity m_j. The breakpoints and their multiplicities define the **knot vector**

$$\Xi = \{\underbrace{\zeta_1, \ldots, \zeta_1}_{m_1 \text{ times}}, \underbrace{\zeta_2, \ldots, \zeta_2}_{m_2 \text{ times}}, \ldots, \underbrace{\zeta_N, \ldots, \zeta_N}_{m_N \text{ times}}\}.$$

IGA basics

Univariate B-Splines

- The **parametric** unit interval is split by the **breakpoints** $\zeta_j \,(j = 1, \dots, N)$ leading to a **parametric mesh**.



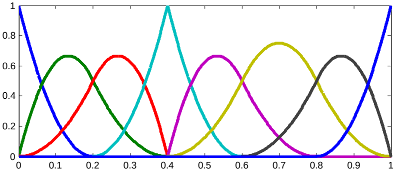
$$U = \{0, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, 1\}$$

- Each breakpoint ζ_j has a multiplicity m_j. The breakpoints and their multiplicities define the **knot vector**

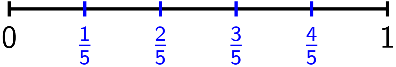
$$\Xi = \{\underbrace{\zeta_1, \dots, \zeta_1}_{m_1 \text{ times}}, \underbrace{\zeta_2, \dots, \zeta_2}_{m_2 \text{ times}}, \dots, \underbrace{\zeta_N, \dots, \zeta_N}_{m_N \text{ times}}\}.$$

- Given a degree p and a knot vector Ξ, the **univariate $(p+1)$-order B-Splines** can be defined recursively.

- **Important properties**



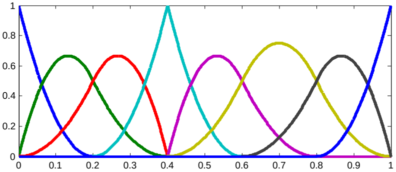
Univariate **quadratique** B-Spline functions defined on the knot vector $\Xi = \{0,\ 0,\ 0,\ \frac{1}{5}, \frac{2}{5}, \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ 1,\ 1,\ 1\}$.



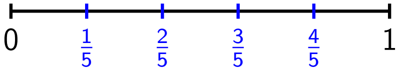
We note the C^0 continuity in $\zeta_j = \frac{2}{5}$.

- Each $\widehat{B}_i^p$ is a piecewise **positive polynomial** of degree p and has a **local support**.

- **Important properties**



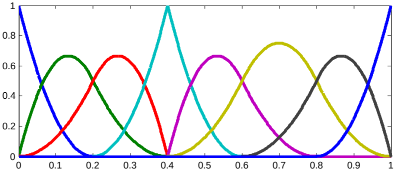
Univariate **quadratique** B-Spline functions defined on the knot vector $\Xi = \{0,\, 0,\, 0,\, \frac{1}{5}, \frac{2}{5}, \frac{2}{5},\, \frac{3}{5},\, \frac{4}{5},\, 1,\, 1,\, 1\}$.



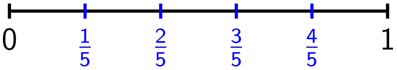
We note the C^0 continuity in $\zeta_j = \frac{2}{5}$.

- Each $\widehat{B}_i^p$ is a piecewise **positive polynomial** of degree p and has a **local support**.
- On each element, at most $(p+1)$ functions have non-zero values.

- **Important properties**



Univariate **quadratique** B-Spline functions defined on the knot vector $\Xi = \{0,\, 0,\, 0,\, \frac{1}{5}, \frac{2}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1,\, 1,\, 1\}$.



We note the C^0 continuity in $\zeta_j = \frac{2}{5}$.

- Each $\widehat{B}_i^p$ is a piecewise **positive polynomial** of degree p and has a **local support**.
- On each element, at most $(p+1)$ functions have non-zero values.
- The inter-element continuity is defined by the breakpoint multiplicity: the basis is $\boldsymbol{C^{p-m_j}}$ **at** ζ_j.

- For any direction δ, we have p_δ, U_δ, Ξ_δ and n_δ. Then we introduce
  - $\mathbf{U} = (U_1 \times U_2 \times \ldots \times U_d)$,
  - $\mathbf{\Xi} = (\Xi_1 \times \Xi_2 \times \ldots \times \Xi_d)$ and
  - $\mathbf{I} = \{\mathbf{i} = (i_1, \ldots, i_d) : 1 \leq i_\delta \leq n_\delta\}$

  to define multivariate B-Splines by a **tensor product** of univariate B-Splines.

- For any direction δ, we have p_δ, U_δ, Ξ_δ and n_δ. Then we introduce
  - $\mathbf{U} = (U_1 \times U_2 \times \ldots \times U_d)$,
  - $\boldsymbol{\Xi} = (\Xi_1 \times \Xi_2 \times \ldots \times \Xi_d)$ and
  - $\mathbf{I} = \{\mathbf{i} = (i_1, \ldots, i_d) : 1 \leq i_\delta \leq n_\delta\}$

  to define multivariate B-Splines by a **tensor product** of univariate B-Splines.
- Given a **set of positive weights** $\{\omega_{\mathbf{i}}, \mathbf{i} \in \mathbf{I}\}$, the **Non-Uniform Rational B-Splines** functions are defined as rational functions of multivariate B-Spline functions by

$$\widehat{N}_{\mathbf{i}}^{p}(\zeta) = \frac{\omega_{\mathbf{i}} \ \widehat{B}_{\mathbf{i}}^{p}(\zeta)}{\sum_{\mathbf{i}\in\mathbf{I}} \omega_{\mathbf{i}} \ \widehat{B}_{\mathbf{i}}^{p}(\zeta)}.$$

- For any direction δ, we have p_δ, U_δ, Ξ_δ and n_δ. Then we introduce
  - $\mathbf{U} = (U_1 \times U_2 \times \ldots \times U_d)$,
  - $\boldsymbol{\Xi} = (\Xi_1 \times \Xi_2 \times \ldots \times \Xi_d)$ and
  - $\mathbf{I} = \{\mathbf{i} = (i_1, \ldots, i_d) : 1 \leq i_\delta \leq n_\delta\}$

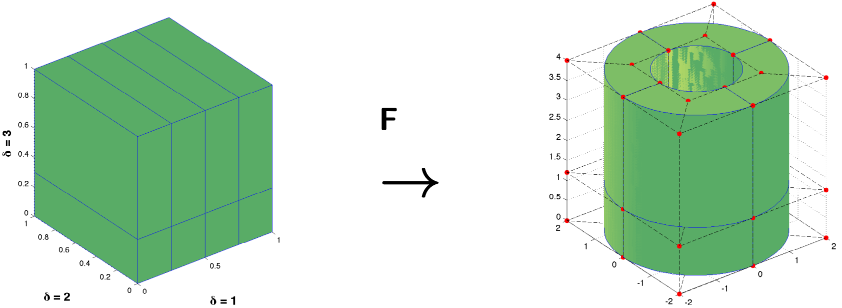
  to define multivariate B-Splines by a **tensor product** of univariate B-Splines.
- Given a **set of positive weights** $\{\omega_{\mathbf{i}}, \mathbf{i} \in \mathbf{I}\}$, the **Non-Uniform Rational B-Splines** functions are defined as rational functions of multivariate B-Spline functions by

$$\widehat{N}_{\mathbf{i}}^{p}(\zeta) = \frac{\omega_{\mathbf{i}}\ \widehat{B}_{\mathbf{i}}^{p}(\zeta)}{\sum_{\mathbf{i}\in\mathbf{I}} \omega_{\mathbf{i}}\ \widehat{B}_{\mathbf{i}}^{p}(\zeta)}.$$

- $S^p(\boldsymbol{\Xi})$ is the **spline** space spanned by the functions $\widehat{S}_{\mathbf{i}}^{p}(\zeta)$, $N^p(\boldsymbol{\Xi})$ the **NURBS** space spanned by the functions $\widehat{N}_{\mathbf{i}}^{p}(\zeta)$.

In CAGD, **NURBS and splines are widely spread** as they are capable of accurately describing various geometries.
Given a set of Control Points $\mathbf{C_i} \in \mathbb{R}^d$, $\mathbf{i} \in \mathbf{I}$, a NURBS parametrization of a curve (d=1), a surface ($d = 2$) or a solid ($d = 3$) is given by

$$\mathbf{F} : \widehat{\boldsymbol{\Omega}} \rightarrow \boldsymbol{\Omega} \qquad \mathbf{F}(\zeta) = \sum_{\mathbf{i} \in \mathbf{I}} \mathbf{C_i}\, \widehat{N}^p_{\mathbf{i}}(\zeta).$$



IGA basics
Geometrical approximation

The computational domain Ω is generally decomposed into $\boldsymbol{K}$ **non-overlapping subdomains**, i.e.,

$$\Omega = \bigcup_{k=1}^{K} \Omega_k, \quad \Omega_i \cap \Omega_j = \emptyset, \quad i \neq j.$$

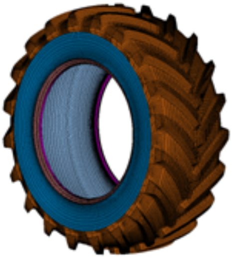
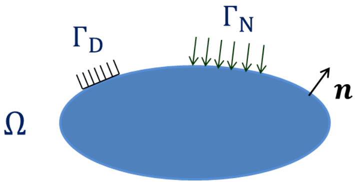


Figure: A tire kindly given by MICHELIN

Some of the reasons:

- Geometrical definition purposes
- Different materials

- Strong formulations



Laplace problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = u_D & \text{on } \Gamma_D, \\ \nabla u \cdot \boldsymbol{n} = l & \text{on } \Gamma_N. \end{cases}$$

Static mechanical balance problem

$$\begin{cases} -\mathbf{div}(\underline{\underline{\sigma}}) = \boldsymbol{f} & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{u_D} & \text{on } \Gamma_D, \\ \underline{\underline{\sigma}} \cdot \boldsymbol{n} = \boldsymbol{l} & \text{on } \Gamma_N. \end{cases}$$

under the following assumptions:

- linear elasticity
  $\underline{\underline{\varepsilon}} = \frac{1}{2}\left(\underline{\underline{\boldsymbol{\nabla} \boldsymbol{u}}} + \underline{\underline{(\boldsymbol{\nabla} \boldsymbol{u})^T}}\right) = \underline{\underline{\boldsymbol{\nabla}^s \boldsymbol{u}}}$,
- small displacement-deformation
  $\underline{\underline{\sigma}} = \lambda_{\text{Lamé}}\ tr(\underline{\underline{\varepsilon}})\ \underline{\underline{\mathbb{I}}} + 2\ \mu_{\text{Lamé}}\ \underline{\underline{\varepsilon}}$.

- Weak formulations

  Find $\boldsymbol{u} \in V_D$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) = f(\boldsymbol{v}), \qquad \forall \boldsymbol{v} \in V_0.$$

Laplace problem

$$a(u, v) = \int_\Omega \nabla u \cdot \nabla v \,\mathrm{dx},$$
$$f(v) = \int_\Omega f \cdot v \,\mathrm{dx} + \int_{\Gamma_N} l \cdot v \,\mathrm{ds}.$$

Static mechanical balance problem

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \lambda_{\text{Lamé}} \; \mathbf{div}(\boldsymbol{u}) \cdot \mathbf{div}(\boldsymbol{v}) + 2\,\mu_{\text{Lamé}} \, \underline{\underline{\boldsymbol{\nabla}^{\boldsymbol{s}} \boldsymbol{u}}} : \underline{\underline{\boldsymbol{\nabla}^{\boldsymbol{s}} \boldsymbol{v}}} \,\mathrm{dx},$$
$$f(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \,\mathrm{dx} + \int_{\Gamma_N} \boldsymbol{l} \cdot \boldsymbol{v} \,\mathrm{ds}.$$

- Each **$\boldsymbol{\Omega_k}$ is parametrised by a NURBS parametrization $\boldsymbol{F_k}$**.

IGA basics
Motivations

- Each $\boldsymbol{\Omega_k}$ **is parametrised by a NURBS parametrization** $\boldsymbol{F_k}$.
- Each **subdomain is discretized independently**
  $\rightarrow$ non-conforming meshes at the subdomain interfaces !

- Each $\boldsymbol{\Omega_k}$ **is parametrised by a NURBS parametrization** $\boldsymbol{F_k}$.
- Each **subdomain is discretized independently**
  $\rightarrow$ non-conforming meshes at the subdomain interfaces !
- The **approximation space** on Ω_k is

$$V_{k,h} = \{\boldsymbol{v_k} = \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \circ \mathbf{F}_k^{-1}, \quad \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \in (N^{p_k}(\boldsymbol{\Xi}_k))^{d_P}\},$$

i.e., a push-forward of a NURBS space.

- Each $\boldsymbol{\Omega_k}$ **is parametrised by a NURBS parametrization** $\boldsymbol{F_k}$.
- Each **subdomain is discretized independently**
  $\rightarrow$ non-conforming meshes at the subdomain interfaces !
- The **approximation space** on Ω_k is

$$V_{k,h} = \{\boldsymbol{v_k} = \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \circ \mathbf{F}_k^{-1}, \quad \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \in (N^{p_k}(\boldsymbol{\Xi}_k))^{d_P}\},$$

  i.e., a push-forward of a NURBS space.
- **at the interfaces ?**

Where we are

1 IGA basics

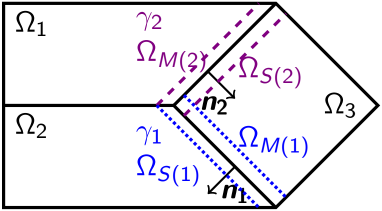
2 Isogeometric mortar methods

3 Applications in contact mechanics

We define the **interface** $\gamma_{k_1 k_2}$ by $\gamma_{k_1 k_2} = \partial\Omega_{k_1} \cap \partial\Omega_{k_2} \quad \forall k_1, k_2$ such that $1 \leq k_1 < k_2 \leq K$.

For each interface, one of the adjacent subdomains is chosen as the **master** side and the other one as the **slave** side.

The **skeleton** Γ is the union of all the interfaces.



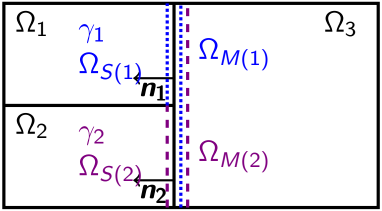


Figure: **Geometrical conforming** case (left) and **slave conforming** case (right)

Mortar method $\Rightarrow$ weak continuity imposition of u at each interface.

Discrete mortar saddle point formulation

Find $(u_h, \lambda_h) \in V_h \times M_h$, such that

$$\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_h) &= f(v_h), & v_h \in V_h, \\ b(u_h, \phi_h) &= 0, & \phi_h \in M_h, \end{aligned}$$

with $V_h = \prod_{k=1}^{K} V_{k,h}$ and $b(u_h, \phi_h) = \int_\Gamma u_h \cdot \phi_h \, \mathrm{d}s.$

Mortar method $\Rightarrow$ weak continuity imposition of u at each interface.

Discrete mortar saddle point formulation

Find $(u_h, \lambda_h) \in V_h \times M_h$, such that

$$\begin{aligned} a(u_h, v_h) + b(v_h, \lambda_h) &= f(v_h), \qquad && v_h \in V_h, \\ b(u_h, \phi_h) &= 0, && \phi_h \in M_h, \end{aligned}$$

with $V_h = \prod_{k=1}^{K} V_{k,h}$ and $b(u_h, \phi_h) = \int_\Gamma u_h \cdot \phi_h \, \mathrm{d}s.$

Choose properly M_h

$\rightarrow$ **Two requirements on each interface to have an optimal method**

- an appropriate approximation order for the multiplier space
- a uniform inf-sup stability between the primal and multiplier spaces

Where we are

1. IGA basics
2. Isogeometric mortar methods
   - The discrete mortar problem
   - Lagrange multiplier spaces
   - A key issue: the evaluation of the mortar integrals
3. Applications in contact mechanics
   - Contact problem definition
   - Example of treatment of deformable-deformable contact in a Lagrange multiplier formulation

- $M_h = \prod_{l=1}^{L} M_{l,h}$ where $M_{l,h}$ is the multiplier space on an interface.
- From now on, we focus on the **one interface** case.
- M_h is generated from a **parametric multiplier** space $\widehat{M}_h$.

  We consider the following choices for this latter one:
  - choice 1: $\widehat{M}_h \subseteq S^{p_s}(\widehat{\Gamma})$,
  - choice 2: $\widehat{M}_h \subseteq S^{p_s-1}(\widehat{\Gamma})$,
  - choice 3: $\widehat{M}_h \subseteq S^{p_s-2}(\widehat{\Gamma})$.

Choice 1

- $\widehat{M}_h \subseteq S^{ps}(\widehat{\Gamma})$
- A boundary modification can be necessary.

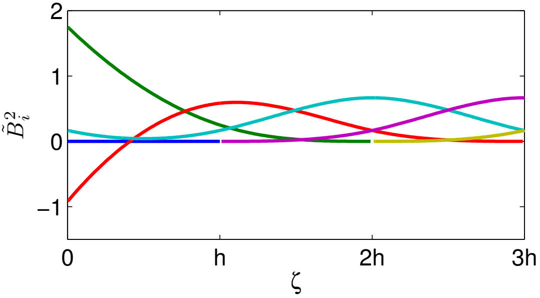
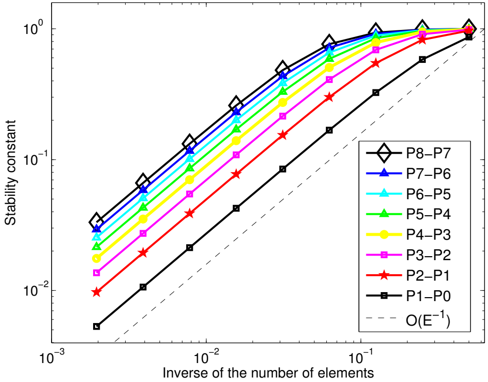


Figure: Left boundary modification of B-Spline functions of degree $p = 3$

$\rightarrow$ **A degree reduction in the boundary elements.**

Choice 2

- $\widehat{M}_h \subseteq S^{p_s-1}(\widehat{\Gamma})$
- Chapelle-Bathe[1] tests to estimate the stability constant



→ **Unstable pairings**

[1] The inf-sup test. D. Chapelle, K. J. Bathe. Computers & Structures, Vol. 47 (1993), No. 4/5, pp. 537-545

Choice 3

- $\widehat{M}_h \subseteq S^{p_S-2}(\widehat{\Gamma})$

Proof for the $p_S/p_S - 2$ pairing in 2D

$$\inf_{\widehat{\phi}\in S^{p_S-2}} \sup_{\widehat{u}\in S_0^{p_S}} \frac{\int_{\widehat{\Gamma}} \widehat{\phi}\,\widehat{u}\,\mathrm{d}\widehat{s}}{\|\widehat{u}\|_{L^2}\|\widehat{\phi}\|_{L^2}} \geq \widehat{\alpha}$$

- Regularity of the mapping
- Quasi-uniformity of the meshes
- Choose $\widehat{u} \in S_0^{p_S}$ such that $\partial_{xx}^2 \widehat{u} = \widehat{\phi}$
- $D : S_0^{p_S} \to S^{p_S-1} \setminus \mathbb{R}$ and $D : S^{p_S-1} \setminus \mathbb{R} \to S^{p_S-2}$ are bijections
- Work with Sobolev norms

$$\inf_{\widehat{\phi}\in S^{p_S-2}} \sup_{\widehat{u}\in S_0^{p_S}} \frac{\int_{\widehat{\Gamma}} \widehat{\phi}\,\widehat{u}\;\mathrm{d}\widehat{s}}{\|\widehat{u}\|_{L^2}\|\widehat{\phi}\|_{L^2}} \geq \widehat{\alpha}$$

$\Downarrow$ From parametric to physical space

$$\inf_{\phi\in M_h} \sup_{u\in W_h} \frac{\int_{\Gamma} \phi\, u\;\mathrm{d}s}{\|u\|_{L^2}\|\phi\|_{L^2}} \geq \alpha$$

- $W_h = \{u_{|_\Gamma}, \quad u \in V_{S,h}\} \cap H_0^1(\Gamma)$
- Variable change: $\int_\Gamma \phi\, u\;\mathrm{d}s = \int_{\widehat{\Gamma}} \widehat{\phi}\,\widehat{u}\,\rho\;\mathrm{d}\widehat{s}$,
- Super-convergence results[a]

$$\Pi : L^2(\widehat{\Gamma}) \to \widehat{M}_h \qquad \|\widehat{\phi}\rho - \Pi(\widehat{\phi}\rho)\|_{L^2(\hat{\Gamma})} \leq Ch\|\widehat{\phi}\|_{L^2(\hat{\Gamma})}$$

[a] Superconvergence in Galerkin Finite Element Methods. L. Wahlbin. Lecture Notes in Mathematics (1995), Vol. 1605, Springer, Berlin

$$\inf_{\widehat{\phi}\in S^{p_S-2}} \sup_{\widehat{u}\in S_0^{p_S}} \frac{\int_{\widehat{\Gamma}} \widehat{\phi}\, \widehat{u}\ \mathrm{d}\widehat{s}}{\|\widehat{u}\|_{L^2}\|\widehat{\phi}\|_{L^2}} \geq \widehat{\alpha}$$

From parametric to physical space

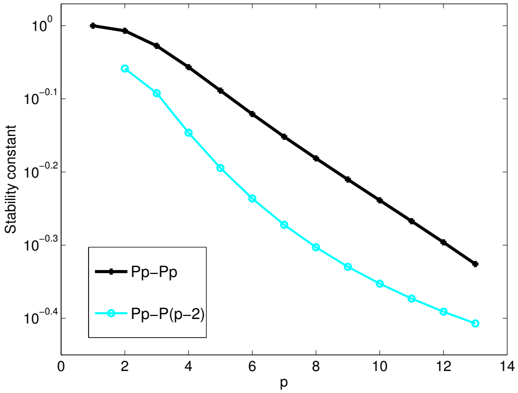
$$\inf_{\phi\in M_h} \sup_{u\in W_h} \frac{\int_{\Gamma} \phi\, u\ \mathrm{d}s}{\|u\|_{L^2}\|\phi\|_{L^2}} \geq \alpha$$

- $W_h = \{u_{|_\Gamma}, \quad u \in V_{S,h}\} \cap H_0^1(\Gamma)$
- Variable change: $\int_\Gamma \phi\, u\ \mathrm{d}s = \int_{\widehat{\Gamma}} \widehat{\phi}\, \widehat{u}\, \rho\ \mathrm{d}\widehat{s}$,
- Super-convergence results[a]

For h small enough, the stability holds !

[a] Superconvergence in Galerkin Finite Element Methods. L. Wahlbin. Lecture Notes in Mathematics (1995), Vol. 1605, Springer, Berlin

Dependence of the stability constant on the degree



$\rightarrow$ **Exponential dependence in p**

For $V_{S,h}$ a push-forward of a NURBS space of degree p_S, and :

- M_h a push-forward of a spline space of degree p_S
  $\Longrightarrow$ problems at cross points.
- M_h a push-forward of a spline space of degree $p_S - 1$
  $\Longrightarrow$ stability is NOK.
- M_h a push-forward of a spline space of degree $p_S - 2$
  $\Longrightarrow$ stability is OK.
  And more generally, for a push-forward of a spline space of degree $p_S - 2k\,(k \in \mathbb{N}$ and $k > 1)$.

	Pairing $p_S - p_S$	Pairing $p_S - (p_S - 2)$
$\|\|u - u_h\|\|_{L^2(\Omega)}$	$p_S + 1$	$p_S + \frac{1}{2}$
$\|\|\lambda - \lambda_h\|\|_{L^2(\Gamma)}$	$p_S - \frac{1}{2}$	$p_S - 1$

Table: Optimal order of convergence of isogeometric mortar methods

Where we are

1. IGA basics
2. Isogeometric mortar methods
   - The discrete mortar problem
   - Lagrange multiplier spaces
   - A key issue: the evaluation of the mortar integrals
3. Applications in contact mechanics
   - Contact problem definition
   - Example of treatment of deformable-deformable contact in a Lagrange multiplier formulation

- We recall the bilinear form

$$b(u,\phi) = \int_{\Gamma_S} \phi \cdot u^S|_{\Gamma_S} \,\mathrm{d}s - \int_{\Gamma_S} \phi \cdot u^M|_{\Gamma_M} \,\mathrm{d}s.$$

- We recall the bilinear form

$$b(u,\phi) = \int_{\Gamma_S} \phi \cdot u^- \quad \mathrm{d}s - \int_{\Gamma_S} \phi \cdot u^+ \quad \mathrm{d}s.$$

$\rightarrow$ A challenge: the evaluation of the second integral due to the **product of functions** which are defined on **different meshes**.

- We recall the bilinear form

$$b(u,\phi) = \int_{\Gamma_S} \phi \cdot u^- \quad \mathrm{d}s - \int_{\Gamma_S} \phi \cdot u^+ \quad \mathrm{d}s.$$

  $\rightarrow$ A challenge: the evaluation of the second integral due to the **product of functions** which are defined on **different meshes**.

- Finite element litterature
  - **Dedicated quadrature rules** have been studied.[a,b]
  - Approximating these integrals taking the quadrature of one side is inducing large (consistency or approximation) errors.

[a] Numerical quadrature and mortar methods. L. Cazabeau, C. Lacour, Y. Maday. Computational Science for the 21st Century, John Wiley and Sons (1997), pp. 119-128

[b] The influence of quadrature formulas in 2D and 3D mortar element methods. Y. Maday, F. Rapetti, B.I. Wohlmuth. Recent developments in domain decomposition methods. Some papers of the workshop on domain decomposition. ETH Zürich, Switzerland, June 7-8. 2001, pp. 203-221. Springer (2002)

Isogeometric mortar methods

A key issue: the evaluation of the mortar integrals

- **Slave integration approach**
  One could expect the sensitivity with respect to the **slave quadrature** to be less than for finite element methods.

$$\begin{aligned} a(\widetilde{u}_h, v_h) + \textstyle\sum_{-}(v_h^- - v_h^+)\, \widetilde{\lambda}_h &= f(v_h), && v_h \in V_h,\\ \textstyle\sum_{-}(\widetilde{u}_h^- - \widetilde{u}_h^+)\, \phi_h &= 0, && \phi_h \in M_h. \end{aligned}$$

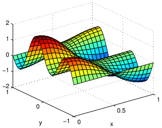
- **Mixed integration approach**
  We also consider another approach which leads to a **non-symmetric saddle point** problem. It is motivated by different requirements for the integration of the primal and multiplier test functions.

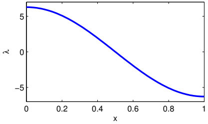
$$\begin{aligned} a(\widetilde{u}_h, v_h) + \textstyle\sum_{-} v_h^- \, \widetilde{\lambda}_h - \textstyle\sum_{+} v_h^+ \, \widetilde{\lambda}_h &= f(v_h), && v_h \in V_h,\\ \textstyle\sum_{-}(\widetilde{u}_h^- - \widetilde{u}_h^+)\, \phi_h &= 0, && \phi_h \in M_h. \end{aligned}$$

Numerical results

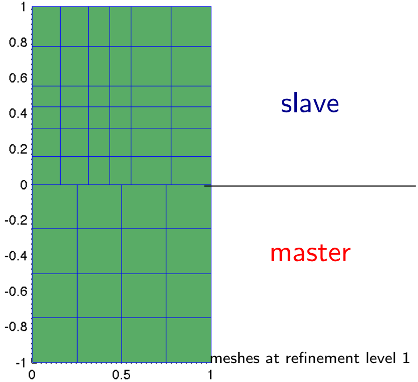
Problem settings

- Standard Poisson problem
$$-\Delta u = f$$
- Analytical solution



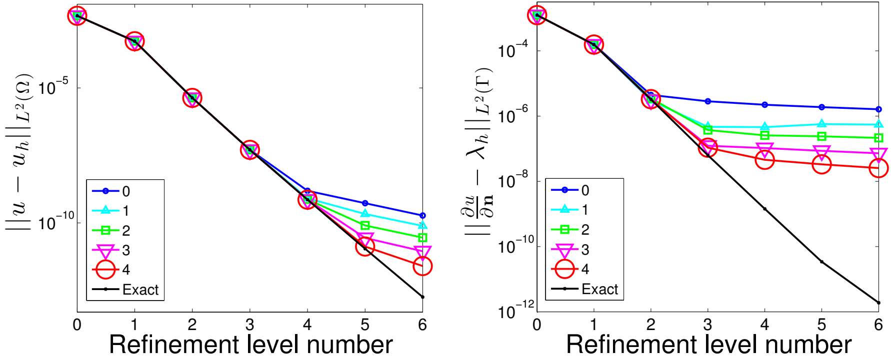


- Boundary conditions imposed such that no need to handle a cross point



Numerical results - Slave integration approach

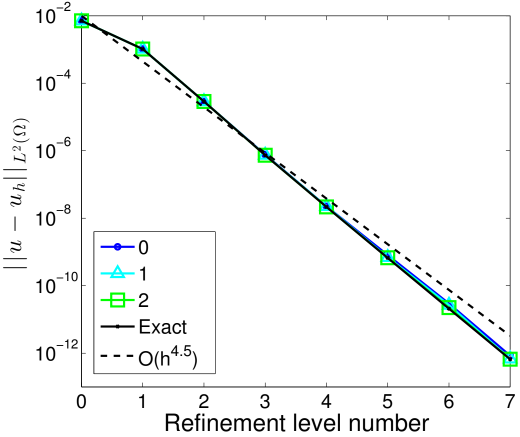
L^2 primal and multiplier errors as a function of the number of additional quadrature points



***P*5 – *P*5**

Numerical results - Mixed integration approach

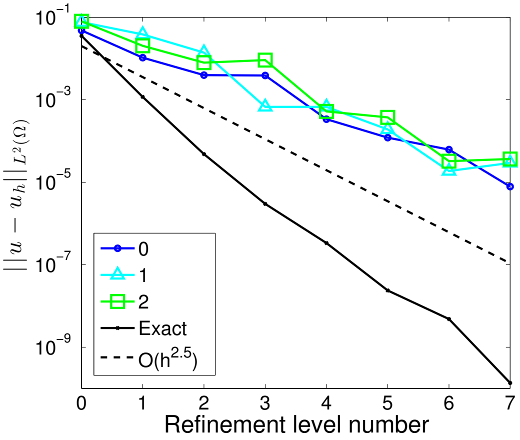
L^2 primal error as a function of the number of additional quadrature points



P4 – P2

Numerical results - Mixed integration approach

L^2 primal error as a function of the number of additional quadrature points



$P4 - P0$

Where we are

1 IGA basics

2 Isogeometric mortar methods

3 Applications in contact mechanics

Applications in contact mechanics

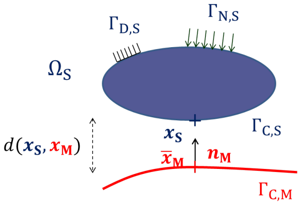
Contact problem definition - Strong formulation

- Assumption: **frictionless** contact
  $\rightarrow$ the **contact variables** are thus the gap g_N and the contact pressure σ_N **defined** as:

$$g_N = (\boldsymbol{x_S} - \overline{\boldsymbol{x}}_{\boldsymbol{M}}) \cdot \boldsymbol{n_M} = [\boldsymbol{x}], \qquad \sigma_N = -(\underline{\underline{\sigma_S}} \cdot \boldsymbol{n_S}) \cdot \boldsymbol{n_M}.$$

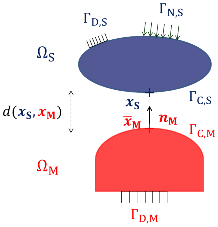
- 2 types

Rigid-Deformable contact



$$g_N = \boldsymbol{u_S} \cdot \boldsymbol{n_M} + (\boldsymbol{X_S} - \overline{\boldsymbol{X}}_{\boldsymbol{M}}) \cdot \boldsymbol{n_M}$$

Deformable-Deformable contact



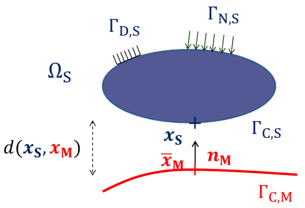
$$g_N = (\boldsymbol{u_S} - \overline{\boldsymbol{u}}_{\boldsymbol{M}}) \cdot \boldsymbol{n_M} + (\boldsymbol{X_S} - \overline{\boldsymbol{X}}_{\boldsymbol{M}}) \cdot \boldsymbol{n_M}$$

- Assumption: **frictionless** contact
  $\rightarrow$ the **contact variables** are thus the gap g_N and the contact pressure σ_N **defined** as:

$$g_N = (\boldsymbol{x_S} - \overline{\boldsymbol{x}}_{\boldsymbol{M}}) \cdot \boldsymbol{n_M} = [\boldsymbol{x}], \qquad \sigma_N = -(\underline{\underline{\sigma_S}} \cdot \boldsymbol{n_S}) \cdot \boldsymbol{n_M}.$$

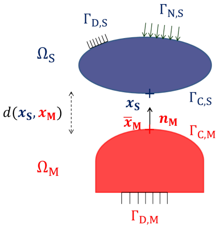
- 2 types

Rigid-Deformable contact



$$g_N = \boldsymbol{u_S} \cdot \boldsymbol{n_M} + g$$

Deformable-Deformable contact



$$g_N = (\boldsymbol{u_S} - \overline{\boldsymbol{u}}_{\boldsymbol{M}}) \cdot \boldsymbol{n_M} + g$$

- Assumption: **frictionless** contact
  $\rightarrow$ the **contact variables** are thus the gap g_N and the contact pressure σ_N **defined** as:

$$g_N = (\boldsymbol{x_S} - \overline{\boldsymbol{x}}_{\boldsymbol{M}}) \cdot \boldsymbol{n_M} = [\boldsymbol{x}], \qquad \sigma_N = -(\underline{\underline{\sigma_S}} \cdot \boldsymbol{n_S}) \cdot \boldsymbol{n_M}.$$

- 2 types

Rigid-Deformable contact **Deformable-Deformable contact**

On $\Omega = \bigcup_k \Omega_k$

with either $k = \{S\}$, or $k = \{S, M\}$,

the mechanical equilibrium with contact conditions is written

$$\begin{cases} -\mathbf{div}(\underline{\underline{\sigma_k}}) & = & \boldsymbol{f_k} & \text{in } \Omega_k, \\ \boldsymbol{u_k} & = & \boldsymbol{u_{D,k}} & \text{on } \Gamma_{D,k}, \\ \underline{\underline{\sigma_k}} \cdot \boldsymbol{n_k} & = & \boldsymbol{l_k} & \text{on } \Gamma_{N,k}, \\ \boldsymbol{g_N \geq 0}, \quad \boldsymbol{\sigma_N} & \leq & \boldsymbol{0}, \quad \boldsymbol{g_N\, \sigma_N = 0} & \text{on } \boldsymbol{\Gamma_{C,S}}. \end{cases}$$

- **Different formulations** to enforce the contact constraints: penalty, Lagrange multiplier, augmented lagrangian, Nitsche's.

- **Different formulations** to enforce the contact constraints: penalty, Lagrange multiplier, augmented lagrangian, Nitsche's.
- In the following, we focus on a **Lagrange multiplier formulation**.

To do so, we introduce a **multiplier space** M defined by

$$M = \{\phi \in H^{-1/2}(\Gamma_{C,s})\},$$

as its subspace

$$M^{-} = \{\phi \in M, \quad \int_{\Gamma_{C,s}} \phi \, [\boldsymbol{u}] \, \mathrm{d}s \leq 0, \quad \forall [\boldsymbol{u}] \in H^{1/2}(\Gamma_{C,s}), \quad [\boldsymbol{u}] \geq 0\}.$$

- **Different formulations** to enforce the contact constraints: penalty, Lagrange multiplier, augmented lagrangian, Nitsche's.
- In the following, we focus on a **Lagrange multiplier formulation**.

We also introduce the following **bilinear and linear forms**:

$$b\colon V \times M \to \mathbb{R}, \qquad b(\boldsymbol{u}, \phi) = \int_{\Gamma_{C,s}} \phi\, [\boldsymbol{u}] \;\mathrm{d}s,$$

$$g\colon M \to \mathbb{R}, \qquad g(\phi) = \int_{\Gamma_{C,s}} \phi\, g \;\mathrm{d}s.$$

- **Different formulations** to enforce the contact constraints: penalty, Lagrange multiplier, augmented lagrangian, Nitsche's.
- In the following, we focus on a **Lagrange multiplier formulation**.

Contact problem - Lagrange multiplier formulation

Find $(\boldsymbol{u}, \lambda) \in V_D \times M^-$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, \lambda) = f(\boldsymbol{v}), & \forall \boldsymbol{v} \in V_0, \\ b(\boldsymbol{u}, \lambda - \phi) \leq -g(\lambda - \phi), & \forall \phi \in M^-. \end{cases}$$

- On each subdomain Ω_k with either $k = \{S\}$ (R-D contact) or $k = \{S, M\}$ (D-D contact), based on its relative NURBS geometrical parametrization, we introduce the **displacement approximation space**

$$V_{k,h} = \{\boldsymbol{v_k} = \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \circ \mathbf{F}_k^{-1}, \quad \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \in (N^{p_k}(\boldsymbol{\Xi}_k))^{d_u}\}.$$

- On each subdomain Ω_k with either $k = \{S\}$ (R-D contact) or $k = \{S, M\}$ (D-D contact), based on its relative NURBS geometrical parametrization, we introduce the **displacement approximation space**

$$V_{k,h} = \{\boldsymbol{v_k} = \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \circ \mathbf{F}_k^{-1}, \quad \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \in (N^{p_k}(\boldsymbol{\Xi}_k))^{d_u}\}.$$

- We note that in the D-D contact case, the discrete product space $V_h = \prod_{k=\{S,M\}} V_{k,h} \subset V$ forms a $(H^1(\Omega_S \cup \Omega_M))^{d_u}$ non-conforming space discontinuous over the potential region of contact $\Gamma_{C,S}$.

- On each subdomain Ω_k with either $k = \{S\}$ (R-D contact) or $k = \{S, M\}$ (D-D contact), based on its relative NURBS geometrical parametrization, we introduce the **displacement approximation space**

$$V_{k,h} = \{\boldsymbol{v_k} = \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \circ \mathbf{F}_k^{-1}, \quad \widehat{\boldsymbol{v}}_{\boldsymbol{k}} \in (N^{p_k}(\boldsymbol{\Xi}_k))^{d_u}\}.$$

- We note that in the D-D contact case, the discrete product space $V_h = \prod_{k=\{S,M\}} V_{k,h} \subset V$ forms a $(H^1(\Omega_S \cup \Omega_M))^{d_u}$ non-conforming space discontinuous over the potential region of contact $\Gamma_{C,S}$.
- Need to complete the definition of the discrete formulation to define **remaining discrete datas**.

- **Discrete multiplier space**

For $V_{S,h}$ a push-forward of a NURBS space of degree p_S and :

- M_h a push-forward of a spline space of degree p_S $\Longrightarrow$ **cross points**
- M_h a push-forward of a spline space of degree $p_S - 2$ $\Longrightarrow$ **stability** OK
  And more generally, for a push-forward of a spline space of degree $p_S - 2k \, (k \in \mathbb{N}$ and $k > 1)$.

- **Discrete gap**
  $\rightarrow$ Use of a lumped L^2-projection $\widetilde{\Pi}$ into M_h defined as

$$\widetilde{\Pi} : \mathbb{R} \rightarrow M_h, \qquad (\widetilde{\Pi} g_N)_j = \frac{\int_{\Gamma_{C,S}} g_N \, B_j^{\phi} \, \mathrm{d}s}{\int_{\Gamma_{C,S}} B_j^{\phi} \, \mathrm{d}s}, \quad \forall j = 1, \dots, n_{M_h}.$$

Where we are

1. IGA basics
2. Isogeometric mortar methods
   - The discrete mortar problem
   - Lagrange multiplier spaces
   - A key issue: the evaluation of the mortar integrals
3. Applications in contact mechanics
   - Contact problem definition
   - Example of treatment of deformable-deformable contact in a Lagrange multiplier formulation

Numerical resolution strategy

The Lagrange multiplier formulation of the contact problem is solved with an **active set strategy** written as:
Find $(\boldsymbol{u_h}, \lambda_h) \in V_{D,h} \times M_h^-$ such that

$$\begin{cases} a(\boldsymbol{u_h}, \boldsymbol{v_h}) + \displaystyle\int_{\mathrm{ACT}} \lambda_h \, [\boldsymbol{v_h}] \;\; \mathrm{d}s = f(\boldsymbol{v_h}), & \forall \boldsymbol{v_h} \text{ in } V_{0,h}, \\ \displaystyle\int_{\mathrm{ACT}} \phi_h \, [\boldsymbol{u_h}] \;\; \mathrm{d}s = -\int_{\mathrm{ACT}} \phi_h \, g \;\; \mathrm{d}s, & \forall \phi_h \text{ in } M_h^-. \end{cases}$$

Numerical resolution strategy

The Lagrange multiplier formulation of the contact problem is solved with an **active set strategy** written as:
Find $(\boldsymbol{u_h}, \lambda_h) \in V_{D,h} \times M_h^-$ such that

$$\begin{cases} a(\boldsymbol{u_h}, \boldsymbol{v_h}) + \displaystyle\int_{\mathrm{ACT}} \lambda_h \, [\boldsymbol{v_h}] \;\mathrm{d}s = f(\boldsymbol{v_h}), & \forall \boldsymbol{v_h} \text{ in } V_{0,h}, \\ \displaystyle\int_{\mathrm{ACT}} \phi_h \, [\boldsymbol{u_h}] \;\mathrm{d}s = -\displaystyle\int_{\mathrm{ACT}} \phi_h \, g \;\mathrm{d}s, & \forall \phi_h \text{ in } M_h^-. \end{cases}$$

Need to focus on

- the **discrete active set region** definition,
- **suitable integration** methods to approximate the **mixed term** contributions.

- Giving a **discrete definition of the dual cone** M_h^-, e.g.,

$$M_{h,E}^- = \{\phi_h \in M_h, \quad \phi_h \leq 0\},$$
$$M_{h,W}^- = \{\phi_h \in M_h, \quad \int_{\Gamma_{C,S}} \phi_h \, B_j \, \mathrm{ds} \leq 0, \quad \forall j = 1, \ldots, n_{M_h}\},$$
$$M_{h,P}^- = \{\phi_h = \sum_k \alpha_k B_k, \quad \alpha_k \leq 0, \quad \forall k = 1, \ldots, n_{M_h}\},$$

$\rightarrow$ consists in giving a discrete definition of the active vs inactive contact region.

Applications in contact mechanics

Example of treatment of deformable-deformable contact in a Lagrange multiplier formulation

- Giving a **discrete definition of the dual cone** M_h^-, e.g.,

$$M_{h,E}^- = \{\phi_h \in M_h, \quad \phi_h \leq 0\},$$

$$M_{h,W}^- = \{\phi_h \in M_h, \quad \int_{\Gamma_{C,S}} \phi_h \, B_j \, \mathrm{ds} \leq 0, \quad \forall j = 1, ..., n_{M_h}\},$$

$$\boldsymbol{M_{h,P}^- = \{\phi_h = \sum_k \alpha_k \, B_k, \quad \alpha_k \leq 0, \quad \forall k = 1, ..., n_{M_h}\}},$$

$\rightarrow$ consists in giving a discrete definition of the active vs inactive contact region.

According to the R-D contact case study, we choose to define it as **the support of active multiplier functions**. This set of functions is denoted $CP_\lambda^{\mathrm{ACT}}$ while the set of inactive ones $CP_\lambda^{\mathrm{INA}}$.

Active set strategy

(i) Initialise $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$

(ii) Compute

$$\begin{bmatrix} K & B_{(\cdot, CP_\lambda^{\mathrm{ACT}})} \\ B^*_{(CP_\lambda^{\mathrm{ACT}}, \cdot)} & 0_{(CP_\lambda^{\mathrm{ACT}}, CP_\lambda^{\mathrm{ACT}})} \end{bmatrix} \begin{bmatrix} U \\ \Lambda_{(CP_\lambda^{\mathrm{ACT}})} \end{bmatrix} = \begin{bmatrix} F \\ G_{(CP_\lambda^{\mathrm{ACT}})} \end{bmatrix}$$

(iii) Check convergence, i.e.,: $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$ stable

(iv) Update $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$ and go to (ii) until convergence is reached

The constraints are checked on the multiplier control values λ_j and the discrete gap control values $(\widetilde{\Pi} g_N)_j$. The 9 possible cases are:

$\lambda_j < 0$,			$\lambda_j > 0$,			$\lambda_j = 0$,		
$(\widetilde{\Pi} g_N)_j = 0$	**Actif**	(a)	$(\widetilde{\Pi} g_N)_j = 0$	**Inactif**	(b)	$(\widetilde{\Pi} g_N)_j = 0$	**Inactif**	(c)
$(\widetilde{\Pi} g_N)_j > 0$	**Actif**	(d)	$(\widetilde{\Pi} g_N)_j > 0$	**Inactif**	(e)	$(\widetilde{\Pi} g_N)_j > 0$	**Inactif**	(f)
$(\widetilde{\Pi} g_N)_j < 0$	**Actif**	(g)	$(\widetilde{\Pi} g_N)_j < 0$	**Actif**	(h)	$(\widetilde{\Pi} g_N)_j < 0$	**Actif**	(i)

Applications in contact mechanics

Example of treatment of deformable-deformable contact in a Lagrange multiplier formulation

Active set strategy

(i) Initialise $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$

(ii) Compute

$$\begin{bmatrix} K & B_{(\cdot, CP_\lambda^{\mathrm{ACT}})} \\ B^*_{(CP_\lambda^{\mathrm{ACT}}, \cdot)} & 0_{(CP_\lambda^{\mathrm{ACT}}, CP_\lambda^{\mathrm{ACT}})} \end{bmatrix} \begin{bmatrix} U \\ \Lambda_{(CP_\lambda^{\mathrm{ACT}})} \end{bmatrix} = \begin{bmatrix} F \\ G_{(CP_\lambda^{\mathrm{ACT}})} \end{bmatrix}$$

(iii) Check convergence, i.e.,: $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$ stable

(iv) Update $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$ and go to (ii) until convergence is reached

$$B = \begin{bmatrix} B^S \\ B^M \end{bmatrix} \qquad B^* = \begin{bmatrix} B^{*S} \\ B^{*M} \end{bmatrix}$$

$$\begin{aligned}
B_{ij}^S &= \int_{\mathrm{ACT}} B_j^\lambda(\boldsymbol{x}^S)\, \boldsymbol{n}_M(\bar{\boldsymbol{x}}^M) \cdot \boldsymbol{B}_i^S(\boldsymbol{x}^S) \;\mathrm{d}s, \\
B_{ij}^{*S} &= \int_{\mathrm{ACT}} B_i^\lambda(\boldsymbol{x}^S)\, \boldsymbol{n}_M(\bar{\boldsymbol{x}}^M) \cdot \boldsymbol{B}_j^S(\boldsymbol{x}^S) \;\mathrm{d}s, \\
B_{ij}^M &= -\int_{\mathrm{ACT}} B_j^\lambda(\boldsymbol{x}^S)\, \boldsymbol{n}_M(\bar{\boldsymbol{x}}^M) \cdot \boldsymbol{B}_i^M(\bar{\boldsymbol{x}}^M) \,\mathrm{d}s, \\
B_{ij}^{*M} &= -\int_{\mathrm{ACT}} B_i^\lambda(\boldsymbol{x}^S)\, \boldsymbol{n}_M(\bar{\boldsymbol{x}}^M) \cdot \boldsymbol{B}_j^M(\bar{\boldsymbol{x}}^M) \,\mathrm{d}s.
\end{aligned}$$

Applications in contact mechanics

Example of treatment of deformable-deformable contact in a Lagrange multiplier formulation

Active set strategy

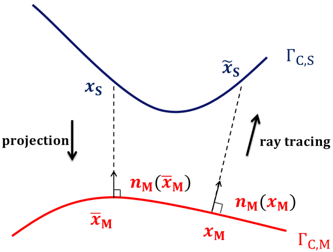
(i) Initialise $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$

(ii) Compute

$$\begin{bmatrix} K & B_{(\cdot, CP_\lambda^{\mathrm{ACT}})} \\ B^*_{(CP_\lambda^{\mathrm{ACT}}, \cdot)} & 0_{(CP_\lambda^{\mathrm{ACT}}, CP_\lambda^{\mathrm{ACT}})} \end{bmatrix} \begin{bmatrix} U \\ \Lambda_{(CP_\lambda^{\mathrm{ACT}})} \end{bmatrix} = \begin{bmatrix} F \\ G_{(CP_\lambda^{\mathrm{ACT}})} \end{bmatrix}$$

(iii) Check convergence, i.e.,: $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$ stable

(iv) Update $CP_\lambda^{\mathrm{ACT}}$ and $CP_\lambda^{\mathrm{INA}}$ and go to (ii) until convergence is reached



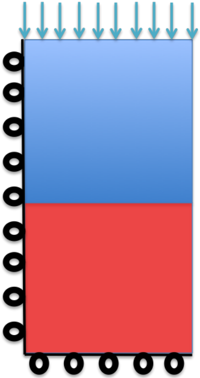
$$B_{ij}^S = \sum_S{}^{\mathrm{ACT}} B_j^\lambda(x^S)\ n_M(\bar{x}^M) \cdot B_i^S(x^S),$$

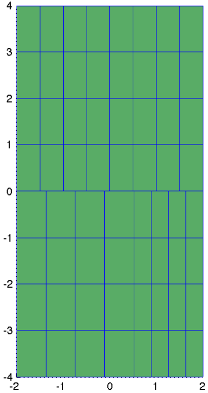
$$B_{ij}^{*S} = \sum_S{}^{\mathrm{ACT}} B_i^\lambda(x^S)\ n_M(\bar{x}^M) \cdot B_j^S(x^S),$$

$$B_{ij}^M = -\sum_M{}^{\mathrm{ACT}} B_j^\lambda(\tilde{x}^S)\ n_M(x^M) \cdot B_i^M(x^M),$$

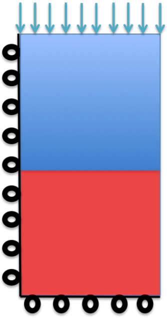
$$B_{ij}^{*M} = -\sum_S{}^{\mathrm{ACT}} B_i^\lambda(x^S)\ n_M(\bar{x}^M) \cdot B_j^M(\bar{x}^M).$$

Numerical results - Test 1 - Transmission of a constant pressure

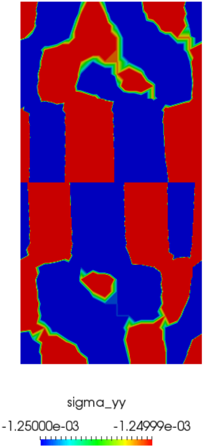




Numerical results - Test 1 - Transmission of a constant pressure



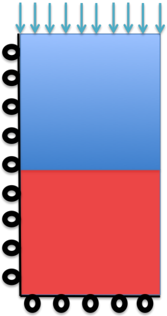
$P2 - P2$,
non-conforming meshes,
4×2 elements for each subdomain



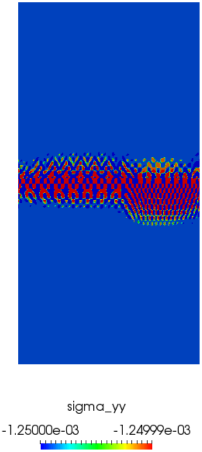
Applications in contact mechanics

Example of treatment of deformable-deformable contact in a Lagrange multiplier formulation

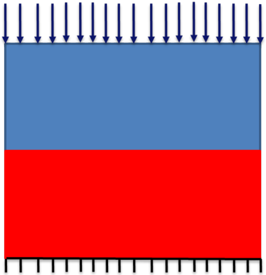
Numerical results - Test 1 - Transmission of a constant pressure

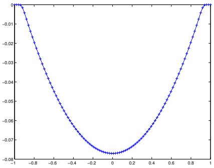


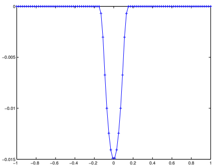
$P2 - P2$,
non-conforming meshes,
64×32 elements for each subdomain



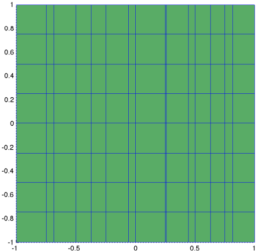
Numerical results - Test 2



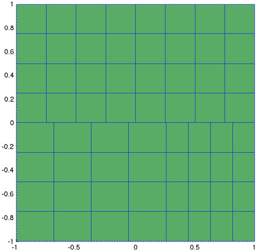




Full contact vs Partial contact



conforming meshes

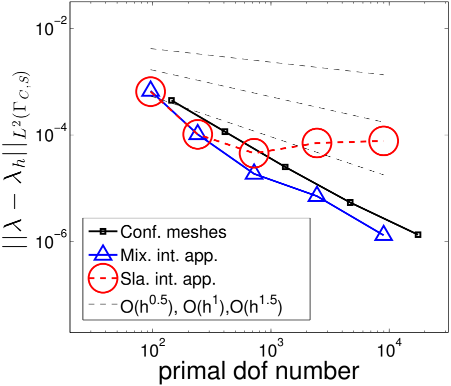


non-conforming meshes

at refinement level 2

Numerical results - Test 2 - Full contact

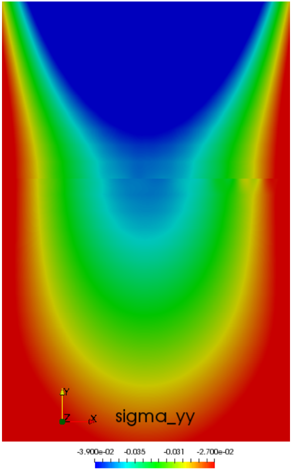
L^2 multiplier errors

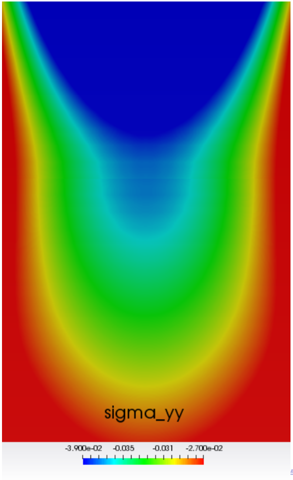


***P*2 − *P*2**

Ref: $P4 - P4$, 511×128 elements for each subdomain

Numerical results - Test 2 - Full contact



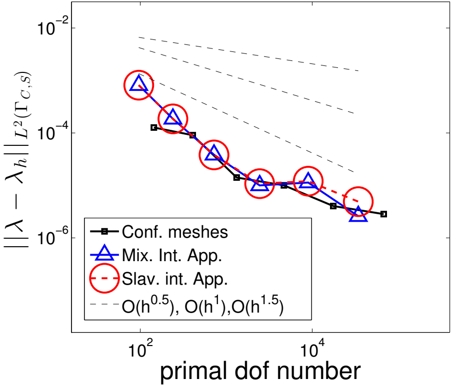


$\boldsymbol{P2} - \boldsymbol{P2}$

Ref. level nb. 3

Numerical results - Test 2 - Partial contact

L^2 multiplier errors



P2 − P2

Ref: $P4 - P4$, 511×128 elements for each subdomain

Where we are

1. IGA basics
2. Isogeometric mortar methods
   - The discrete mortar problem
   - Lagrange multiplier spaces
   - A key issue: the evaluation of the mortar integrals
3. Applications in contact mechanics
   - Contact problem definition
   - Example of treatment of deformable-deformable contact in a Lagrange multiplier formulation

Conclusion & perspectives

- Theoretical and numerical study of **isogeometric mortar** methods.
- Study of **alternative integration methods** to alleviate the construction of the merged mesh.
- From these former results, proposition of **contact methods** for rigid-deformable and deformable-deformable contact.
- Additionally, work on a **segmentation process** for IGA mortar applications. It has been thought to be suitable for mortar-like contact methods.
- **Perspectives**: generalisation of the proposed segmentation process, extension of the methods to the large deformation cases, consideration of contact problems with friction.

Conclusion & perspectives

- Isogeometric mortar methods. E. Brivadis, A. Buffa, B. Wohlmuth and L. Wunderlich. Comput. Methods Appl. Mech. Eng. 284 (2015), pp. 292-319.
- The Influence of Quadrature Errors on Isogeometric Mortar Methods. E. Brivadis, A. Buffa, B. Wohlmuth and L. Wunderlich. Isogeometric Analysis and Applications 2014. Ed. by B. Jttler and B. Simeon. Vol. 107. 2015, pp. 33-50.
- Isogeometric contact mechanics applications. E. Brivadis, A. Buffa. In preparation.

Thanks for your attention