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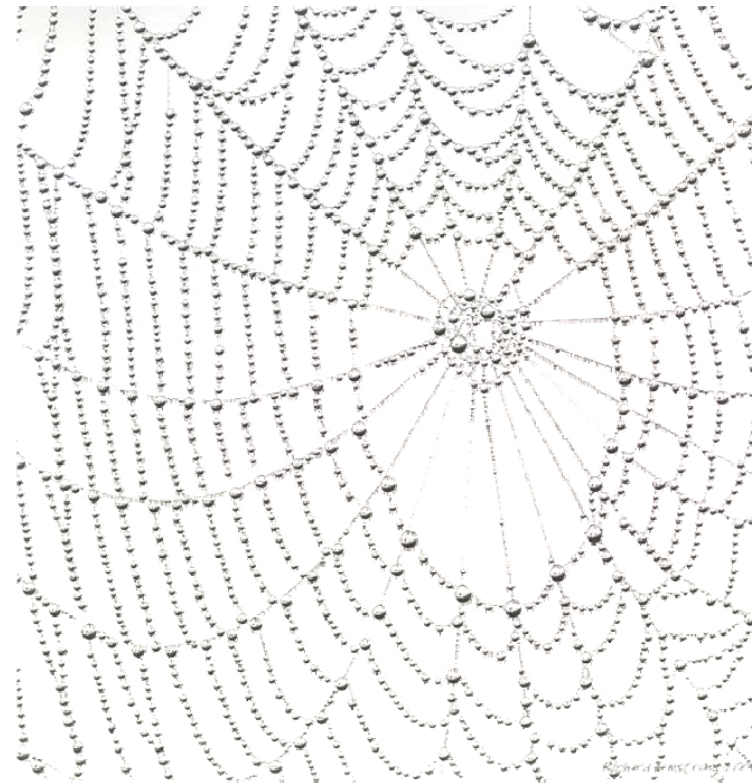
Fakultät Bau und-
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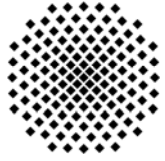


On the Sensitivity of Finite Elements to Mesh Distortions

Manfred Bischoff, Benjamin Schneider
Deepak Ganapayya, Nagesh Chilakunda

Institute of Structural Mechanics
University of Stuttgart
bischoff@ibb.uni-stuttgart.de





motivation

- locking phenomena in structural and solid finite elements
- “optimal” technologies available for rectangular element shapes
- different technologies are identical for rectangular element shapes

⇒ **sensitivity to mesh distortions is one of the last open problems
(further challenges in non-linear analysis and 3d-shells)**



How to Deal with this Problem

1. avoid mesh distortions

complex geometries
adaptive (re-) meshing



How to Deal with this Problem

1. avoid mesh distortions

complex geometries
adaptive (re-) meshing



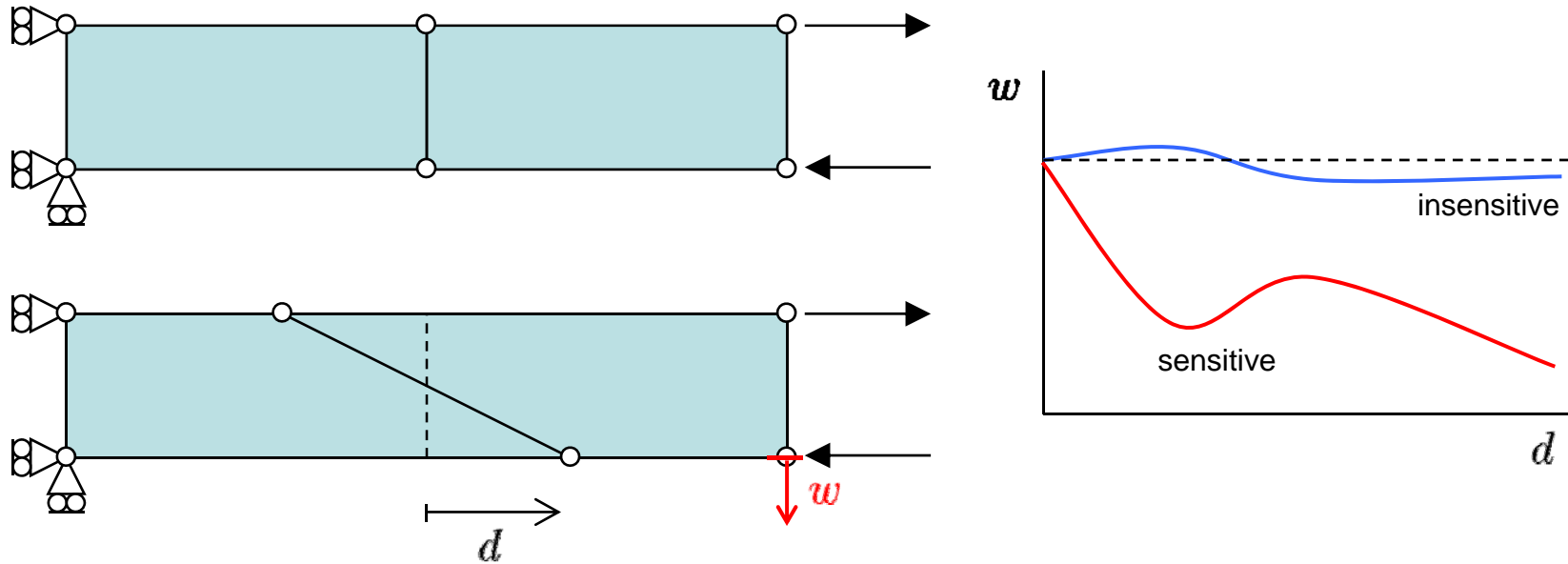
2. develop finite elements which are insensitive to mesh distortions

two-dimensional elasticity
four-node elements



Measuring Distortion Sensitivity

the most popular test: bending of a cantilever



various versions in the literature



Motivation



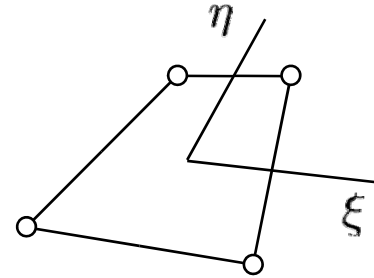
Element Types under Investigation

four-node plane stress elements

Q1

standard Galerkin formulation

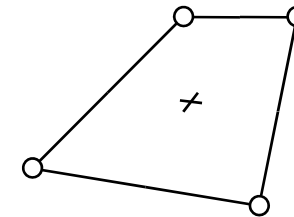
suffers from shear locking and volumetric locking



Q1-SRI

selective reduced integration of shear part

no shear locking, o.k. for small Poisson's ratio

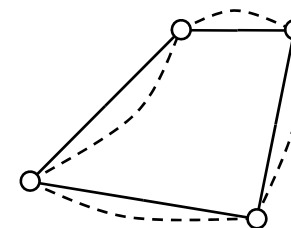


Qm6

method of incompatible modes (= Q1-E4)

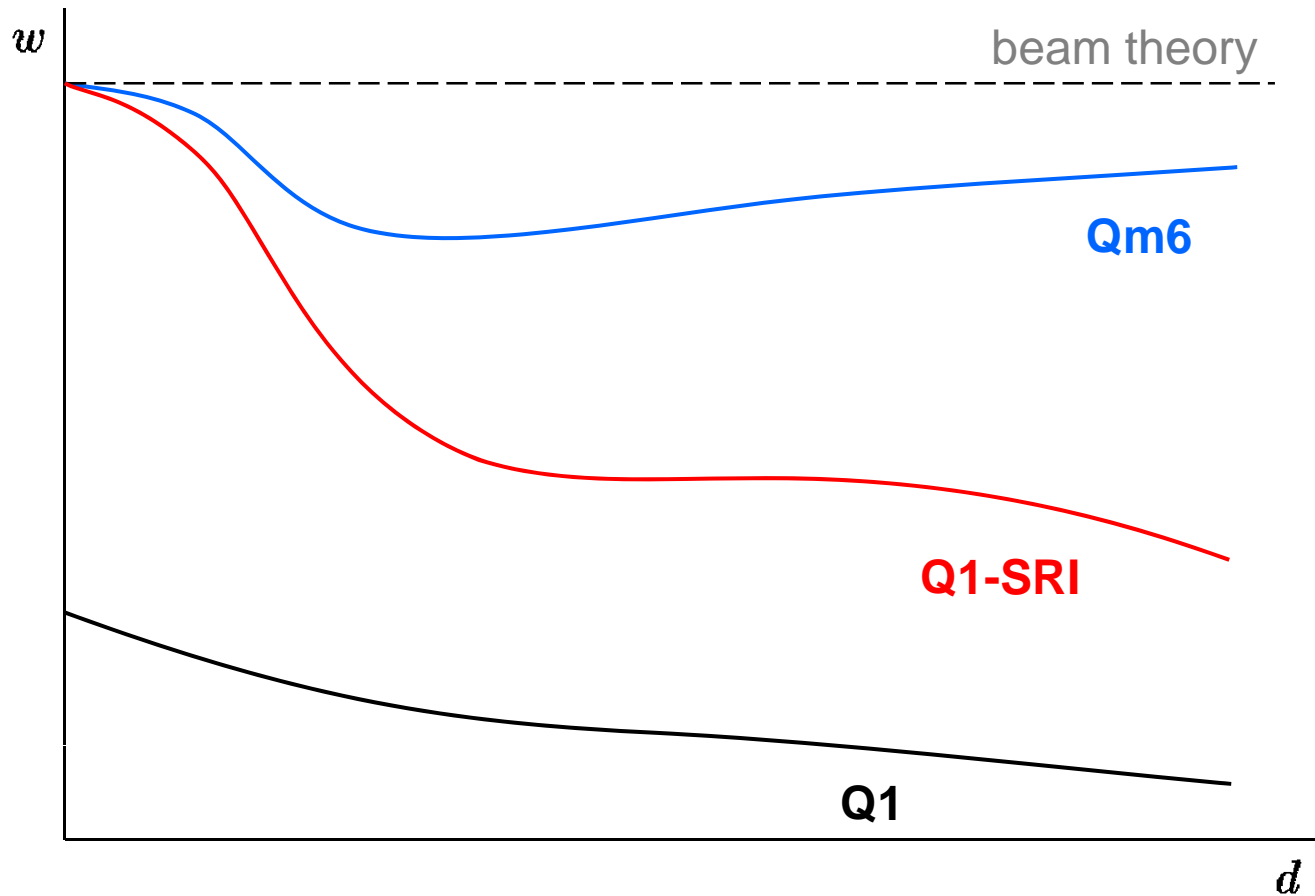
locking-free (?)

Taylor, Wilson (1973, 1976), Simo et al. (1990)



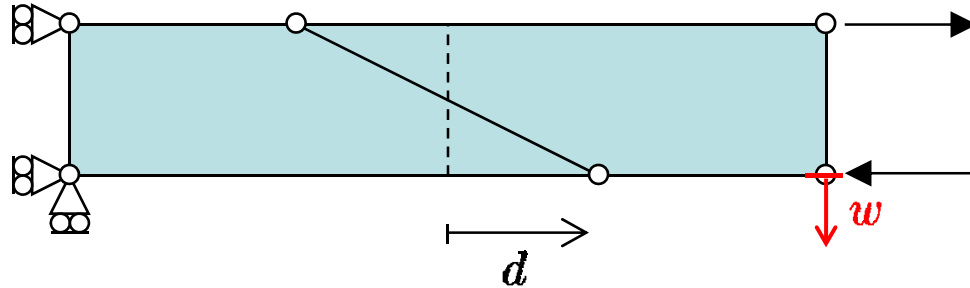
Two Element Bending Test

typical results

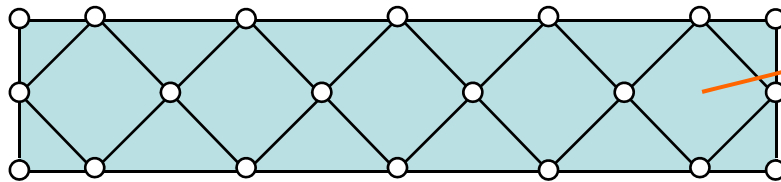


Two Element Bending Test

how meaningful is this test setup?

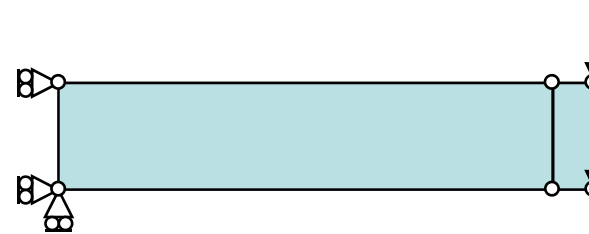
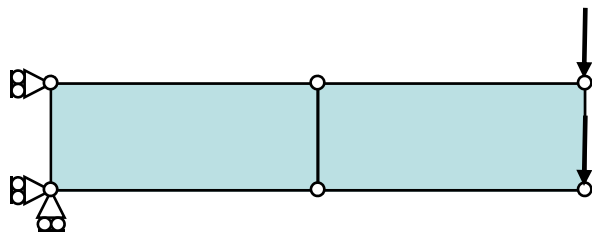


is this a distorted mesh?



stresses in quadrilaterals are identically zero!

undistorted elements – different results

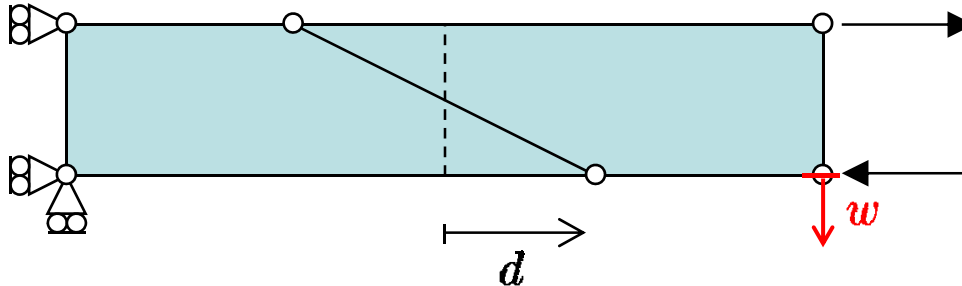


Measuring Distortion Sensitivity



Two Element Bending Test

how meaningful is this test setup?



numerical experiment is restricted to

- specific loads and boundary conditions
- constant-linear stress distribution
- principal stresses aligned to edges
- “exact” solution = beam solution

may be a **hint** toward optimization of element technology
but not a **guideline**



Element Quality \leftrightarrow Tendency to Locking

eigenvalues as “objective” measure of element quality

$$\mathbf{K} \cdot \mathbf{D} = \mathbf{F}$$

$$\mathbf{K} \cdot \mathbf{D}_i = \lambda_i \cdot \mathbf{D}_i \quad (\text{no sum on } i)$$

$$\mathbf{K} \cdot \mathbf{D}_i - \lambda \cdot \mathbf{D}_i = (\mathbf{K} - \lambda_i \cdot \mathbf{I}) \cdot \mathbf{D}_i = \mathbf{0}$$

eigenvalue λ = stiffness

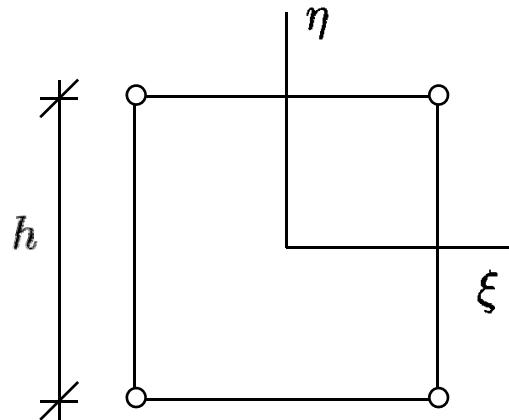
eigenvector \mathbf{D} = deformation mode

- no need to choose specific loads and boundary conditions
- eigenvalue spectrum = element deformation spectrum



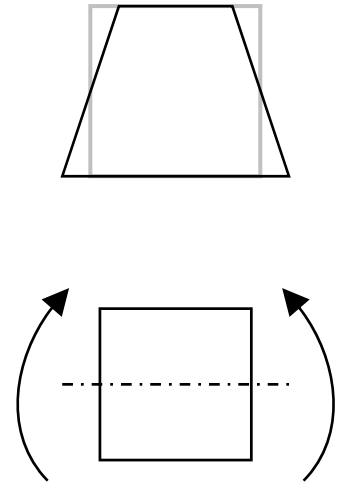
Eigenvalue Analysis

shear locking



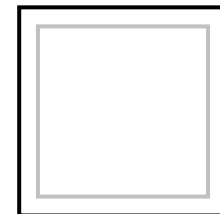
trapezoidal mode $D_i = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}$

optimal: $\lambda_i = c \cdot h^3$



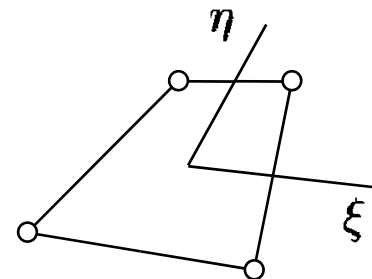
volumetric locking

optimal: only **one** eigenvalue with $\lim_{\nu \rightarrow 0.5} \lambda_i \rightarrow \infty$



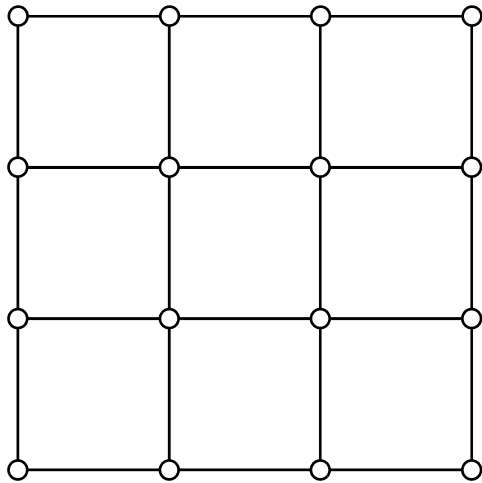
distortion sensitivity

what are the “correct” values of $\lambda_1 \dots \lambda_8$ for arbitrary element shapes?



Considering a Patch of Finite Elements

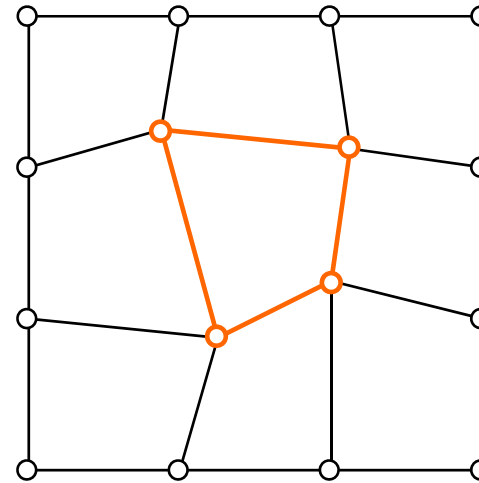
9 elements, 16 nodes, 32 d.o.f. \Rightarrow spectrum of 32 eigenvalues



undistorted, “optimal” mesh
provides reference solution

$$\lambda_1 \dots \lambda_{32}$$

$$D_1 \dots D_{32}$$



distorted mesh
perturbed eigenvalue spectrum

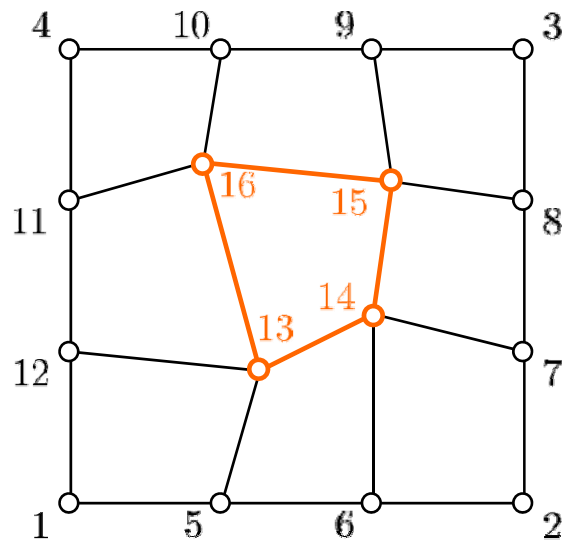
$$\tilde{\lambda}_1 \dots \tilde{\lambda}_{32}$$

$$\tilde{D}_1 \dots \tilde{D}_{32}$$

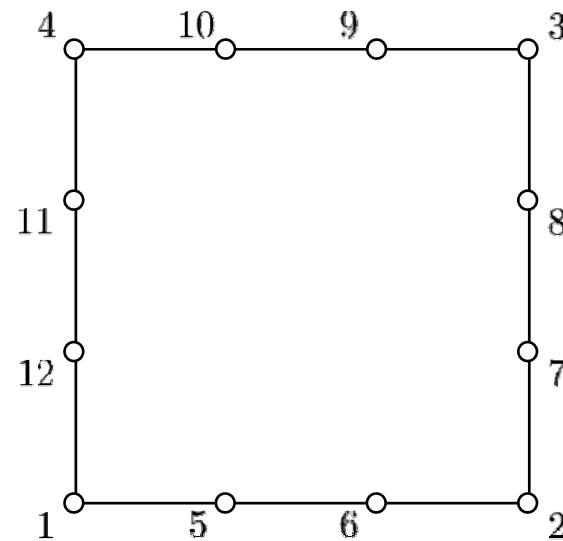
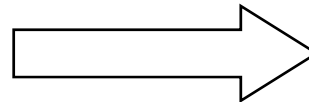


Distortion Patch Test

removing internal d.o.f. via static condensation



static condensation of
node 13 – 16



$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{(1..24),(1..24)} & \mathbf{K}_{(1..24),(25..32)} \\ \mathbf{K}_{(25..32),(1..24)} & \mathbf{K}_{(25..32),(25..32)} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} \\ \mathbf{K}_{21} & \mathbf{K}_{22} \end{bmatrix}$$

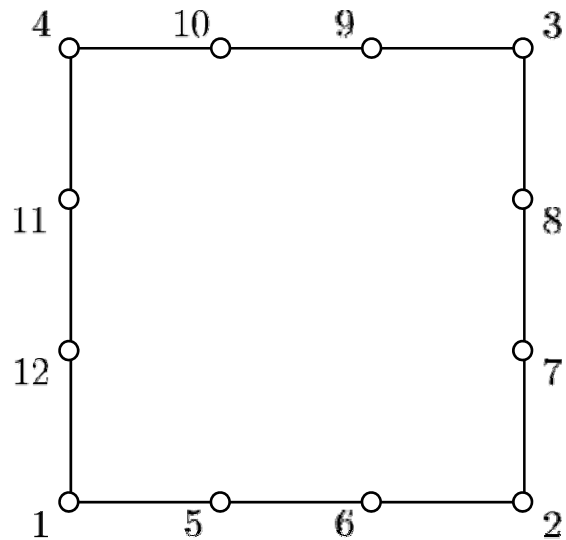
macro element with

$$\mathbf{K}_{red} = \mathbf{K}_{11} - \mathbf{K}_{12} \cdot \mathbf{K}_{22}^{-1} \cdot \mathbf{K}_{21}$$



Distortion Patch Test

properties of macro element

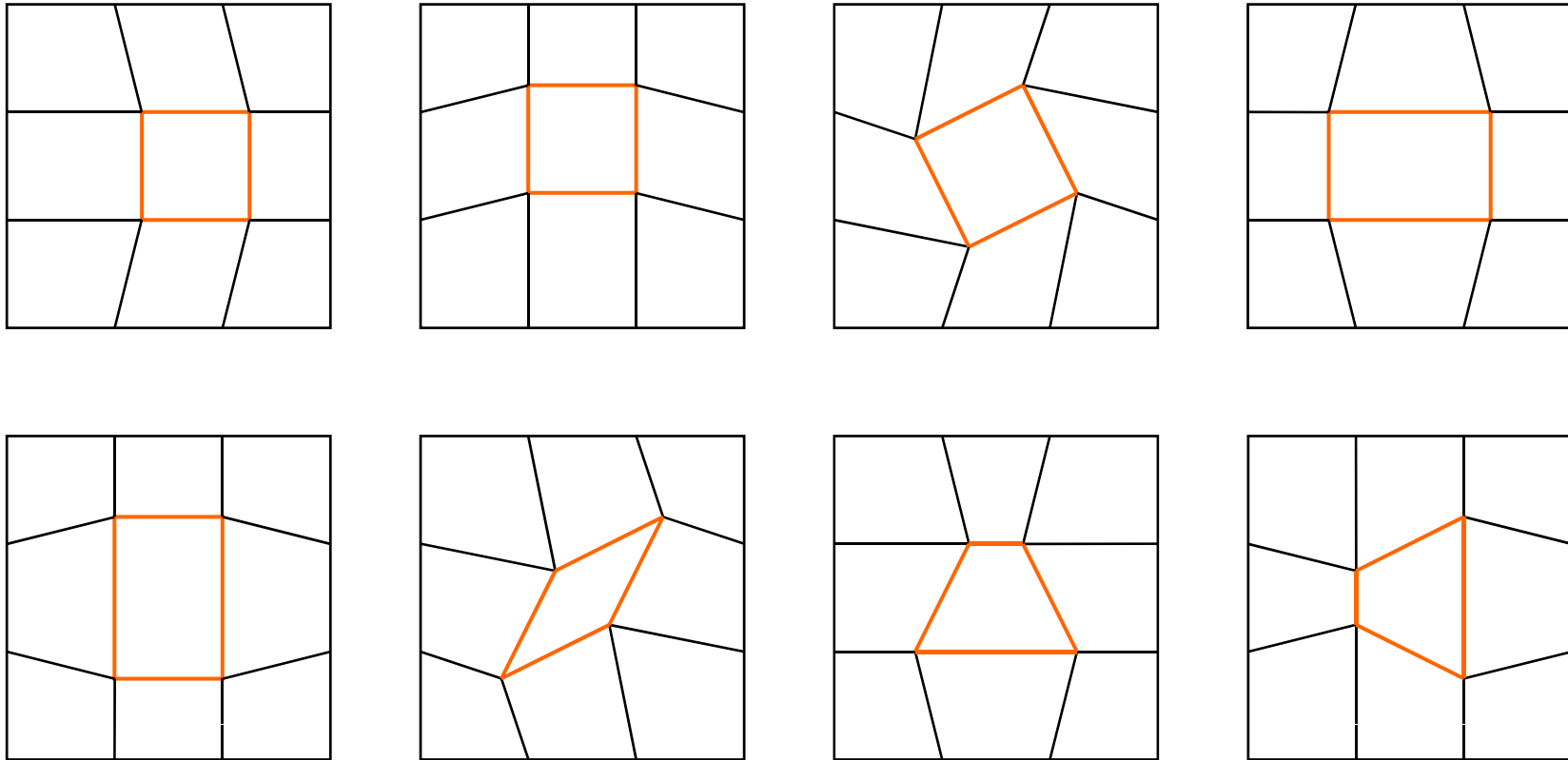


- 24 eigenpairs $\lambda_1 \dots \lambda_{24}, \mathbf{D}_1 \dots \mathbf{D}_{24}$
- $\tilde{\lambda}_1 \dots \tilde{\lambda}_{24}, \tilde{\mathbf{D}}_1 \dots \tilde{\mathbf{D}}_{24}$ depend on locations of “invisible” nodes 13 – 16
- comparison of $\tilde{\lambda}_i$ and λ_i yields objective measure for distortion sensitivity



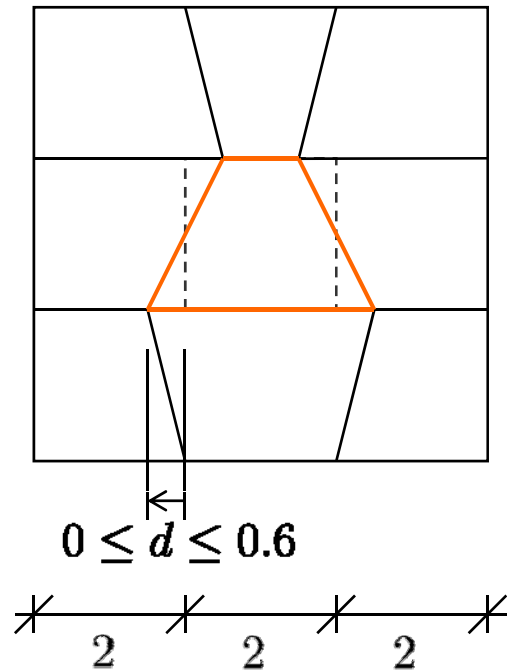
Element Distortion

8 generic distortion modes = 8 single element modes



Application of Distortion Patch Test

numerical experiments with Q1, Q1-SRI and Qm6

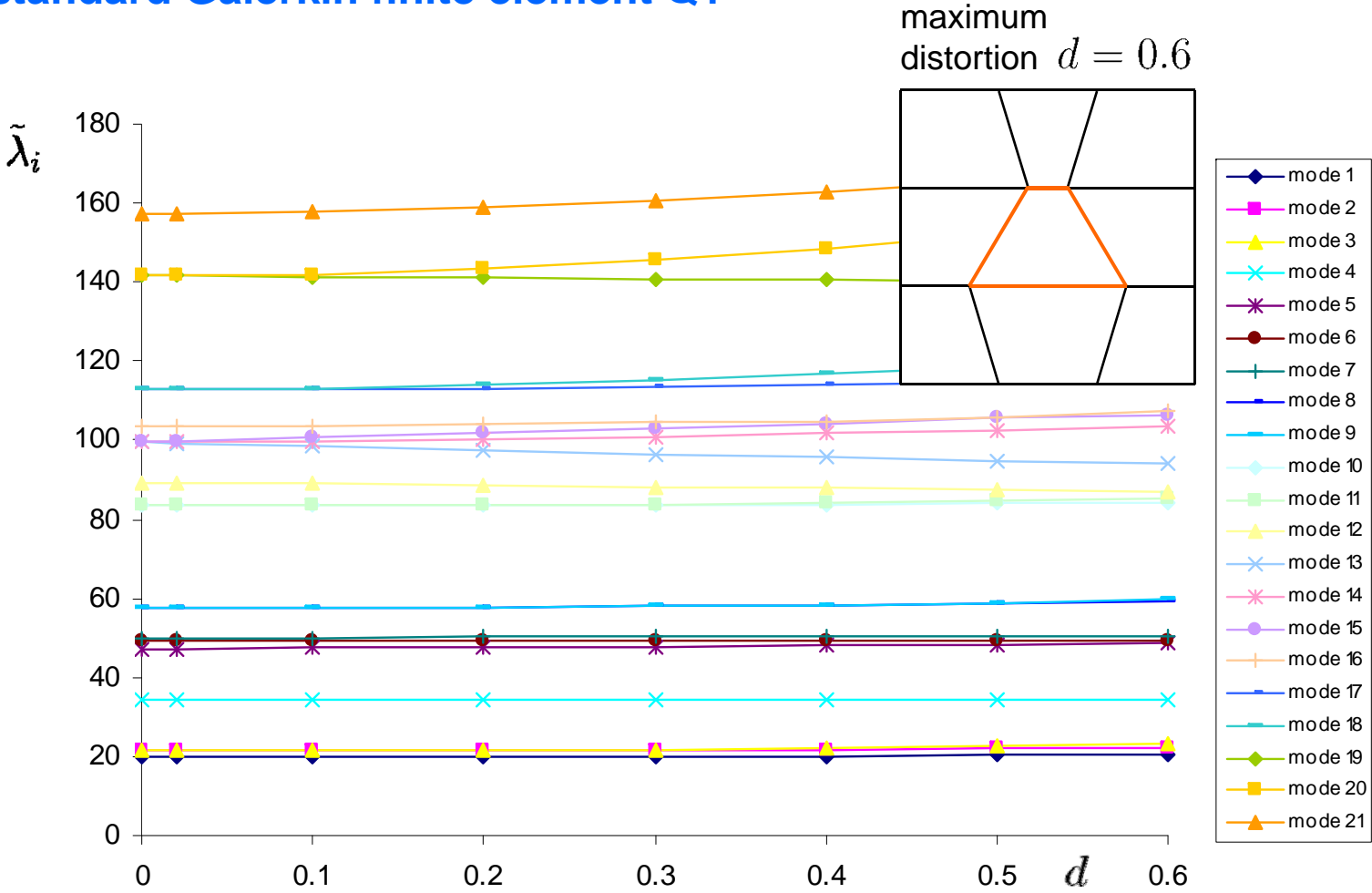


- trapezoidal type of distortion
- plotting $\tilde{\lambda}_i$ versus d
- comparing eigenvalue spectra for different values of d



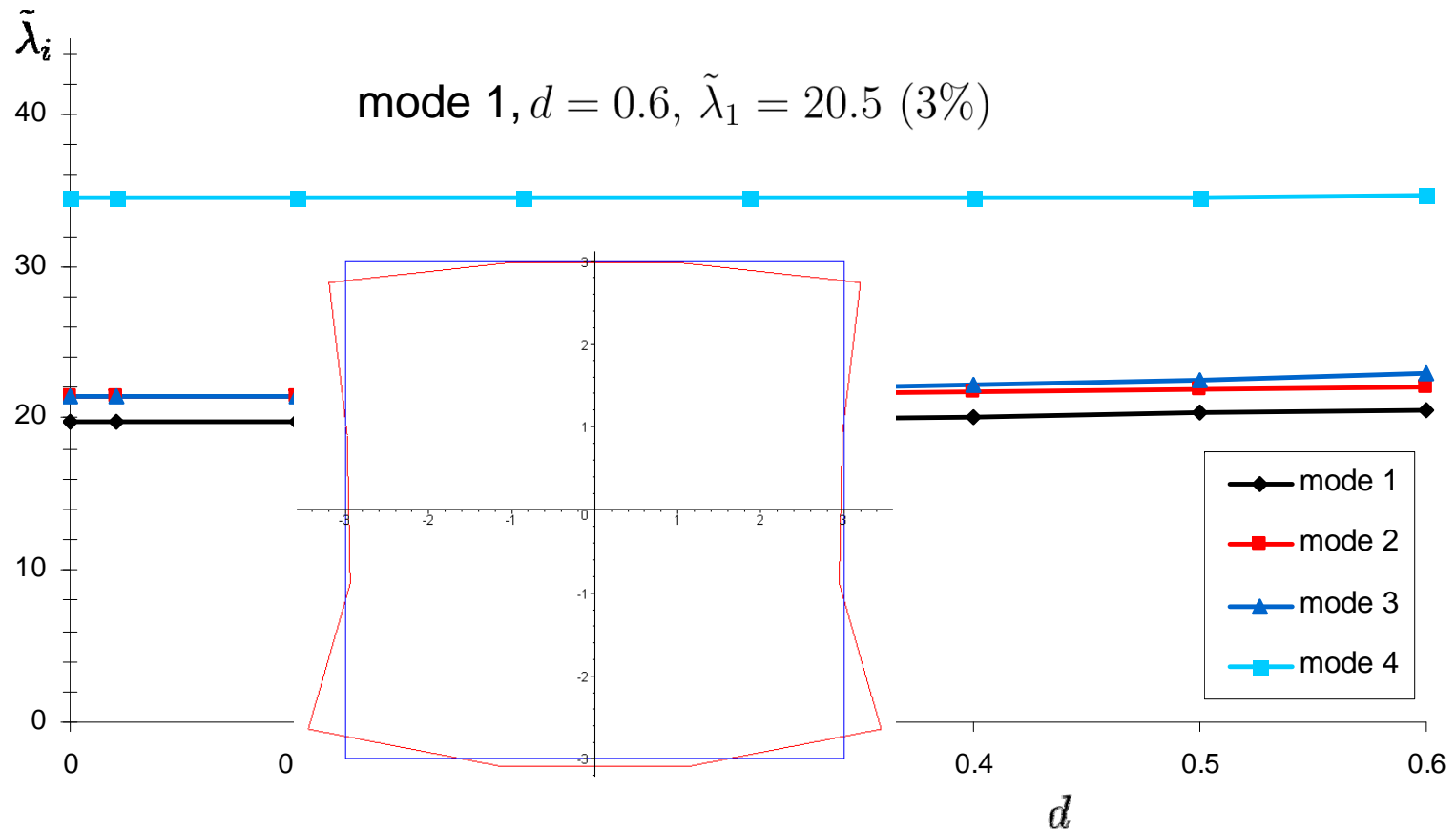
Eigenvalues versus Distortion Parameter

standard Galerkin finite element Q1



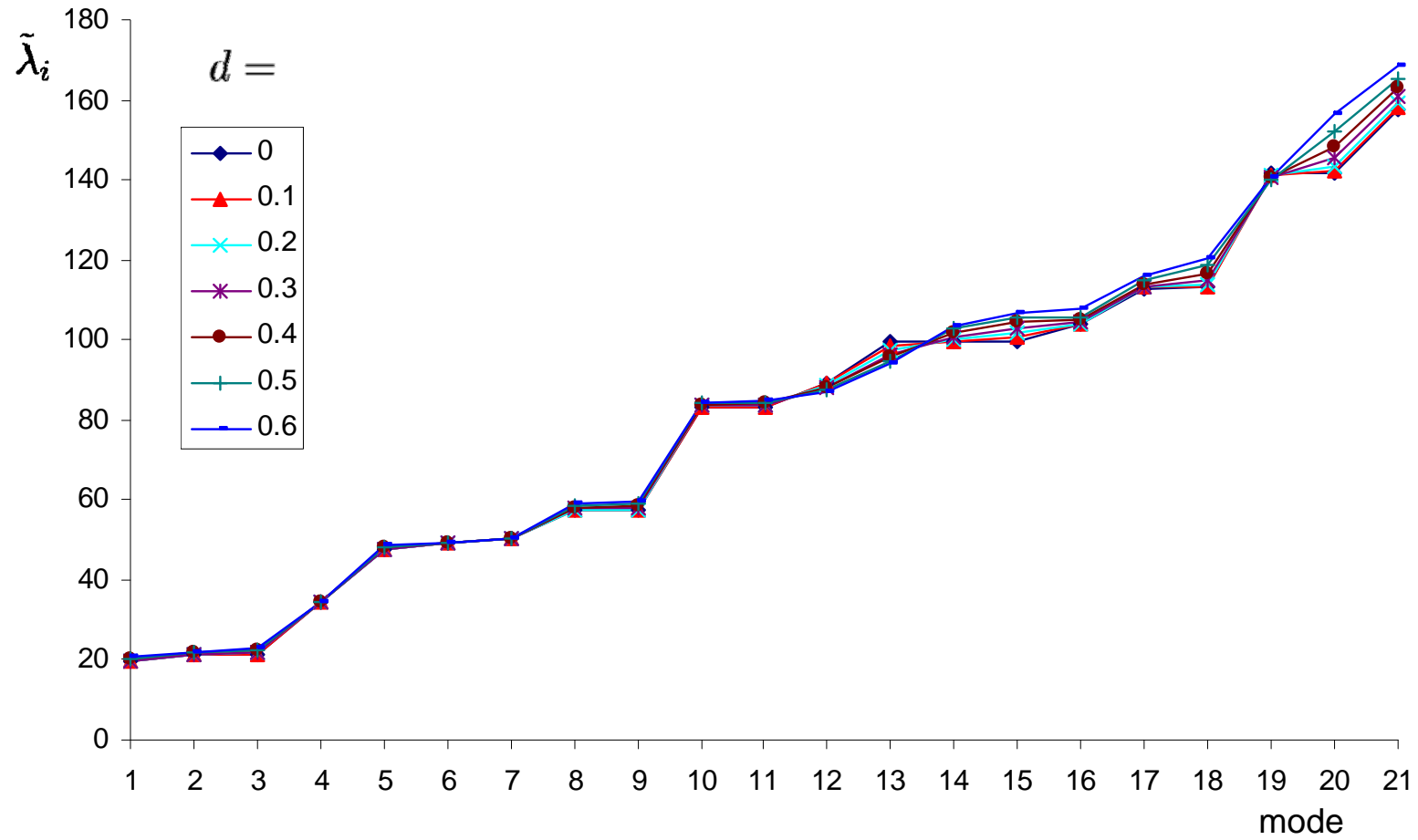
Eigenvalues versus Distortion Parameter

modes 1 – 4



Eigenvalue Spectra

standard Galerkin finite element Q1

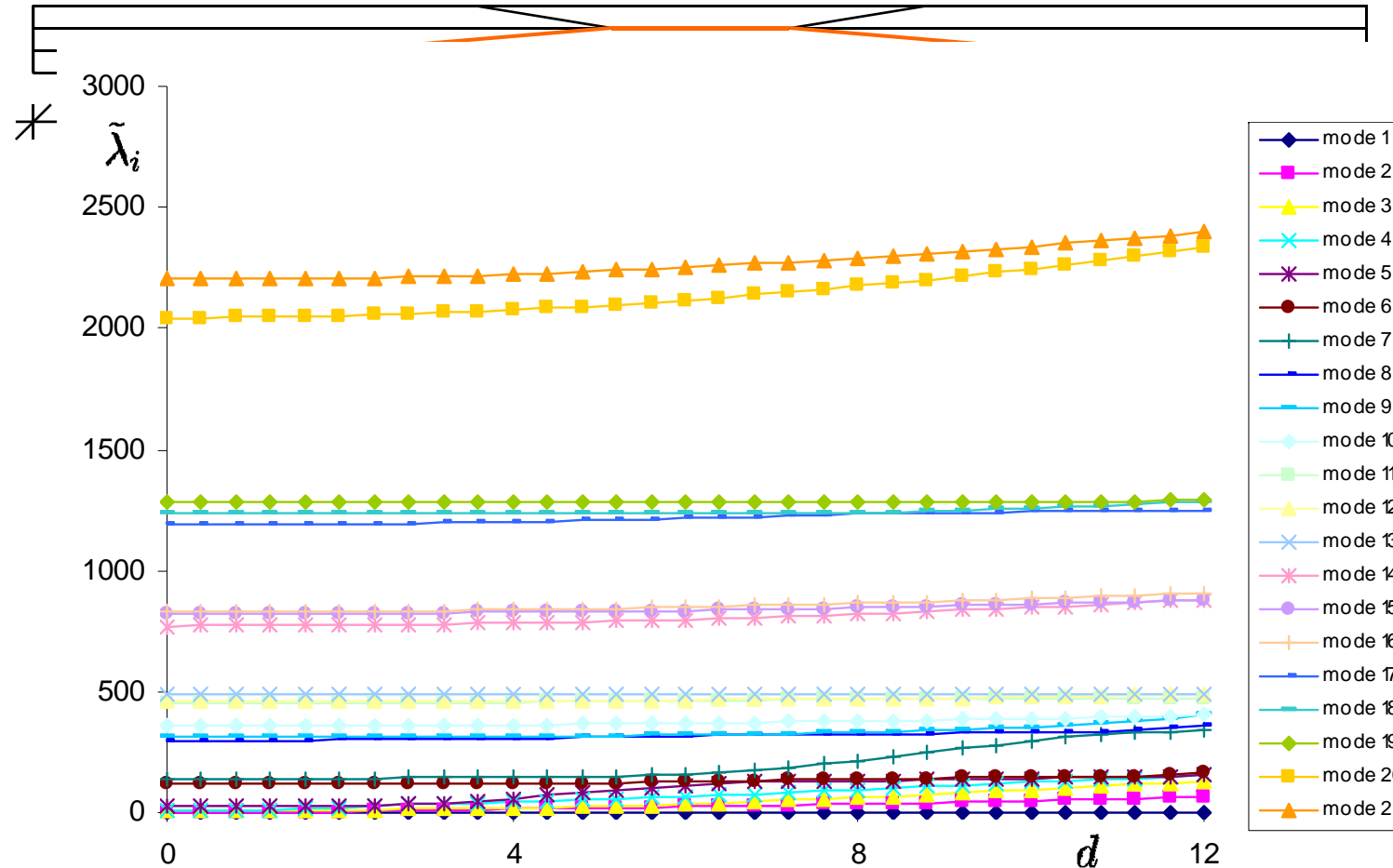


Element Type Q1



Application to “Thin” Structure

more sensitive to locking

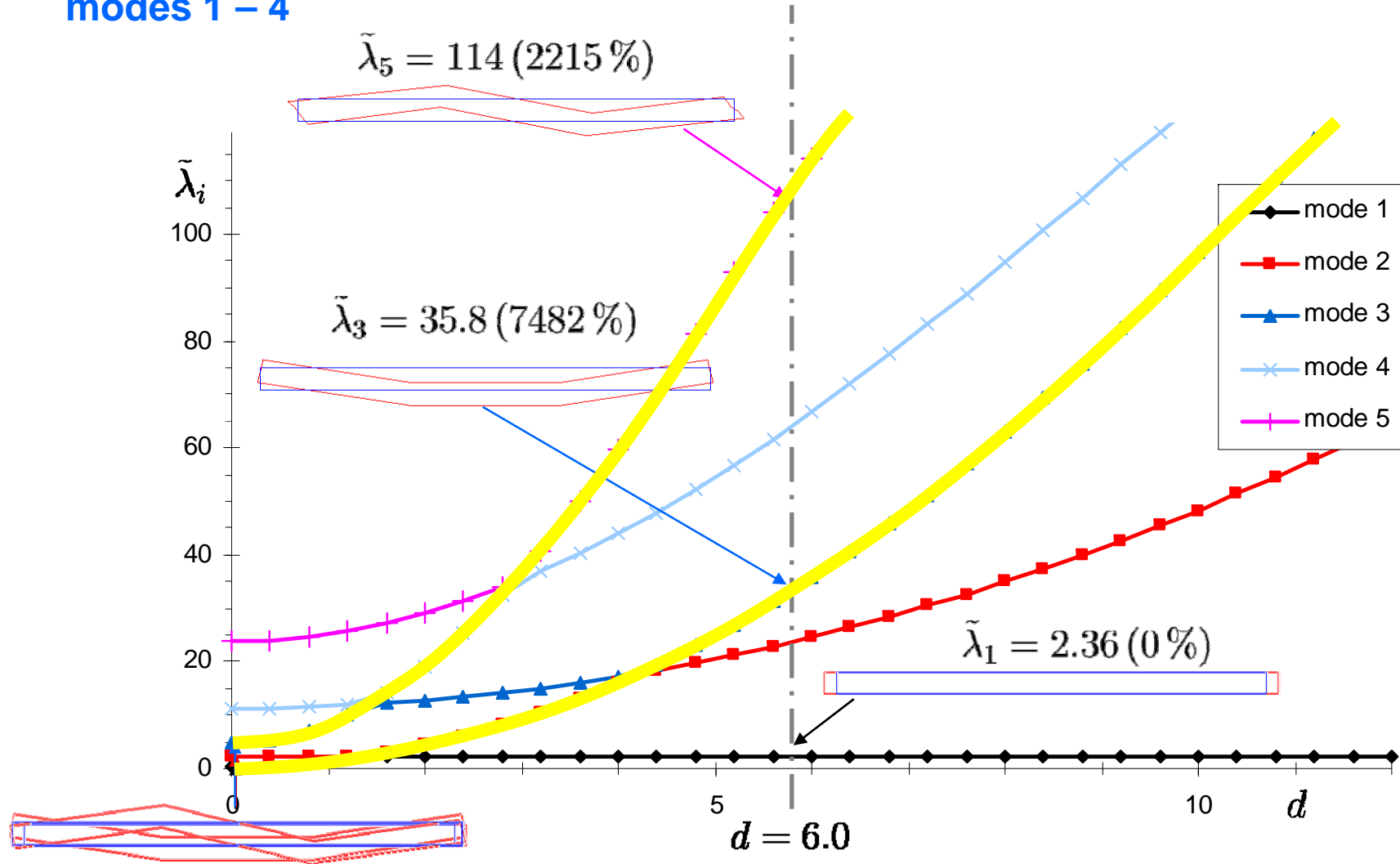


Element Type Q1



Application to “Thin” Structure

modes 1 – 4

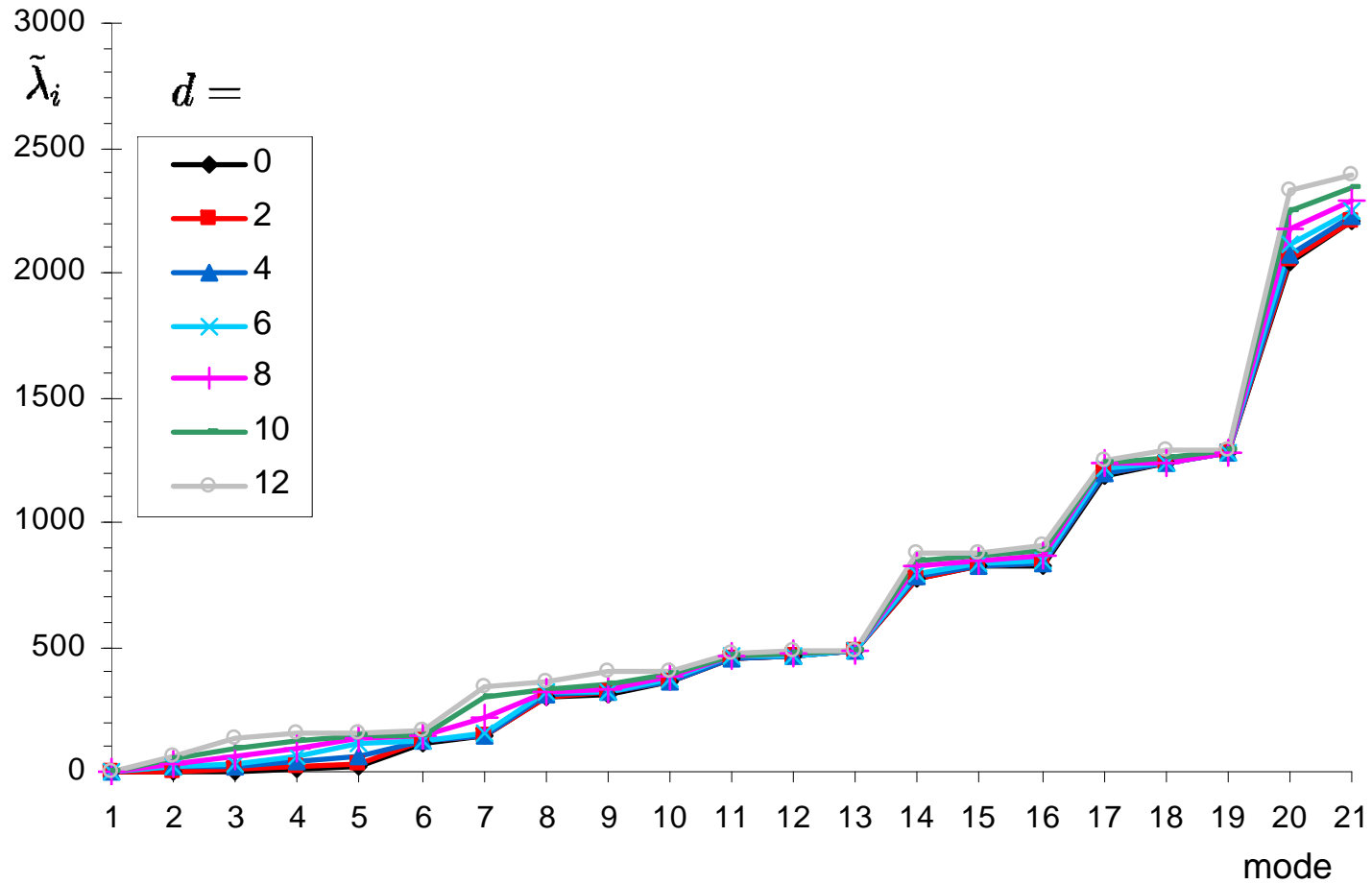


— Element Type Q1



Eigenvalue Spectrum of Thin Structure

standard Galerkin finite element Q1

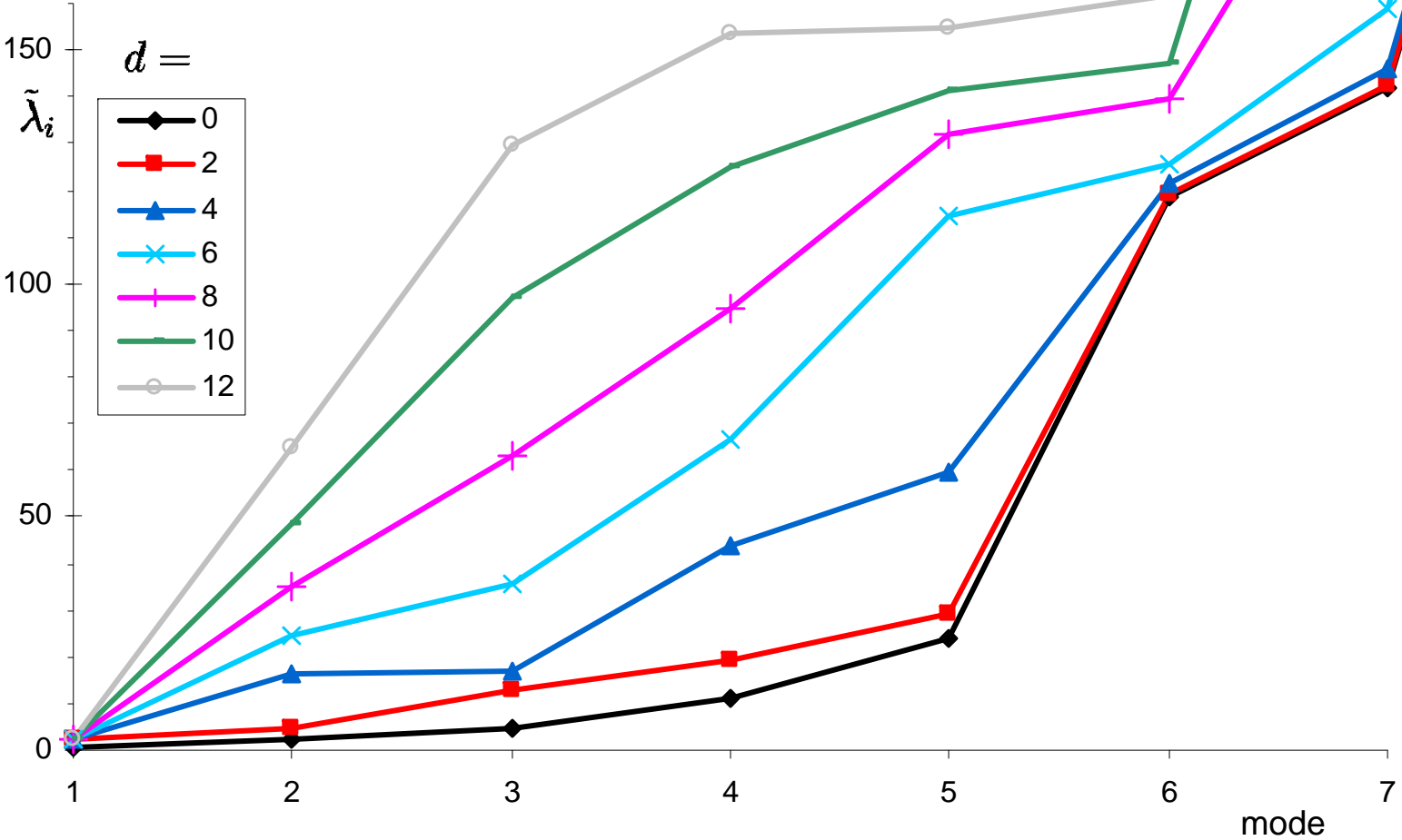


Element Type Q1



Eigenvalue Spectrum of Thin Structure

standard Galerkin finite element Q1

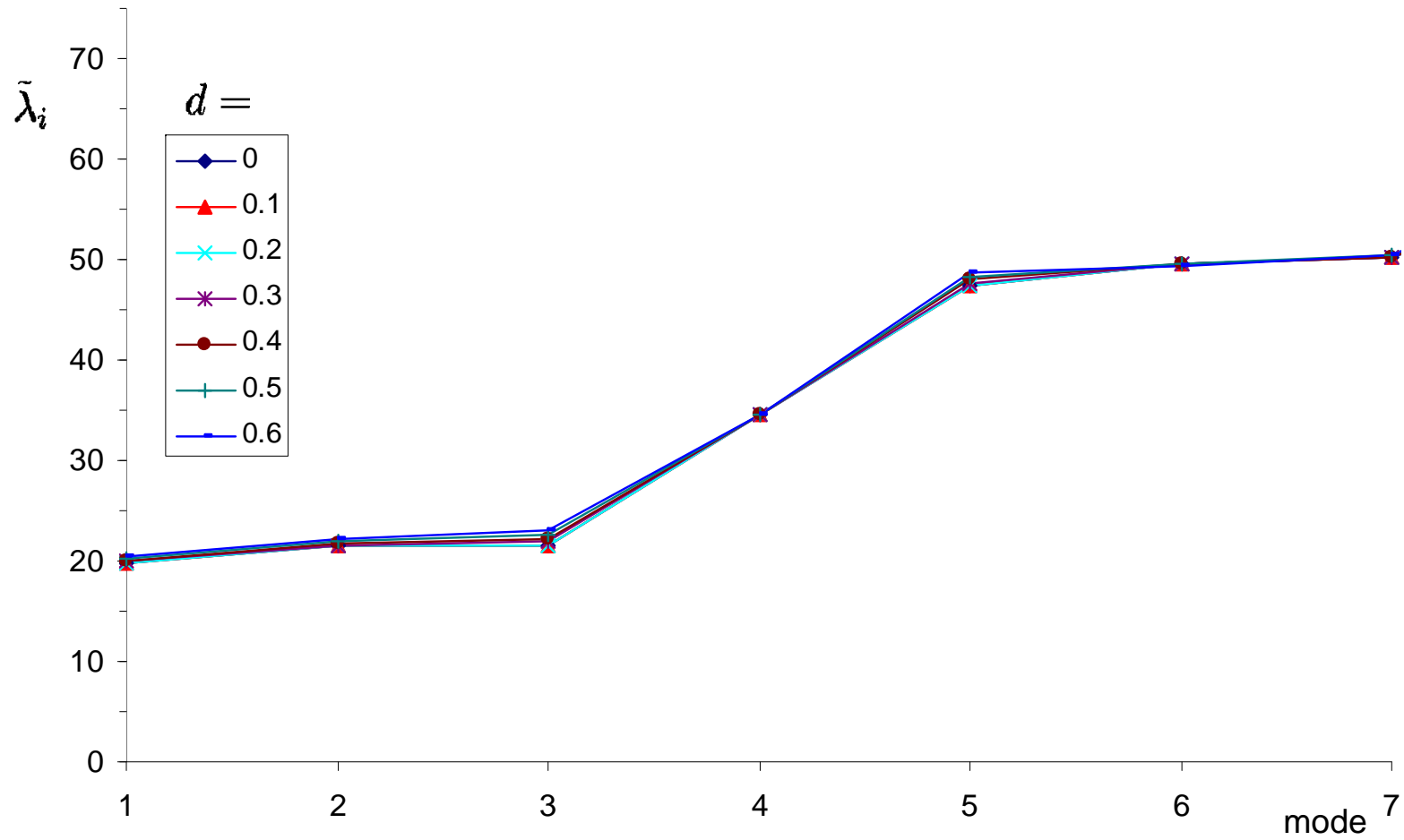


Element Type Q1



Eigenvalue Spectrum of Thick Structure

for comparison

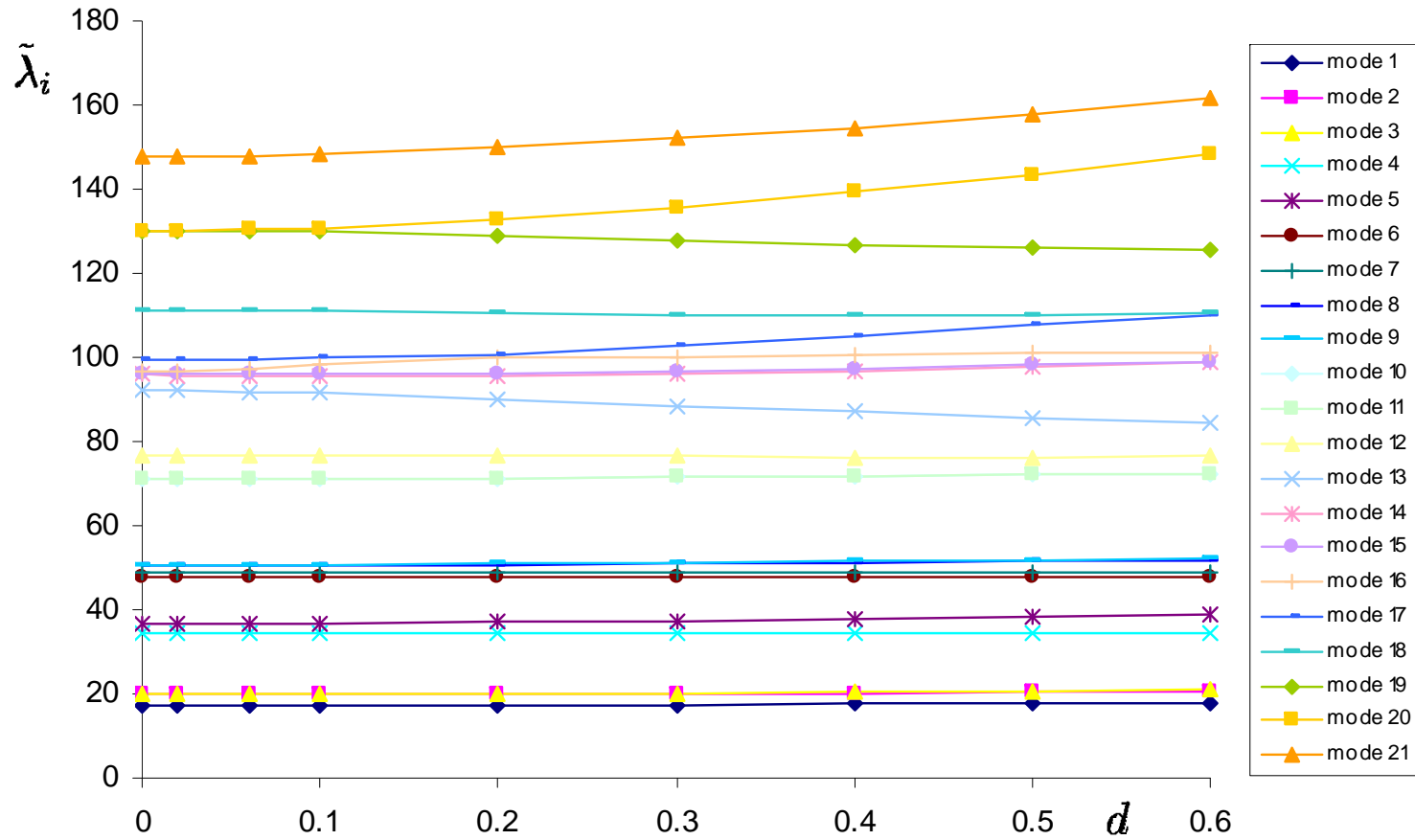


Element Type Q1



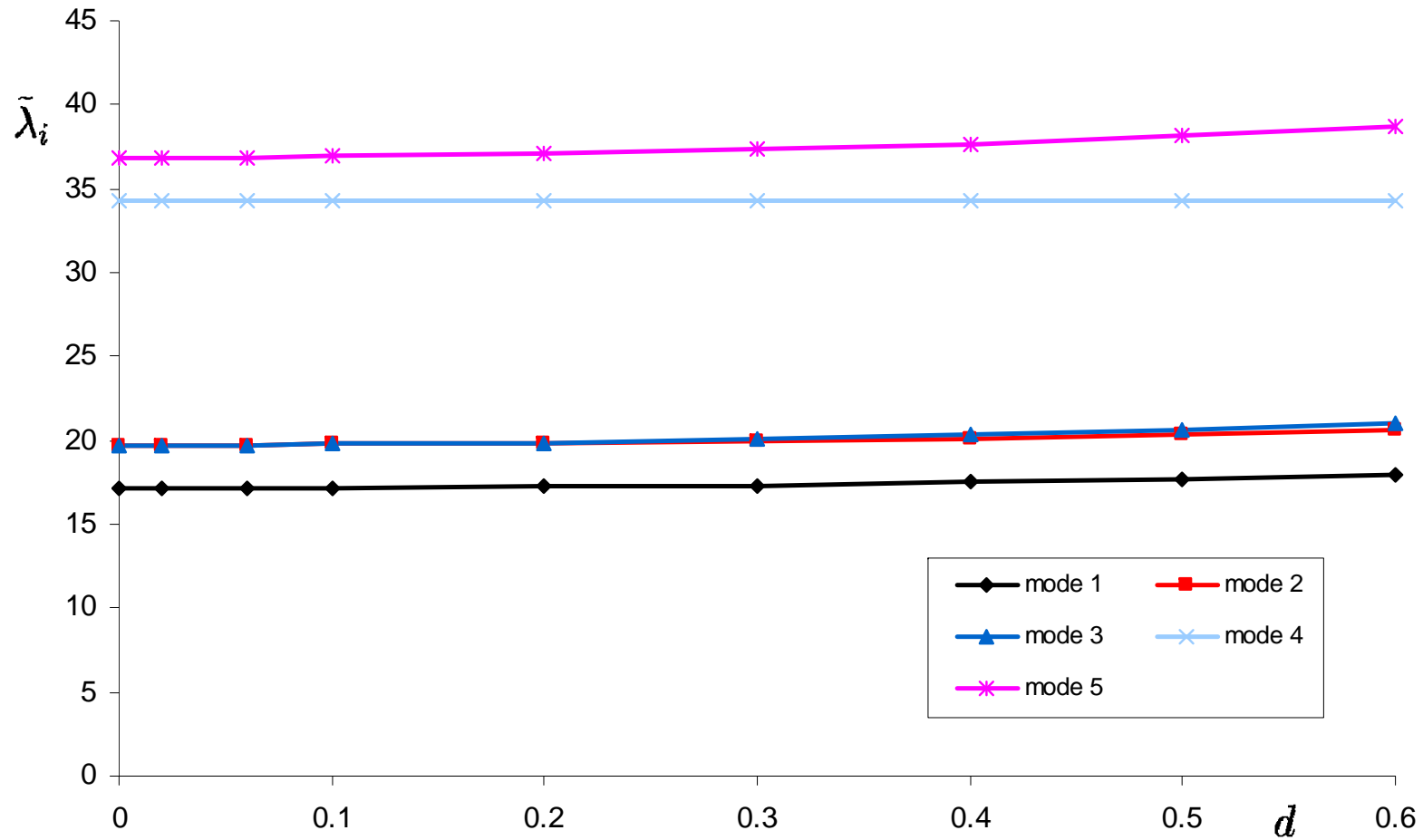
Selective Reduced Integration, Q1-SRI

eigenvalues for thick structure



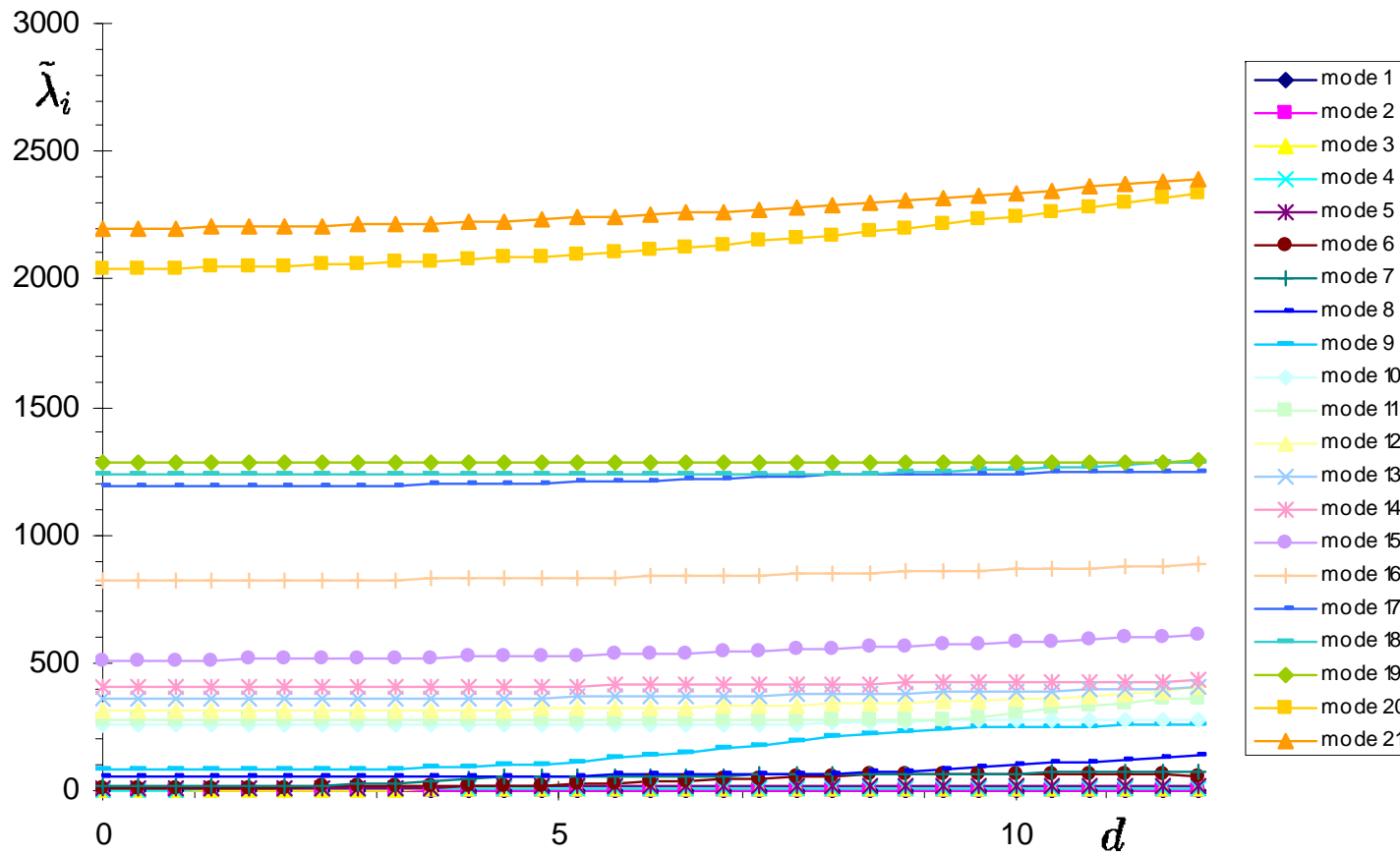
Selective Reduced Integration, Q1-SRI

eigenvalues for thick structure



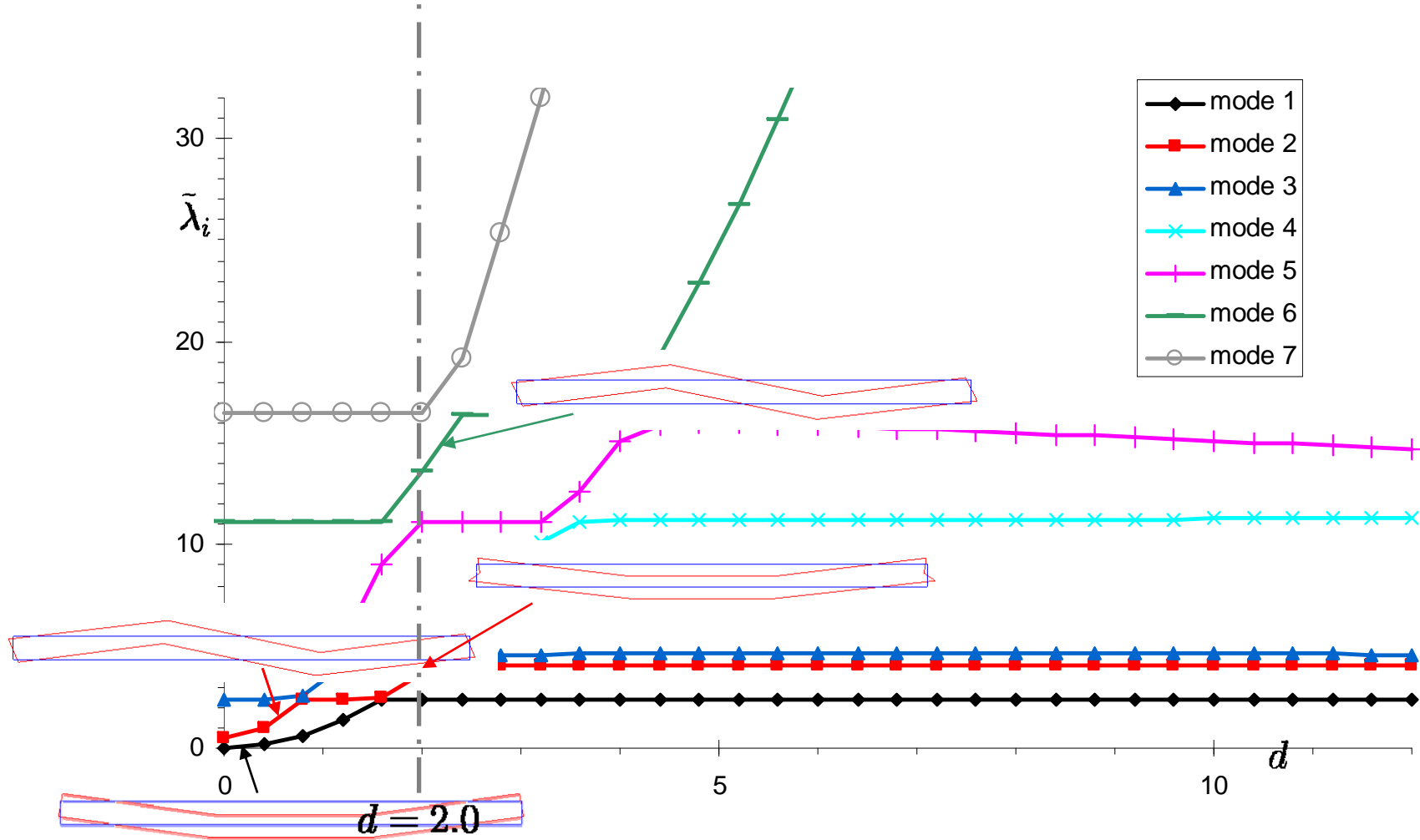
Selective Reduced Integration, Q1-SRI

eigenvalues for thin structure



Selective Reduced Integration, Q1-SRI

eigenvalues for thin structure

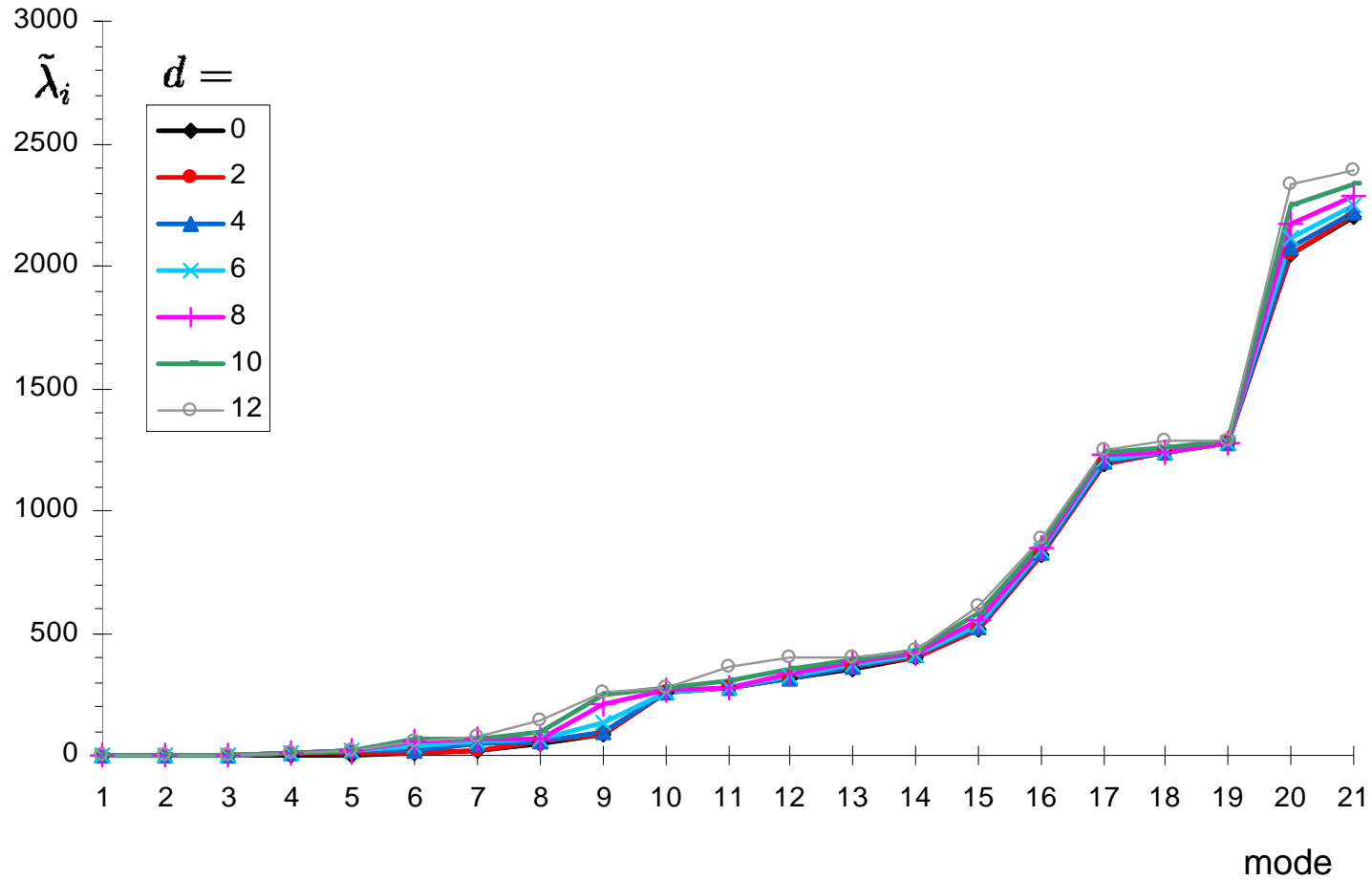


Element Type Q1-SRI



Selective Reduced Integration, Q1-SRI

eigenvalue spectrum

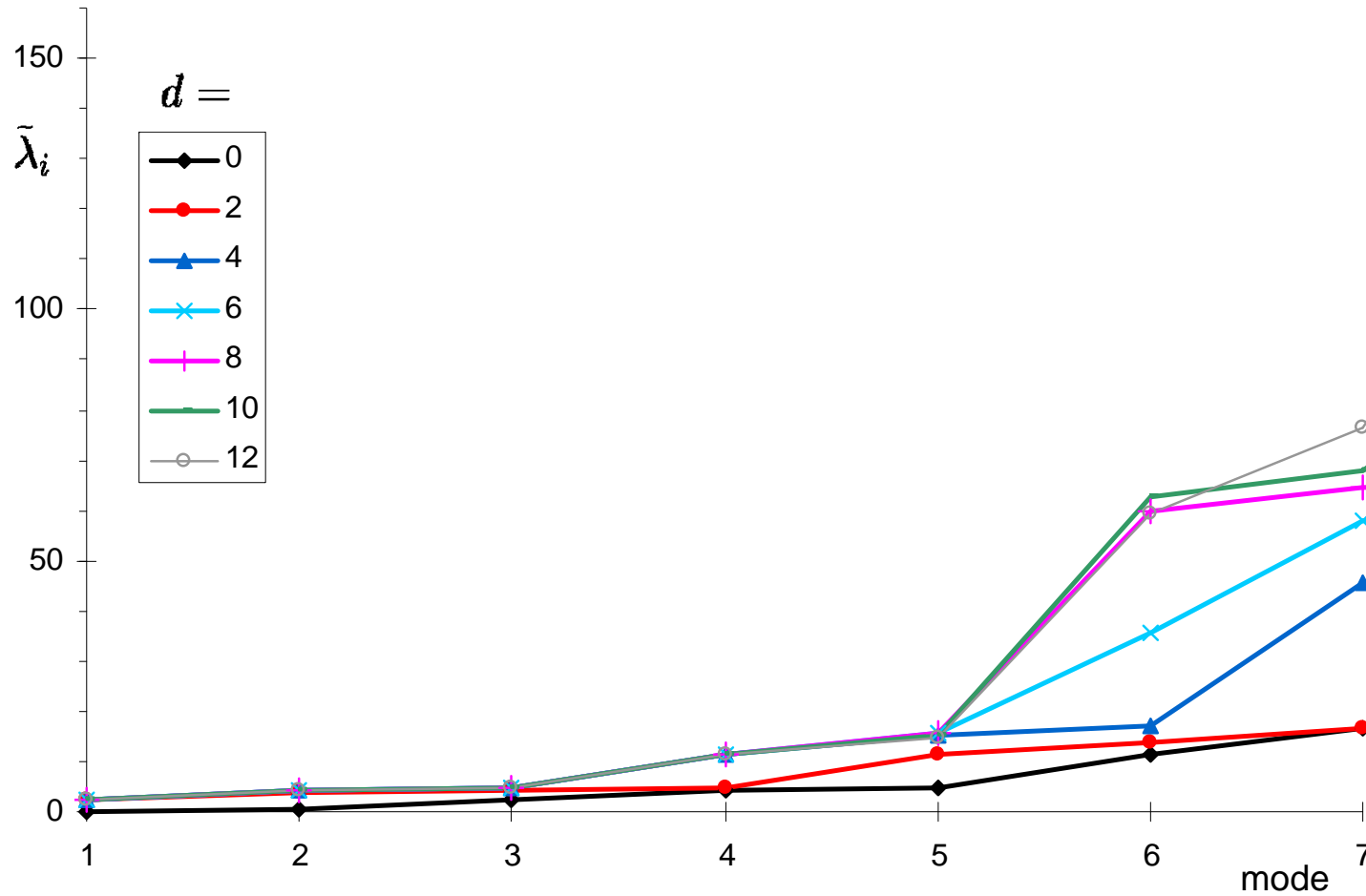


Element Type Q1-SRI



Selective Reduced Integration, Q1-SRI

eigenvalue spectrum

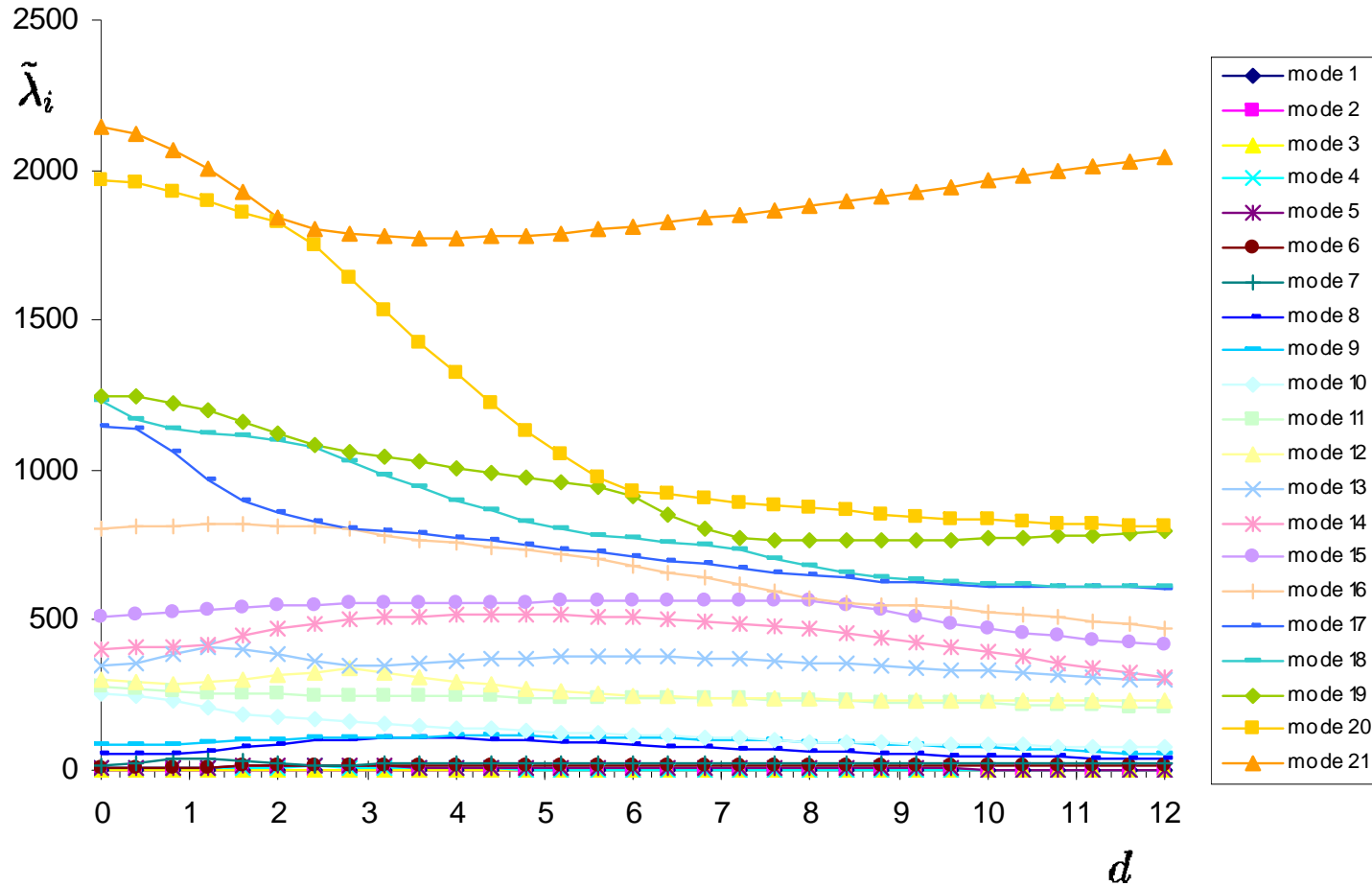


Element Type Q1-SRI



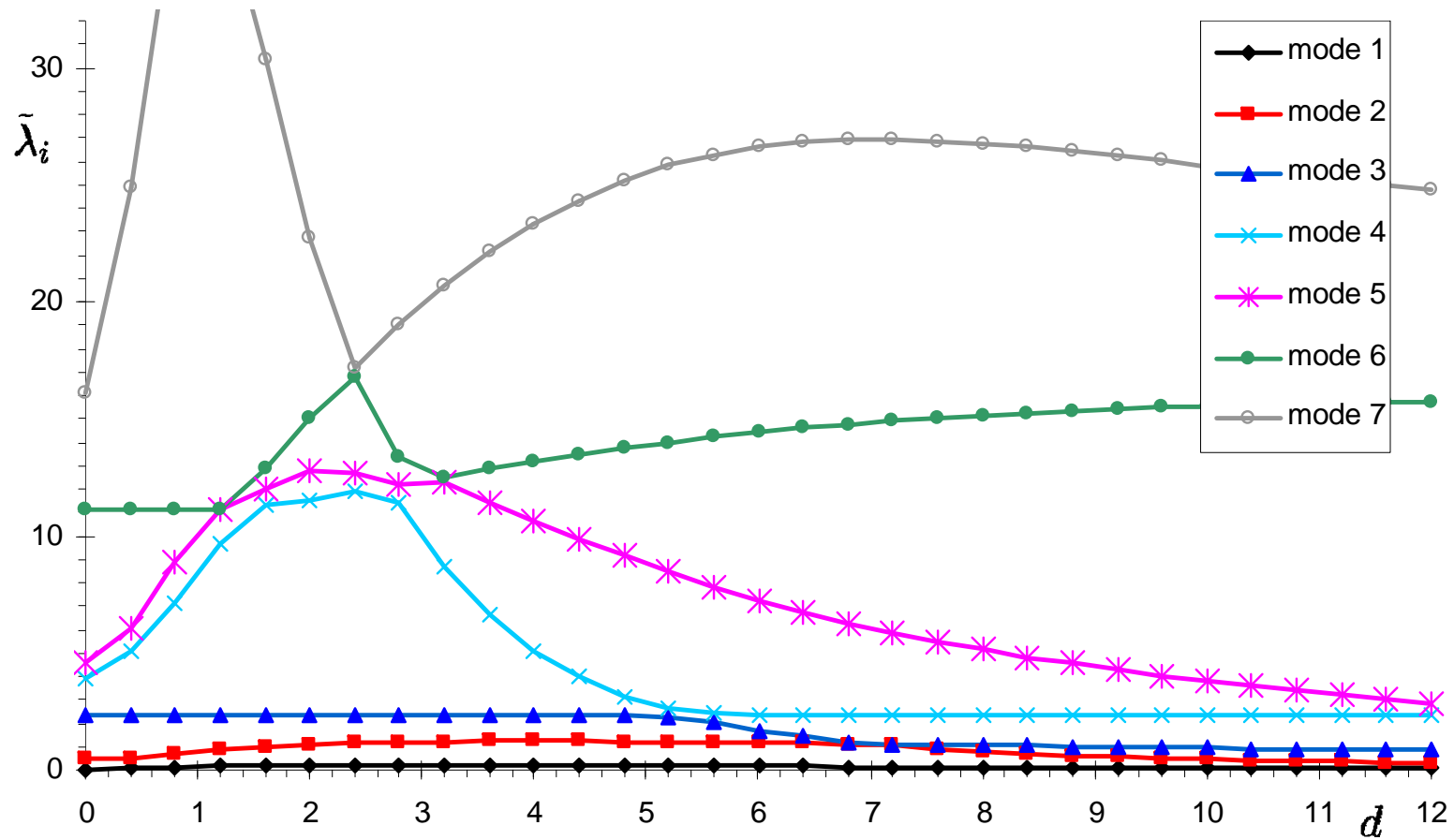
Method of Incompatible Modes, Qm6 (=Q1-E4)

eigenvalues for thin structure



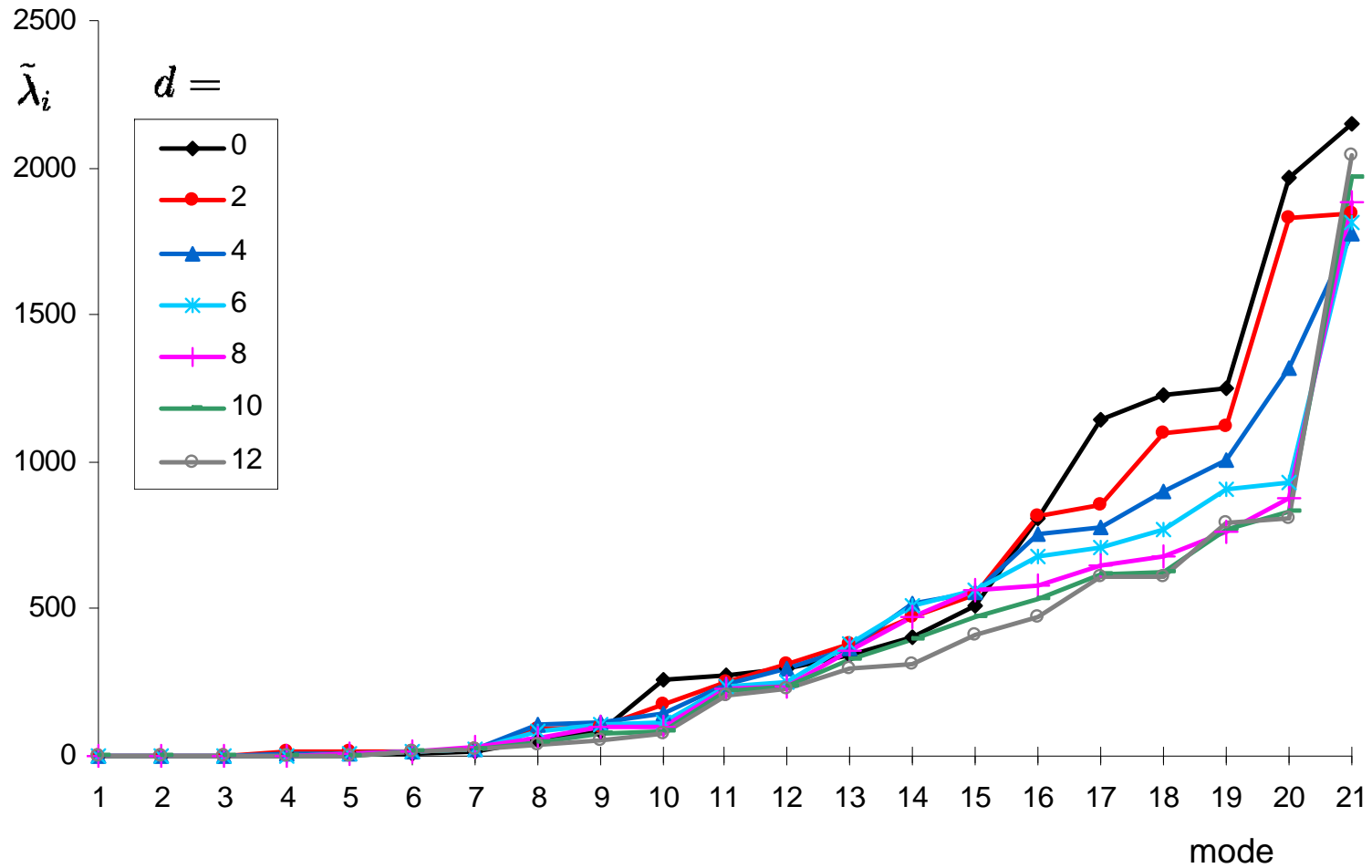
Method of Incompatible Modes, Qm6 (=Q1-E4)

eigenvalues for thin structure



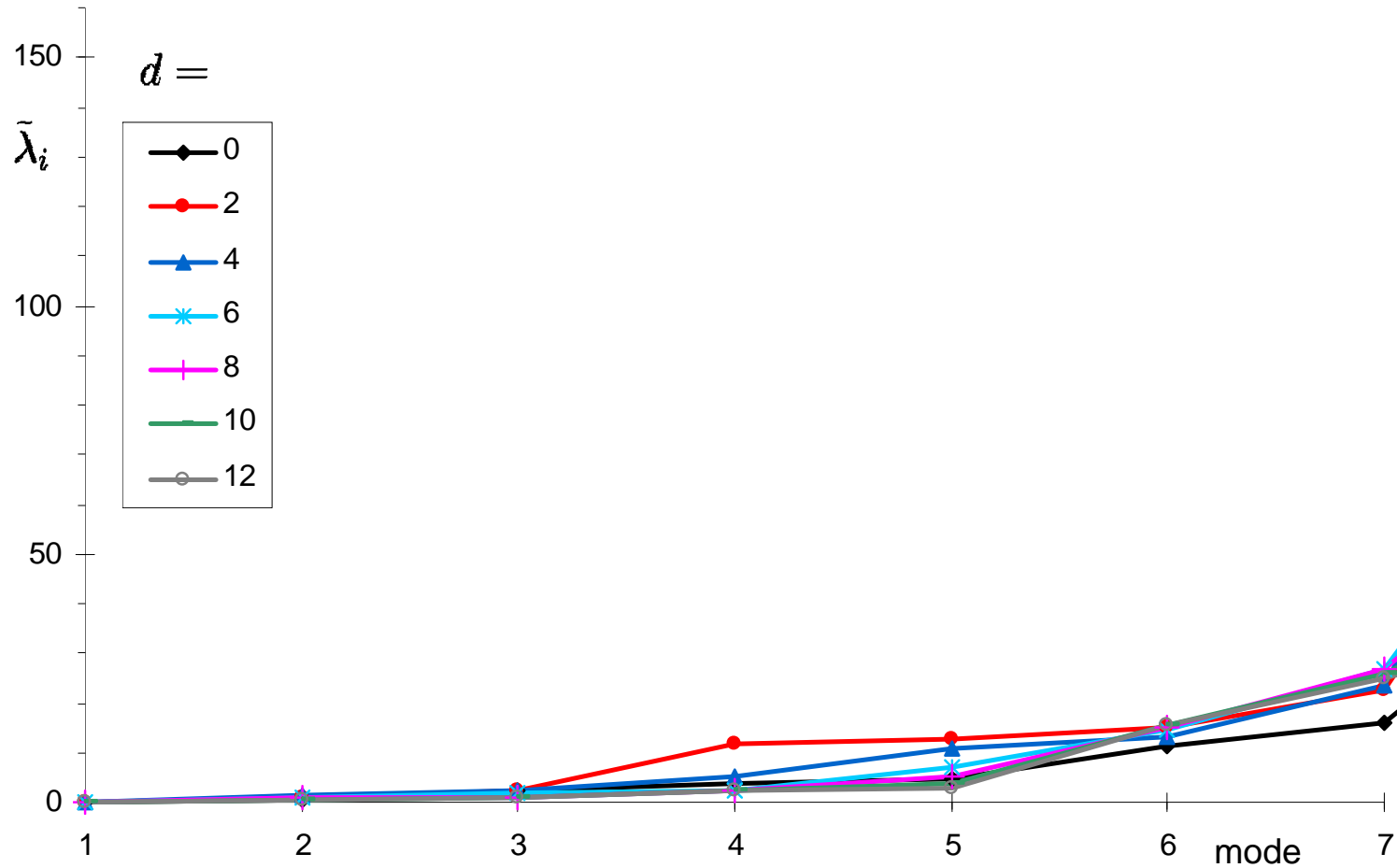
Method of Incompatible Modes, Qm6 (=Q1-E4)

eigenvalues for thin structure



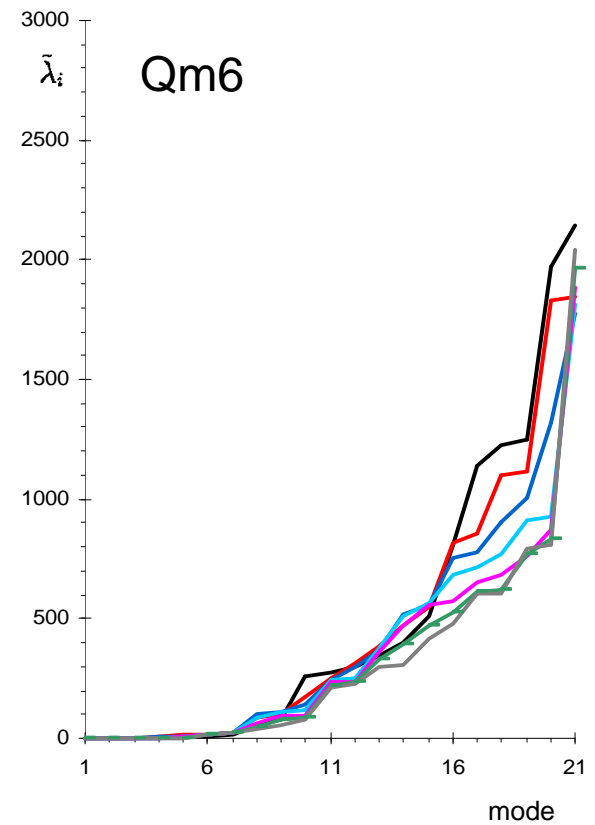
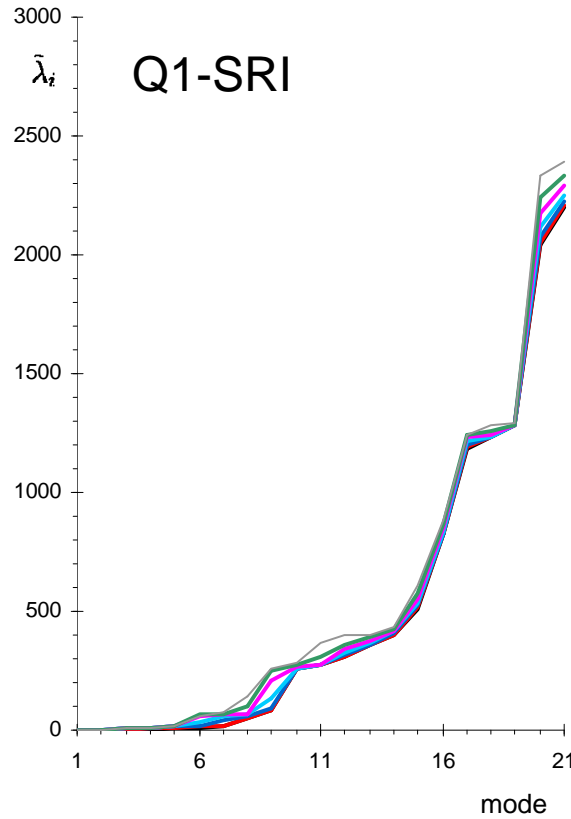
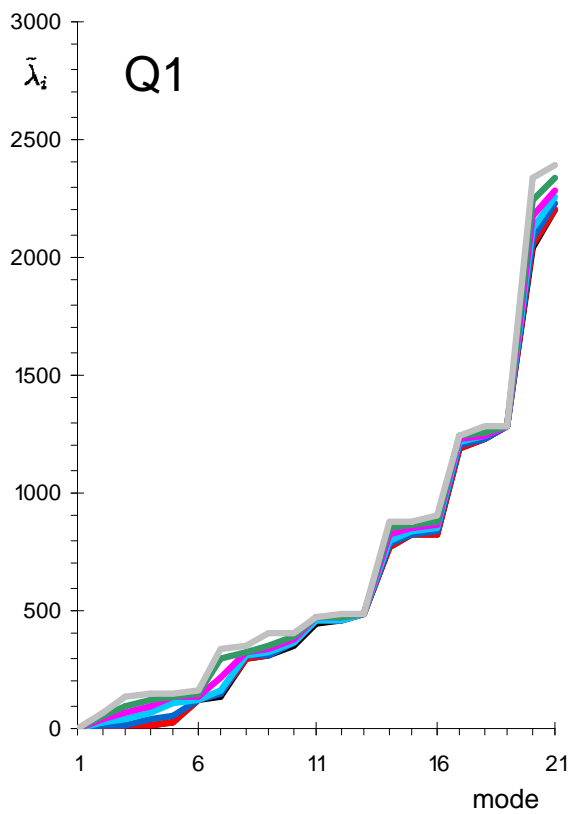
Method of Incompatible Modes, Qm6 (=Q1-E4)

eigenvalues for thin structure



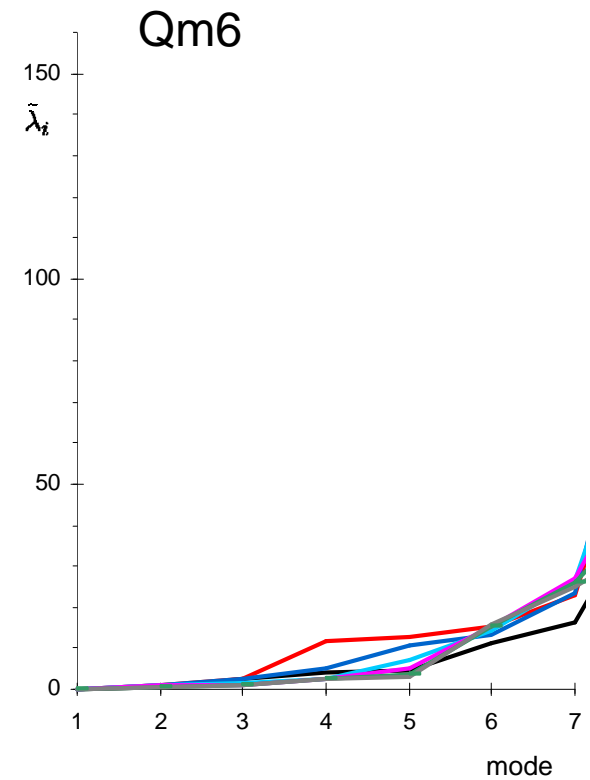
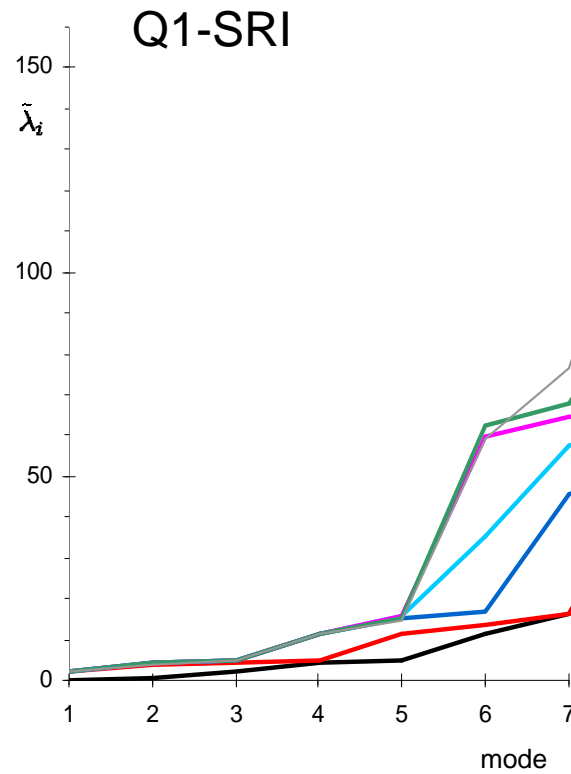
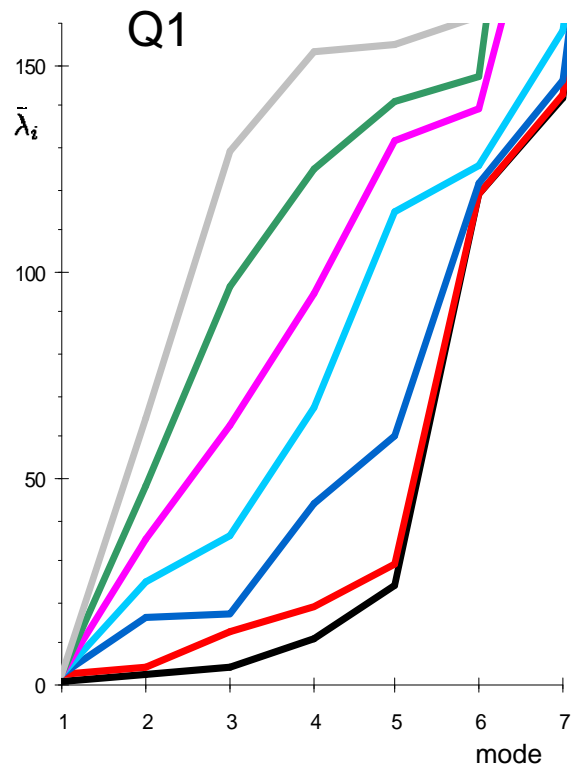
Summary of Results

eigenvalue spectra



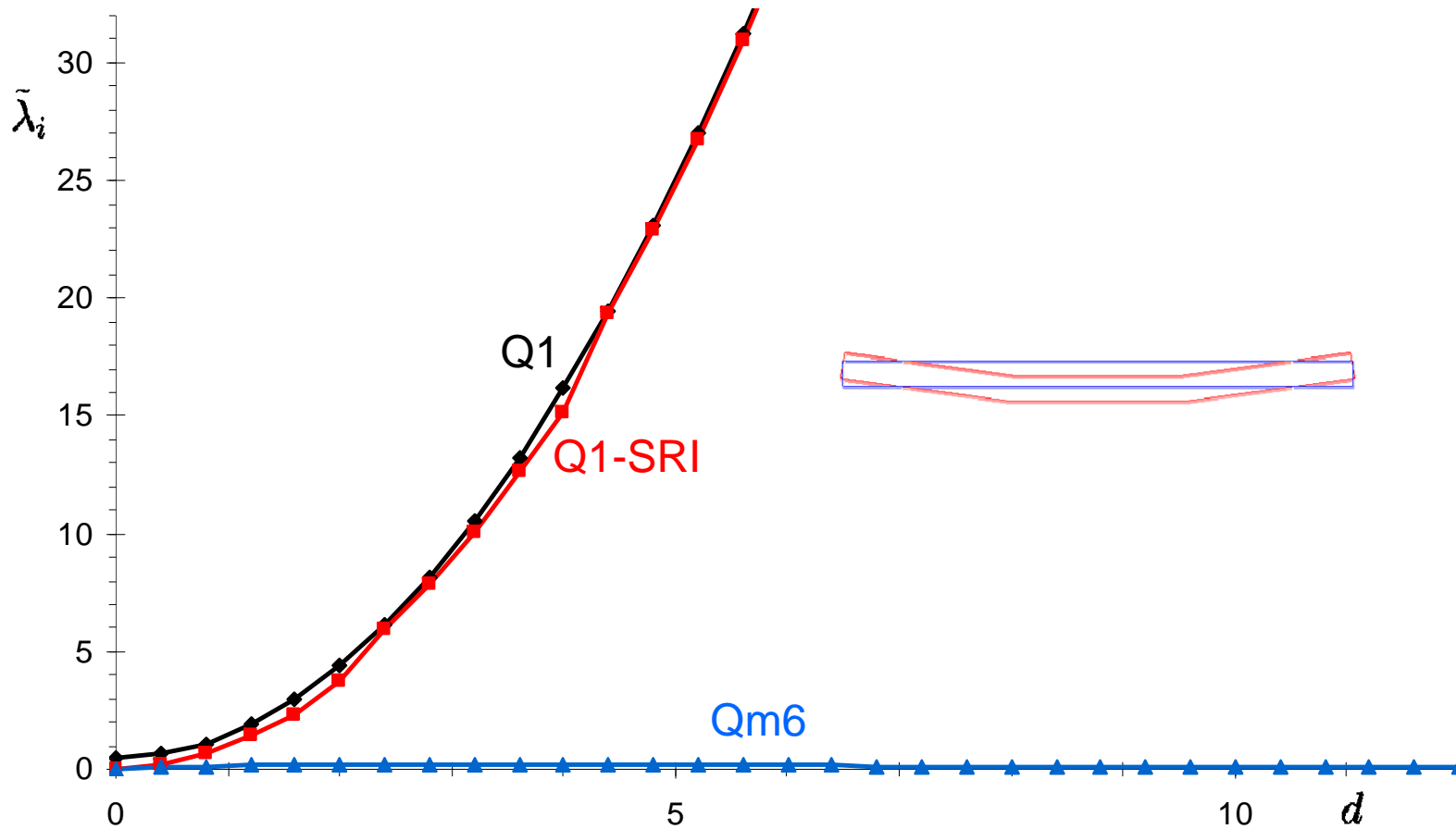
Summary of Results

eigenvalue spectra



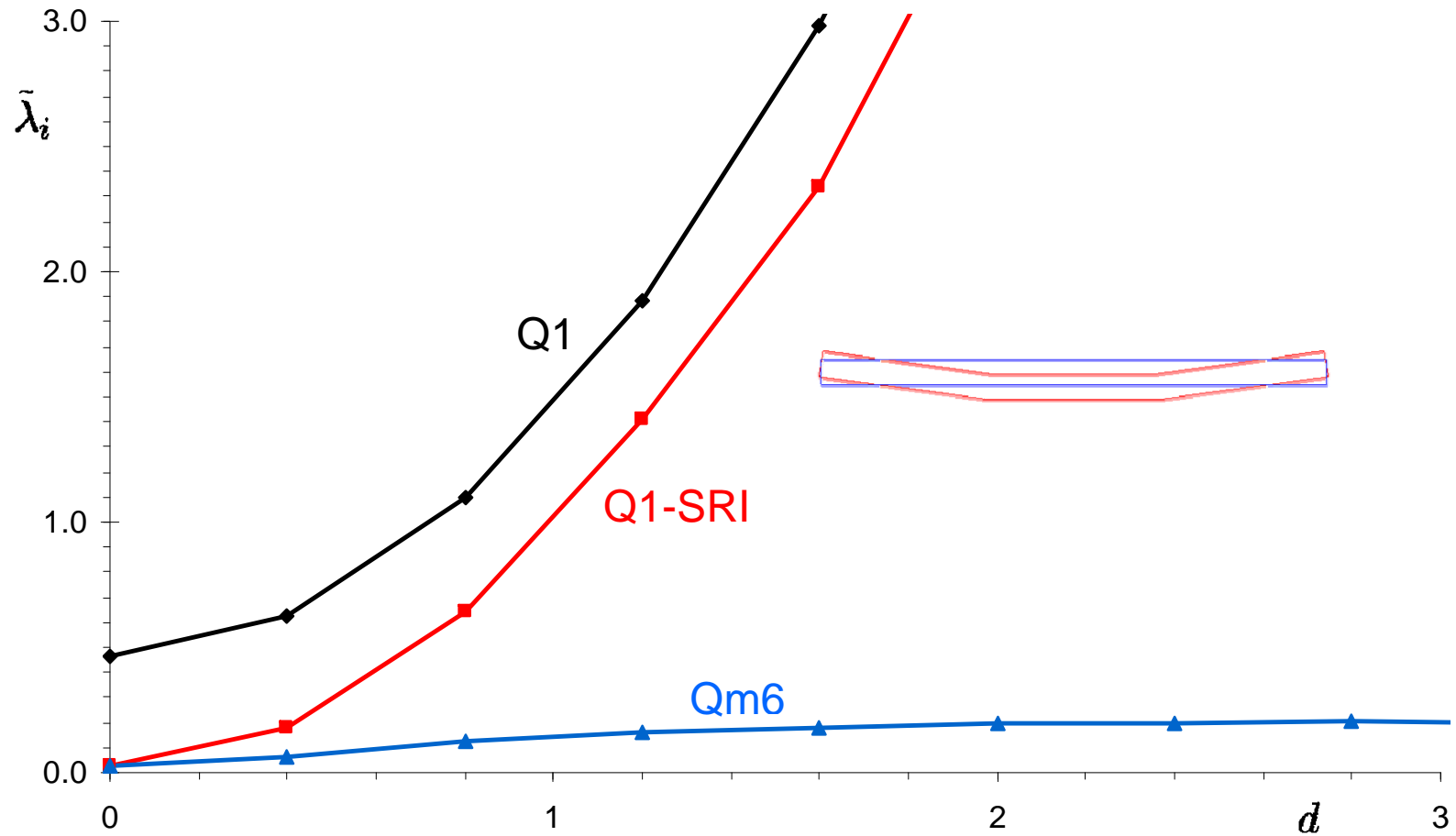
Summary of Results

eigenvalues of mode 1 in dependence of mesh distortion d



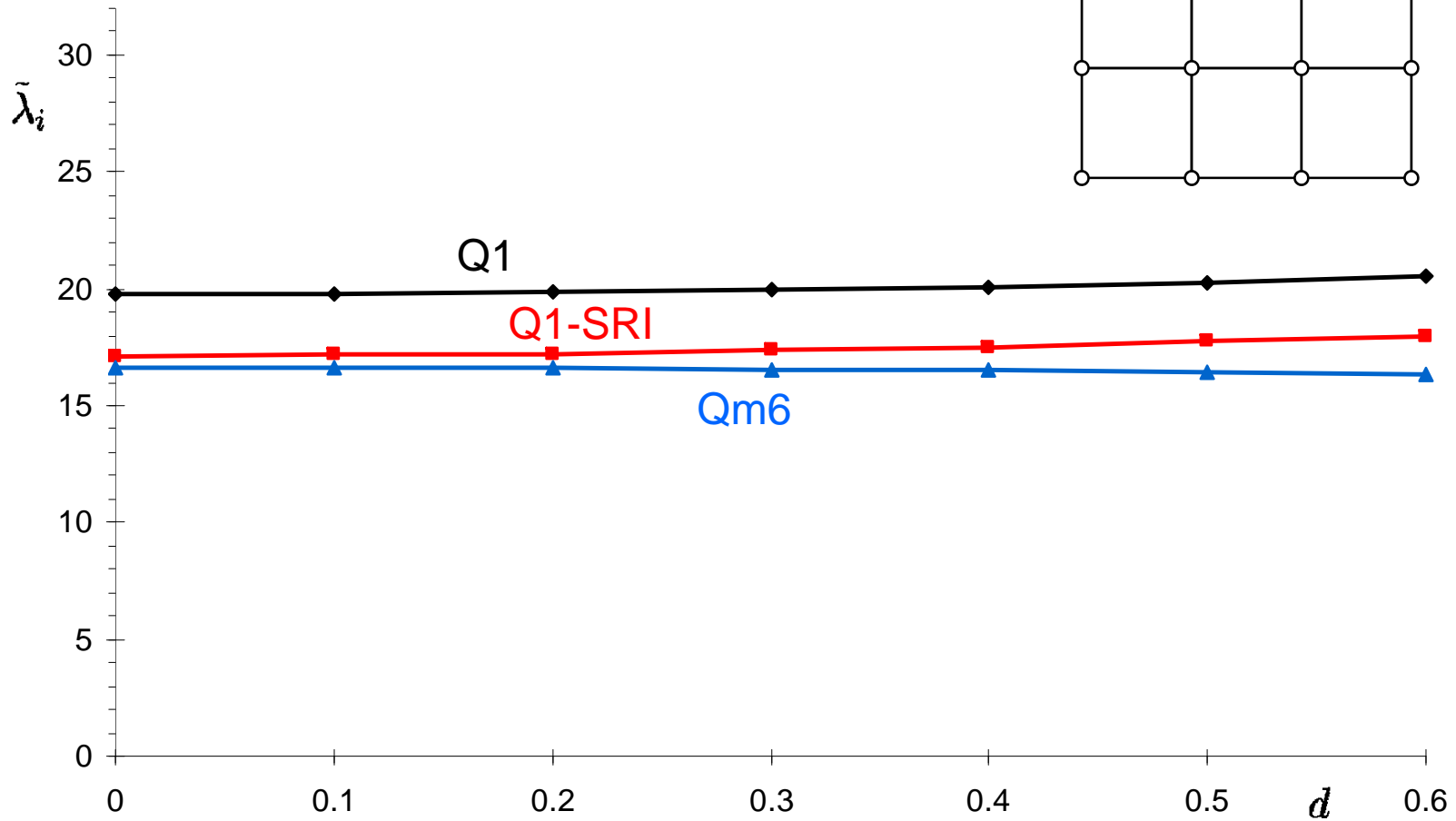
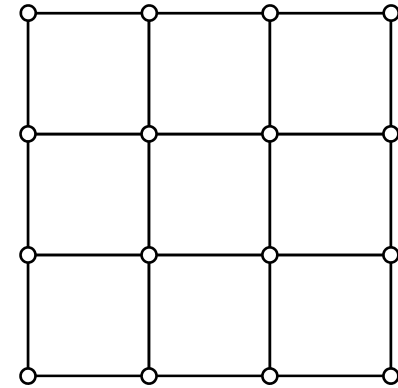
Summary of Results

eigenvalues of mode 1 in dependence of mesh distortion d



Summary of Results

eigenvalues of mode , thick structure (square)



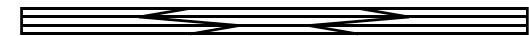
Conclusions

measuring distortion sensitivity

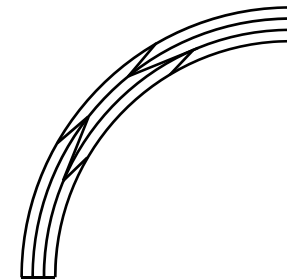
- bending test is not so bad
- eigenvalues may provide objective measure

newly proposed distortion patch test

- objective measure for distortion sensitivity
- universal locking test
- applicable to arbitrary problems
(thin and curved structures, near incompressibility, etc.)



$\nu \rightarrow 0.5$



numerical results

- distortion sensitivity is related to locking
- Q1 and Q1-SRI equally sensitive to distortion
- Qm6 **significantly better** than Q1-SRI
(more than bending test implies)

