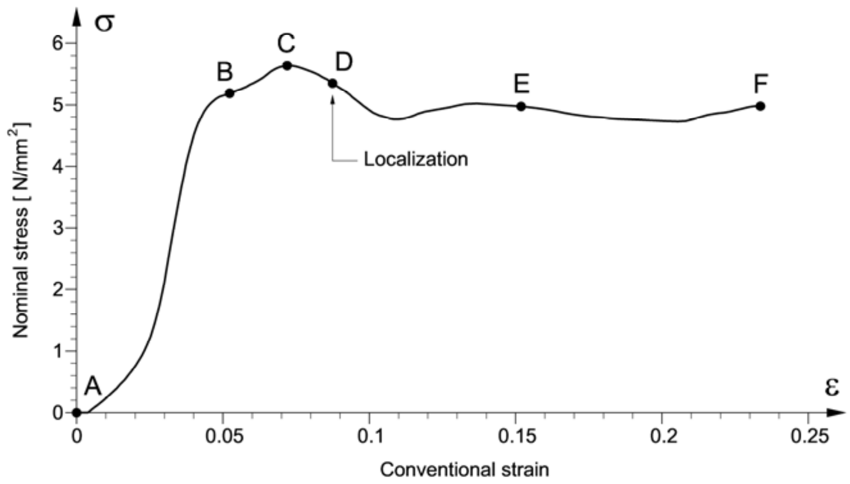
22nd INTERNATIONAL CONGRESS OF
THEORETICAL & APPLIED MECHANICS
24th – 30th August, 2008 - Adelaide, Australia

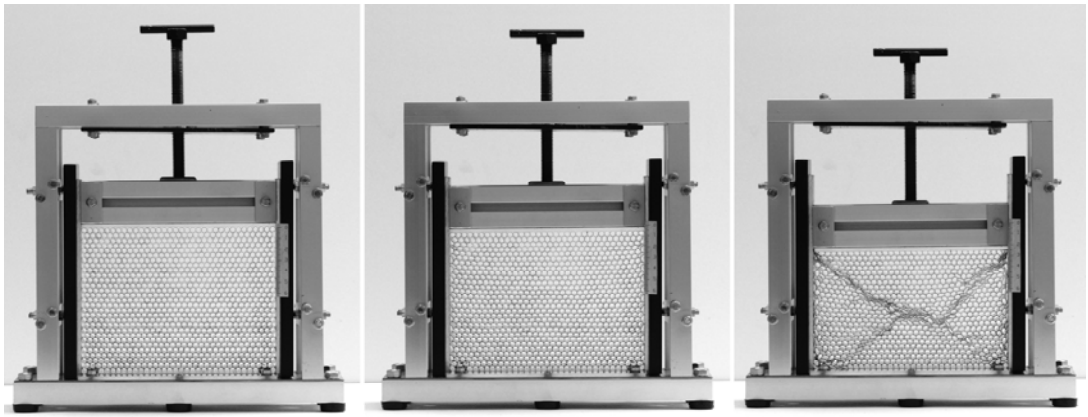
Material instabilities in elastic and plastic solids (the perturbative approach)

Davide Bigoni

Department of Mechanical and Structural Engineering
University of Trento, Italy

MATERIALS ON THE VERGE OF A BREAKDOWN

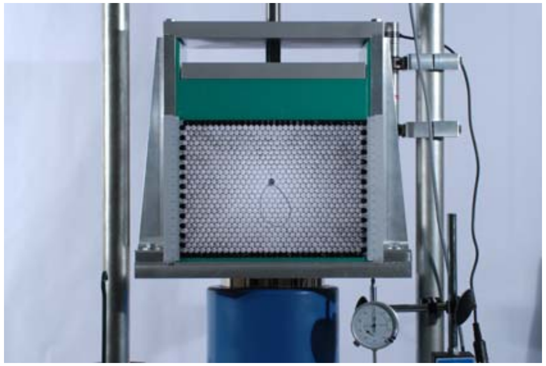




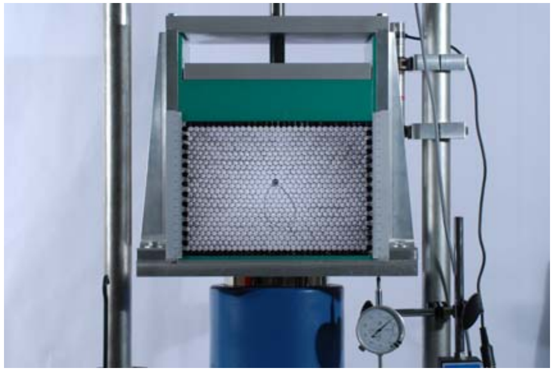
A unloaded | C on the verge | E collapsed

Can we decide that C is ‘on the verge’, without loading to E ?

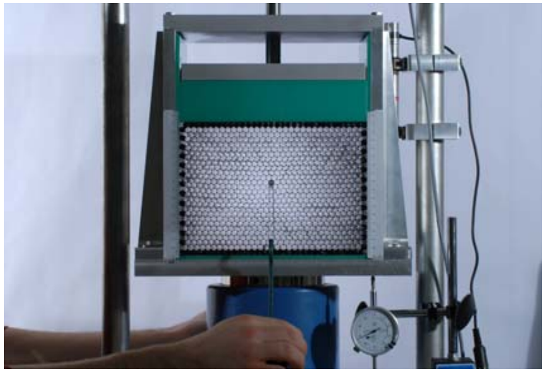
THE PERTURBATIVE APPROACH



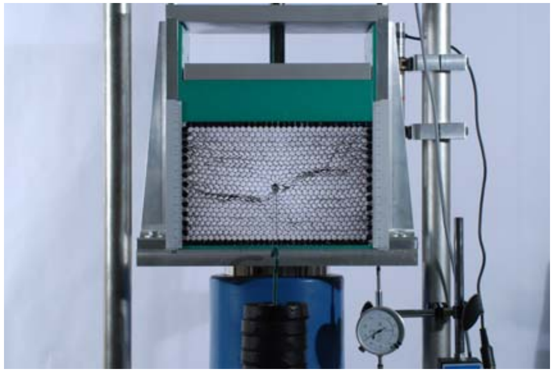
A unloaded



C ‘on the verge’



C is perturbed



shear bands appear!

HOW TO RATIONALIZE THIS? THE PERURBATIVE APPROACH

$$\boldsymbol{G}(\boldsymbol{x}) = -\frac{1}{4\pi^2}\sum_{N=1}^{2}\int_0^{\pi}[2\cos(rk_N|\cos\alpha|)\mathrm{Ci}(rk_N|\cos\alpha|)$$

$$+2\sin(rk_N|\cos\alpha|)\mathrm{Si}(rk_N|\cos\alpha|) - i\pi\cos(rk_N|\cos\alpha|)]\frac{\boldsymbol{v}_N\otimes\boldsymbol{w}_N}{\rho\, c_N^2}d\alpha$$

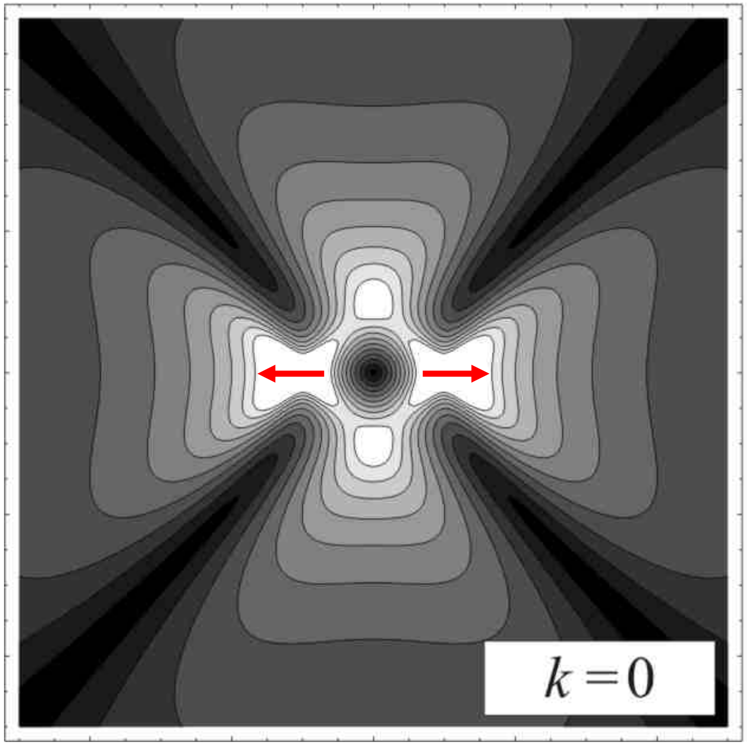
$$k_N = \omega/c_N$$

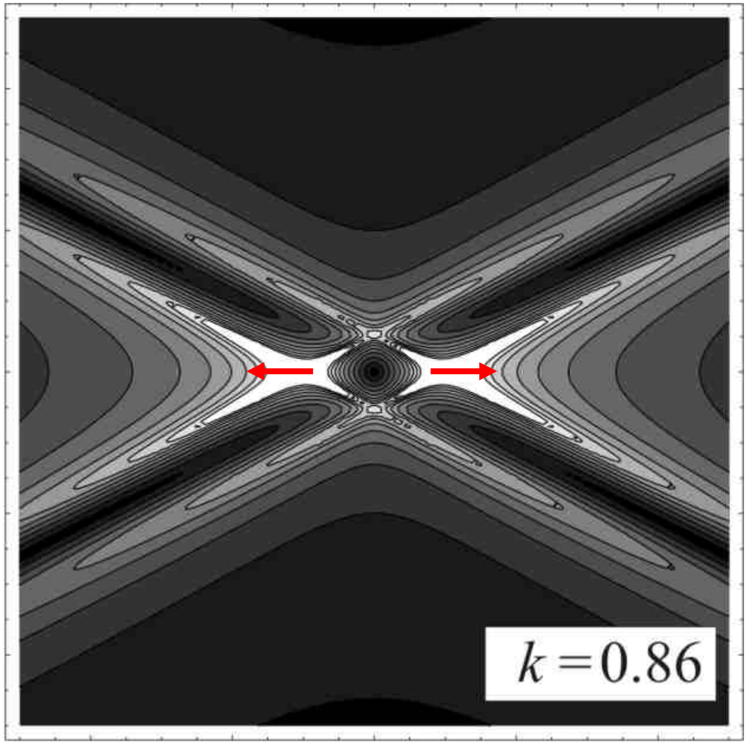
eigenvalues right and left eigenvectors of the acoustic tensor

Green's function for incremental deformation of a prestressed body

Bigoni & Capuani	JMPS 2002
Bigoni & Capuani	JMPS 2005
Piccolroaz, Bigoni & Willis	JMPS 2006

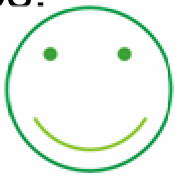
CONCENTRATED FORCES IN A PRESTRESSED MATERIAL





far from the elliptic boundary:
the usual elastic solution

near the elliptic boundary:
strain localization emerges!



An example ?

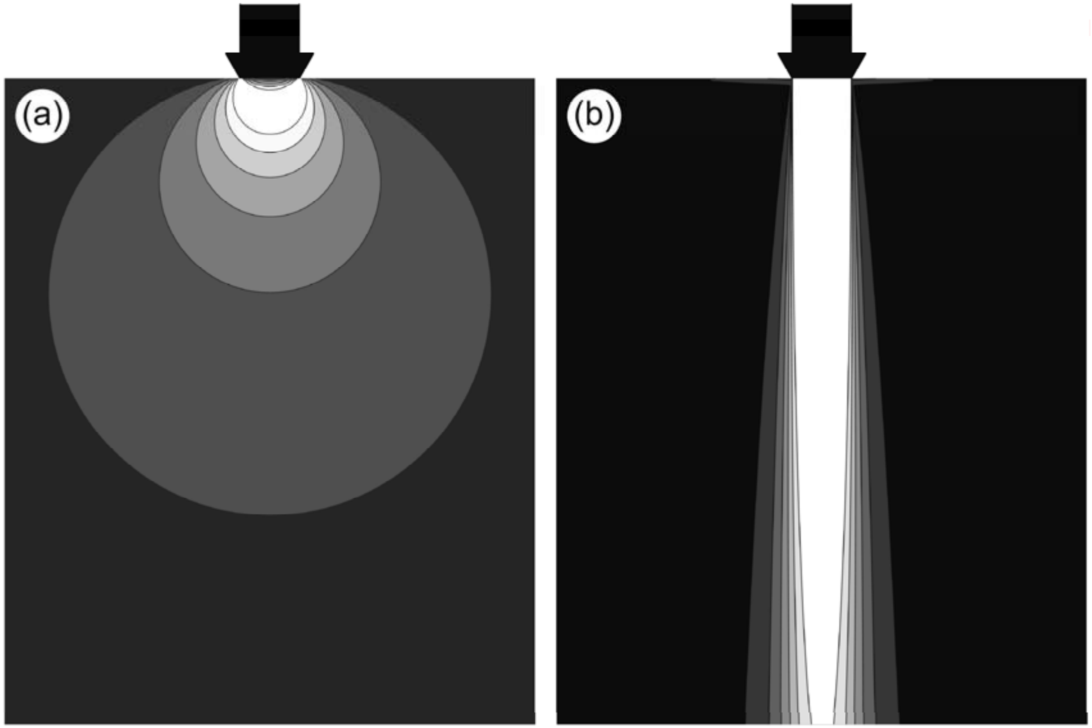
... a model of a dry masonry wall

DRY MASONRY WALL IS ON THE VERGE OF INSTABILITY

photoelastic model of a dry masonry wall

the material has an extreme orthotropy & is near the elliptic boundary

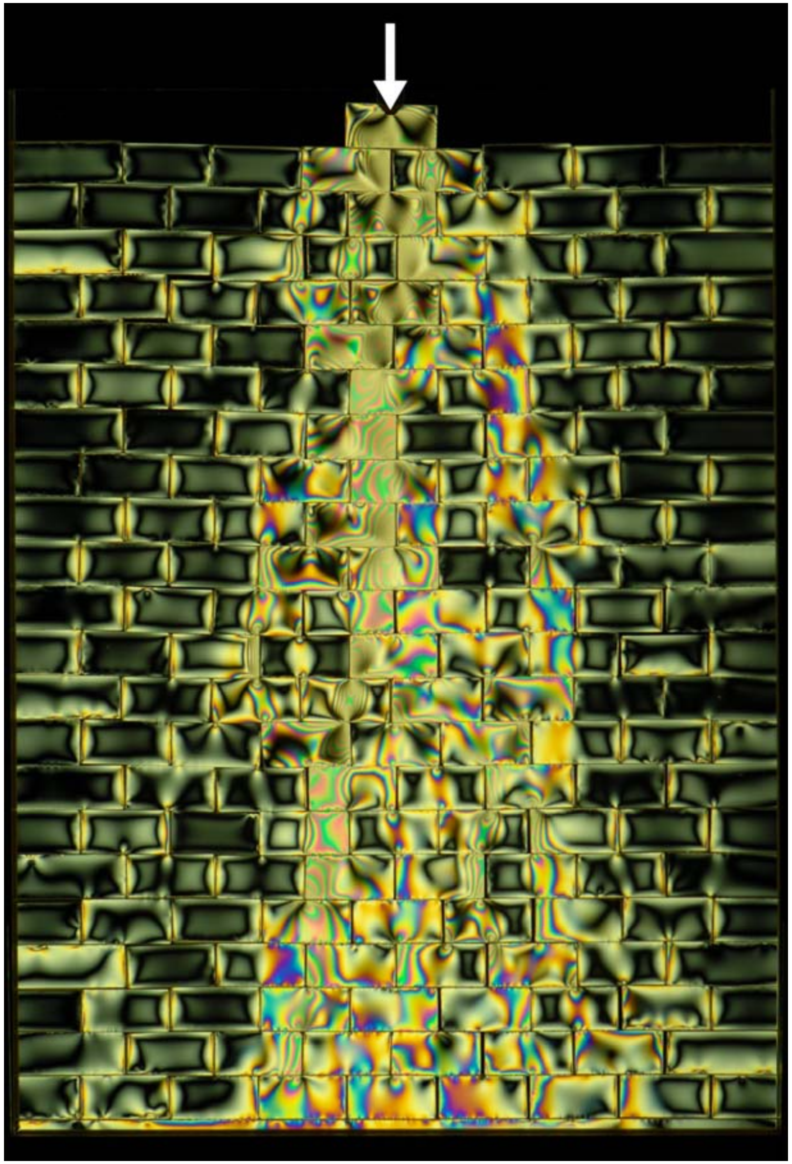
LOADING OF AN ELASTIC HALF SPACE



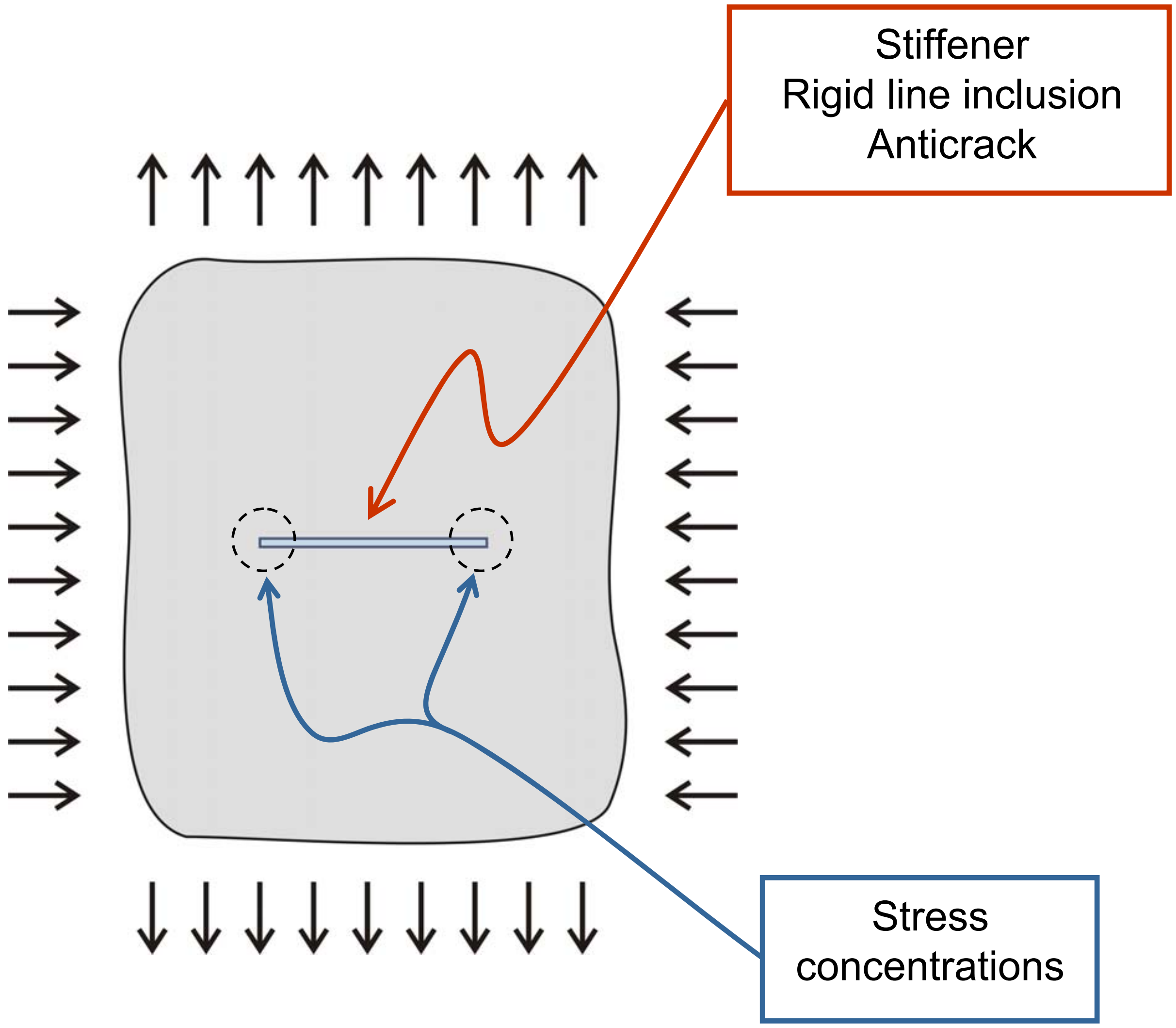
isotropy
far from ellipticity loss

extreme orthotropy
near ellipticity loss

STRESS PERCOLATION IN A DRY MASONRY WALL



A DIFFERENT PERTURBATION: A RIGID LINE INCLUSION



IS A THIN, RIGID INCLUSION ONLY A MATHEMATICAL MODEL?

A thin, rigid inclusion model poses new problems:

- Is the inclusion sufficiently thin to induce a stress ‘singularity’?
- Is the inclusion bonding enough strong to prevent de-cohesion?
- Unlike cracks, stiffeners produce a singularity also for compressive stresses. Is this true in practice?
- Is the hoop-stress criterion of fracture mechanics valid for a stiffener?

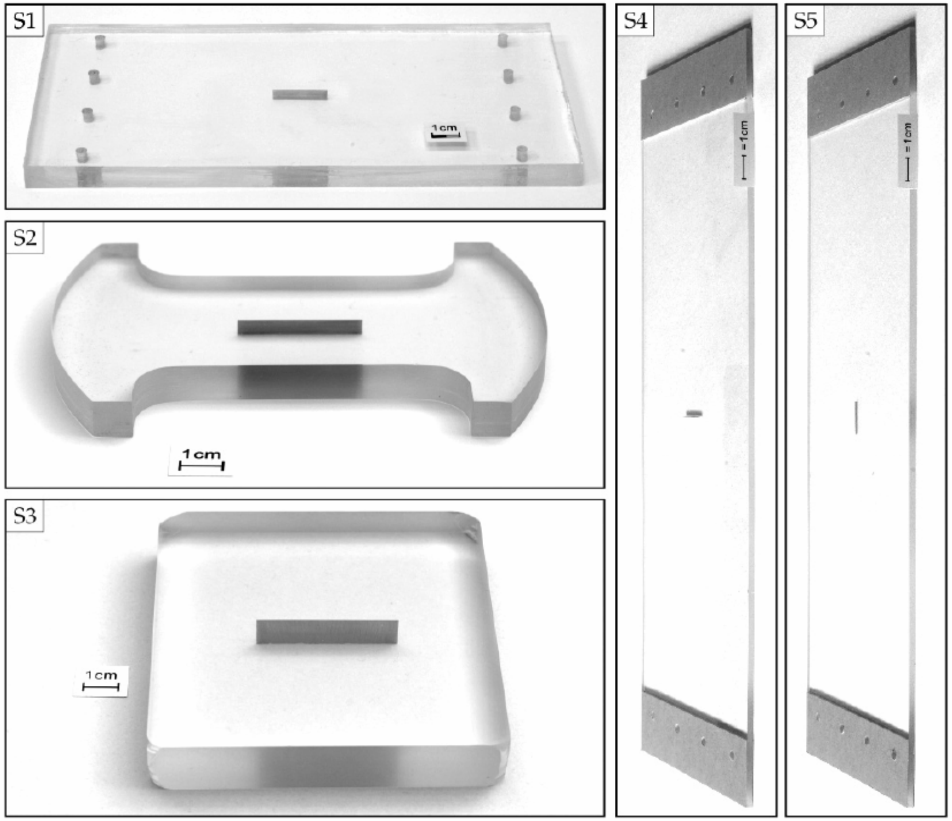
→ NO ANSWER PUBLISHED UNTIL NOW ...

LET’S PERFORM OUR OWN EXPERIMENTS!

EXPERIMENTS

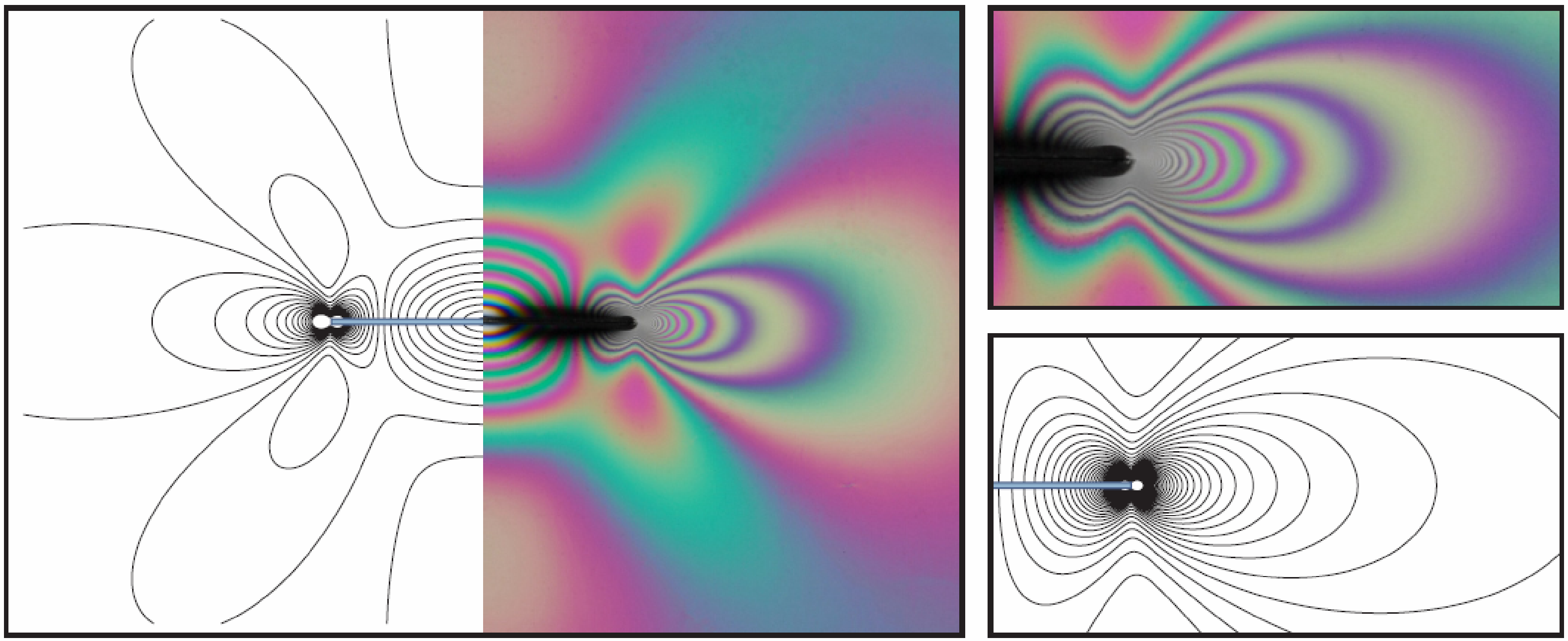
Matrix: transparent two-part epoxy resin

Inclusion: aluminum sheet (0.3 mm thick) with improved superficial rugosity



PHOTOELASTIC EXPERIMENTS MATCH ANALYTICAL RESULTS

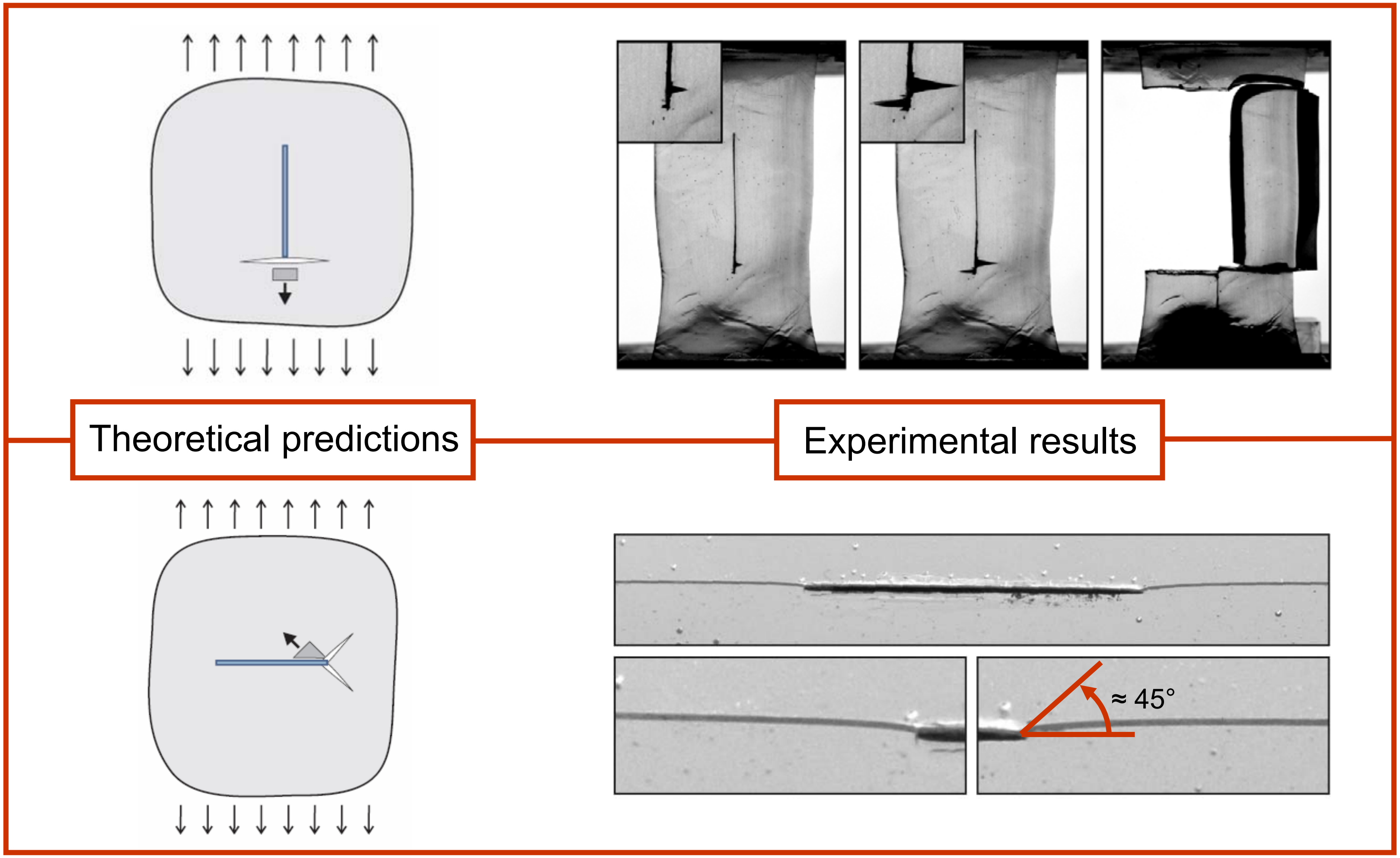
A stiff, thin inclusion has finite thickness and stiffness, and adhesion at the stiffener/matrix contact is necessarily imperfect but…



Comparison between analytical (black and white) and photoelastic results

…STIFFENER IS A SOUND MODEL!!

STRANGE FAILURE MODES OF A STIFFENER IN A BRITTLE MATERIAL



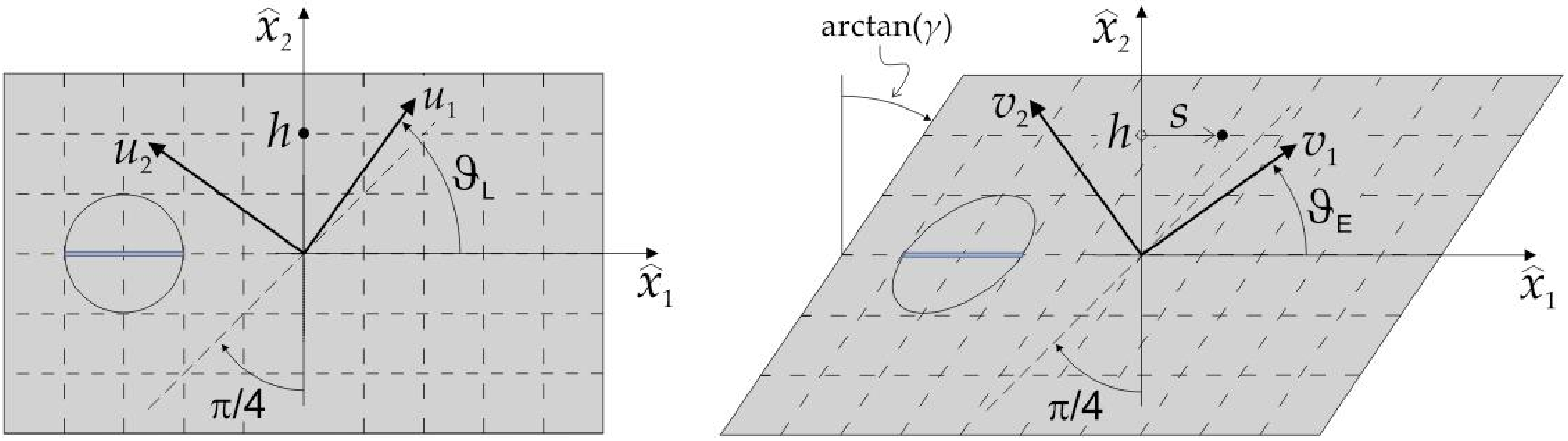
THE HOOP STRESS CRITERION

–TYPICAL OF FRACTURE MECHANICS–

IS NOT VALID!!

INCREMENTAL MODE I PERTURBATION SUPERIMPOSED TO SHEAR

Under a simple shear deformation, stiffener does not perturb the homogeneous state of stress



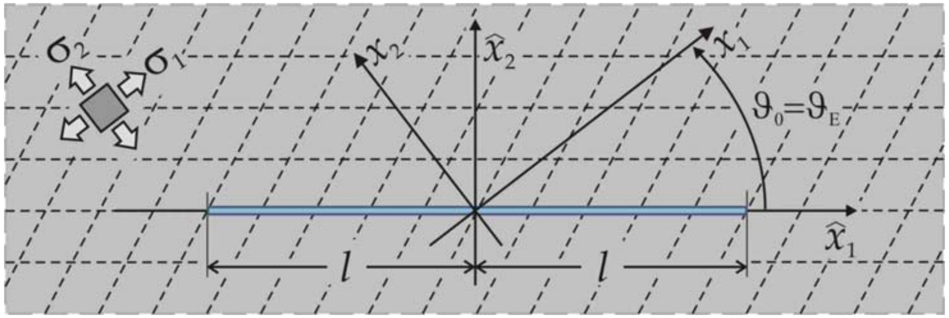
The principal axes of prestress are inclined w.r.t. the stiffener line

$$\vartheta_E = \frac{1}{2}\arctan\left(\frac{2}{\gamma}\right), \qquad \vartheta_L = \frac{\pi}{2} - \vartheta_E$$

INCOMPRESSIBLE INCREMENTAL NONLINEAR ELASTICITY

Linear incremental incompressible constitutive equations (Biot, 1965):

$$
\begin{aligned}
\dot{t}_{11} &= \mu(2\xi - k - \eta)v_{1,1} + \dot{p} \\
\dot{t}_{22} &= \mu(2\xi + k - \eta)v_{2,2} + \dot{p} \\
\dot{t}_{12} &= \mu[(1 + k)v_{2,1} + (1 - \eta)v_{1,2}] \\
\dot{t}_{21} &= \mu[(1 - \eta)v_{2,1} + (1 - k)v_{1,2}]
\end{aligned}
$$



$\dot{t}_{ij}$: nominal stress increment $\quad t_{ij} = JF^{-1}_{ik}\sigma_{kj}$

v_i : incremental displacement

$$\xi = \frac{\mu_*}{\mu}$$

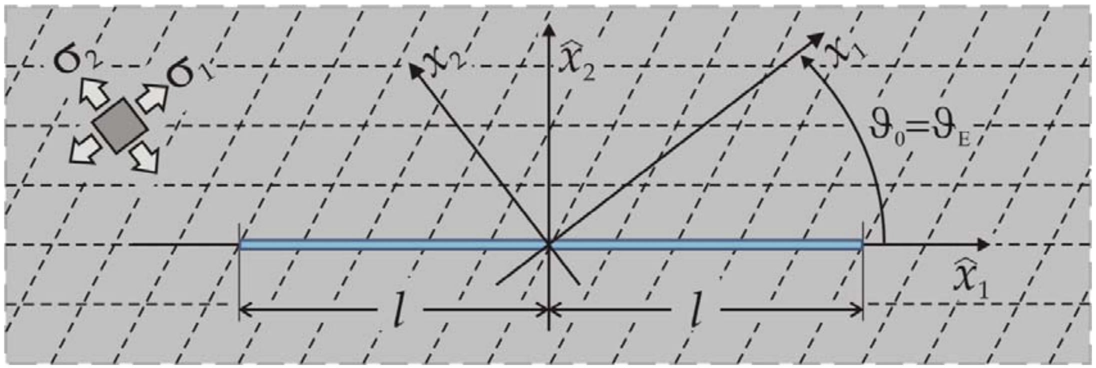
Incremental modulus for a shear inclined at π/4 to the principal axes x_1–x_2

Incremental modulus for a shear parallel to the principal axes x_1–x_2

$$\eta = \frac{p}{\mu} = \frac{\sigma_1 + \sigma_2}{2\mu}$$

$$k = \frac{\sigma_1 - \sigma_2}{2\mu}$$

"INCLINED" STIFFENER



$$\hat{\mathbf{t}} = \widehat{\mathbb{K}}[\hat{\nabla}\hat{\mathbf{v}}^T] + \dot{p}\,\mathbf{I}$$

$$\hat{\mathbf{t}} = \mathbf{Q}^T\dot{\mathbf{t}}\mathbf{Q}, \qquad \hat{\nabla}\hat{\mathbf{v}} = \mathbf{Q}^T\nabla\mathbf{v}\mathbf{Q}, \qquad \widehat{\mathbb{K}}_{ijhk} = Q_{li}Q_{mj}\mathbb{K}_{lmno}Q_{nh}Q_{ok}$$

$$\hat{z}_j = \hat{x}_1 + W_j\,\hat{x}_2, \qquad W_j = \frac{\sin\vartheta_0 + \Omega_j\cos\vartheta_0}{\cos\vartheta_0 - \Omega_j\sin\vartheta_0}, \qquad (j = 1, ..., 4) \qquad [\mathbf{Q}] = \begin{bmatrix} \cos\vartheta_0 & \sin\vartheta_0 \\ -\sin\vartheta_0 & \cos\vartheta_0 \end{bmatrix}$$

$$\hat{v}_1 = \frac{\partial\widehat{\psi}}{\partial\hat{x}_2}, \qquad \hat{v}_2 = -\frac{\partial\widehat{\psi}}{\partial\hat{x}_1}$$

FULL-FIELD SOLUTION FOR A STIFFENER IN A PRESTRESS. MATER.

$$\psi = \psi^{(\circ)} + \psi^{(\infty)} = -\frac{v_{2,2}^{(\infty)}}{4\,\alpha}\left\{\mathrm{Re}\left[\beta^2\frac{z_1^2 - z_2^2}{\alpha^2+\beta^2} - z_1\sqrt{z_1^2 - l^2} + z_2\sqrt{z_2^2 - l^2} + l^2\ln\left(\frac{z_1 + \sqrt{z_1^2 - l^2}}{z_2 + \sqrt{z_2^2 - l^2}}\right)\right] + \mathrm{Im}\left[\alpha\beta\frac{z_1^2 + z_2^2}{\alpha^2+\beta^2}\right]\right\}$$

$$\dot{p} - \dot{p}^{(\infty)} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{-\alpha[2(1-k)\beta^2 + k]\,\mathrm{Re}\left[2 - \frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - \beta[2(1-k)\alpha^2 - k]\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right]\right\},$$

$$\dot{t}_{11} - \dot{p}^{(\infty)} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{(\beta\delta + \chi\alpha)\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} + \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - (\alpha\delta - \chi\beta)\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - 2\alpha[2(1-k)\beta^2 + k]\right\},$$

$$\dot{t}_{22} - \dot{p}^{(\infty)} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{(\beta\delta - \chi\alpha)\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} + \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - (\alpha\delta + \chi\beta)\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - 2\alpha[2(1-k)\beta^2 + k]\right\},$$

$$\dot{t}_{12} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{(\chi\beta^2 - \chi\alpha^2 + 2\alpha\beta\delta)\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - (\delta\alpha^2 - \delta\beta^2 + 2\alpha\beta\chi)\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} + \frac{z_2}{\sqrt{z_2^2 - l^2}}\right]\right\},$$

$$\dot{t}_{21} = -\frac{\mu\, v_{2,2}^{(\infty)}}{2\,\alpha}\left\{\chi\,\mathrm{Re}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} - \frac{z_2}{\sqrt{z_2^2 - l^2}}\right] - \delta\,\mathrm{Im}\left[\frac{z_1}{\sqrt{z_1^2 - l^2}} + \frac{z_2}{\sqrt{z_2^2 - l^2}}\right]\right\},$$

$$\dot{p}^{(\infty)} = \frac{\dot{t}_{11}^{(\infty)}(2\xi + k - \eta) + \dot{t}_{22}^{(\infty)}(2\xi - k - \eta)}{2(2\xi - \eta)}$$

Coefficients depending on prestress

$$\chi = 2\xi - \eta, \quad \delta = 2(1-k)\alpha\beta$$

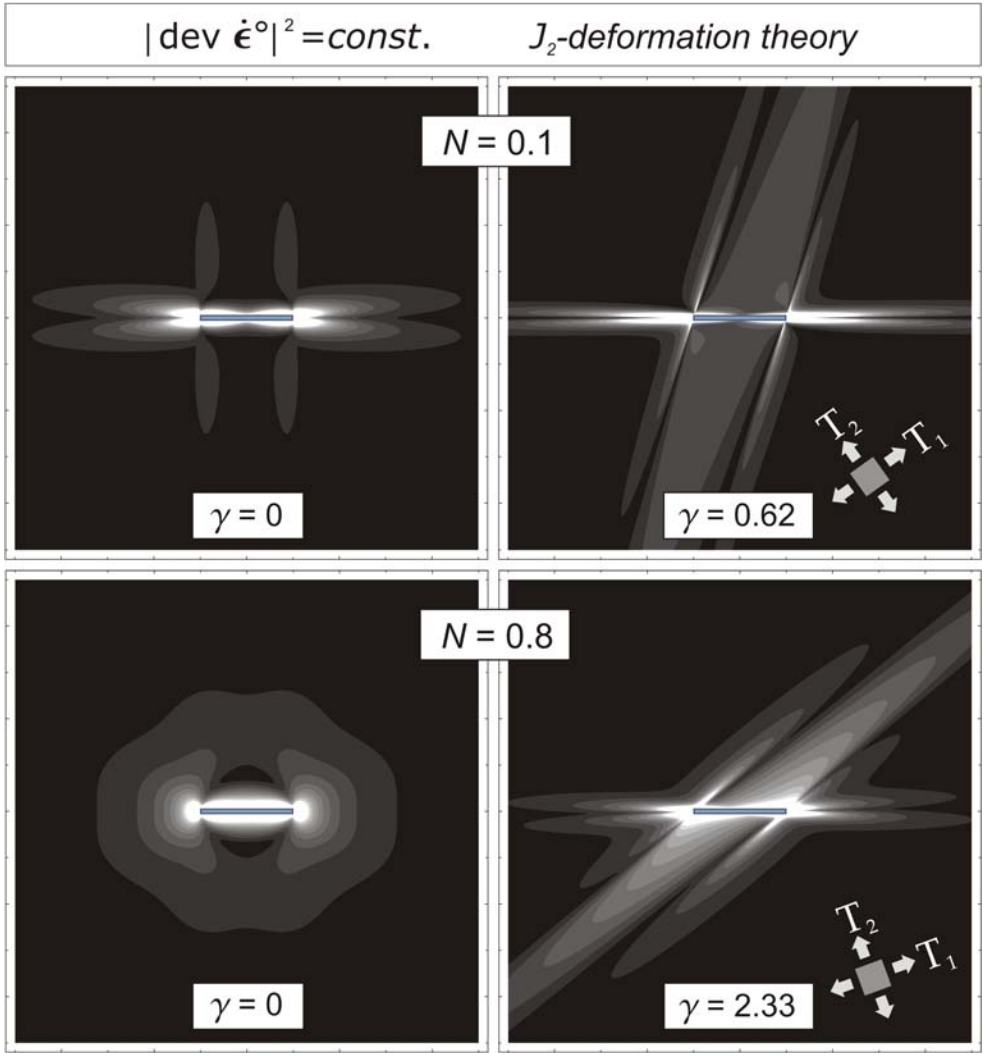
When $z_i \to \pm l$ (stiffener tips) $\Rightarrow \quad \dot{t}_{ij} \sim \dfrac{1}{\sqrt{r}}$

For a uniform incremental strain $v_{2,2}^{(\infty)}$ at infinity:

$$\dot{K}_{(\epsilon)\,I} = \lim_{r\to 0} 2\mu\sqrt{2\pi r}\, v_{2,2}(r, \vartheta = 0) = 2\,\mu\, v_{2,2}^{(\infty)}\sqrt{\pi l}$$

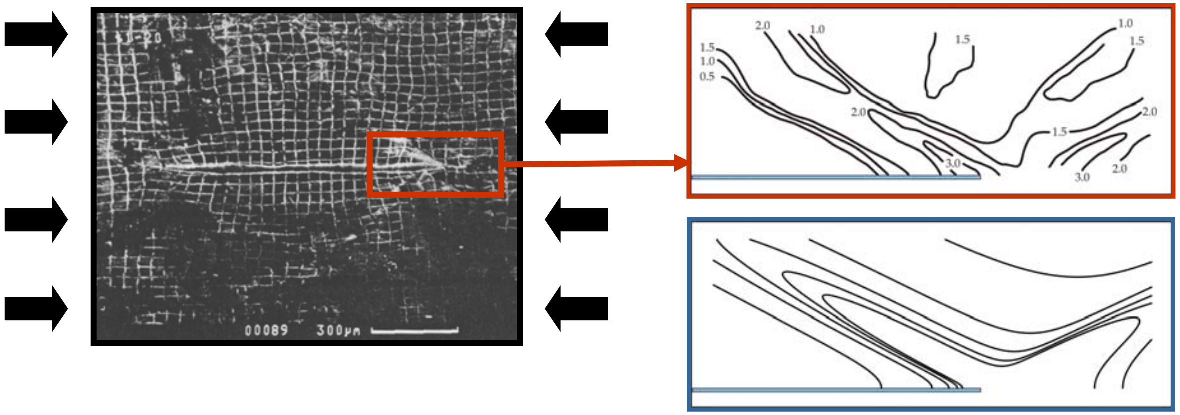
(same value of incompressible classical elasticity)

INCREMENTAL MODE I PERTURBATION SUPERIMPOSED TO FINITE SIMPLE SHEAR

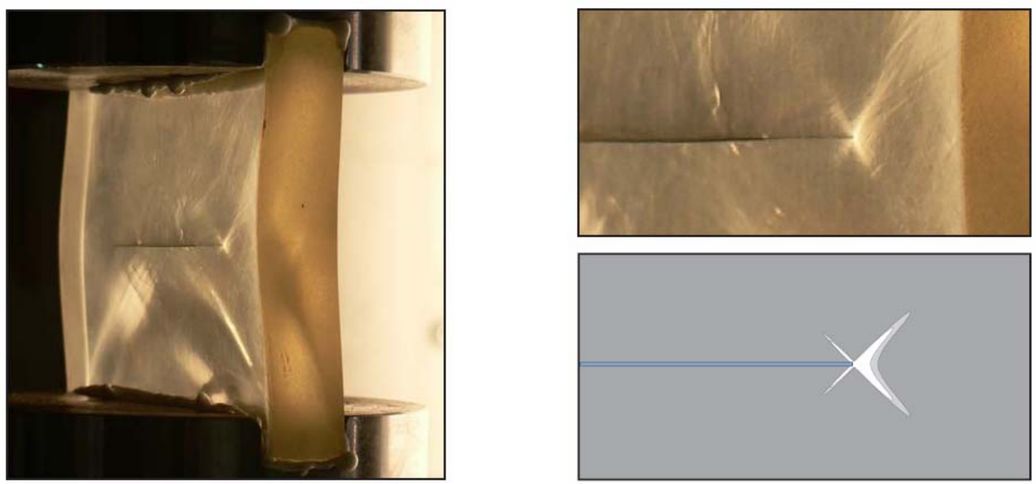


THE SHEAR BAND CLOSEST TO THE STIFFENER IS PRIVILEGED!!

EXPERIMENTAL EVIDENCE [1]

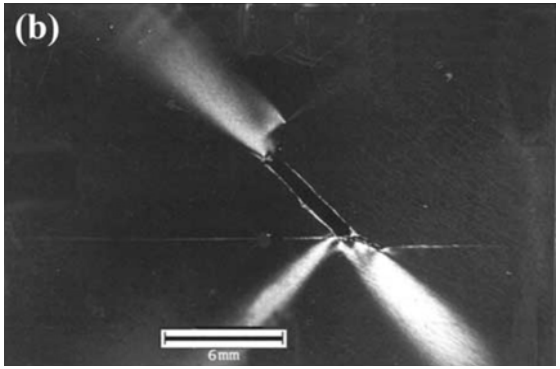


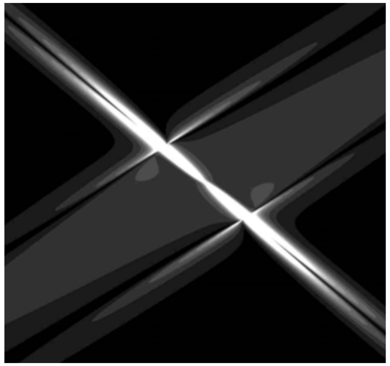
Cu-W laminates subject to large strain (Öztürk et al., 1991)



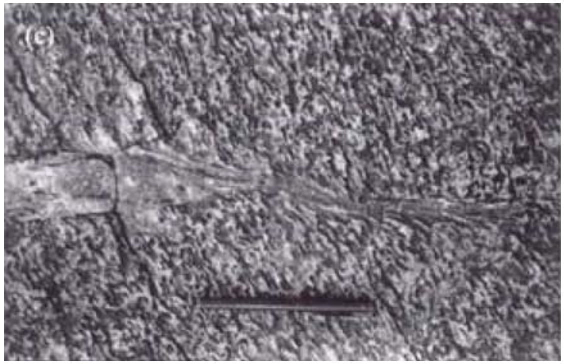
Compression test on an epoxy resin matrix containing an aluminum lamellar inclusion

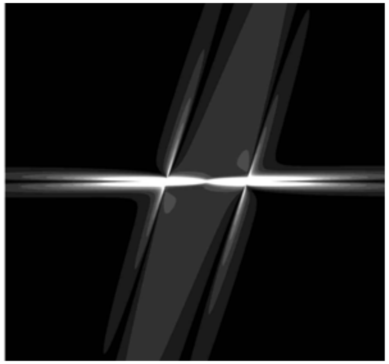
EXPERIMENTAL EVIDENCE [2]





Compression test on a PMMA block
containing a thin steel inclusion
(Misra and Mandal, 2007)





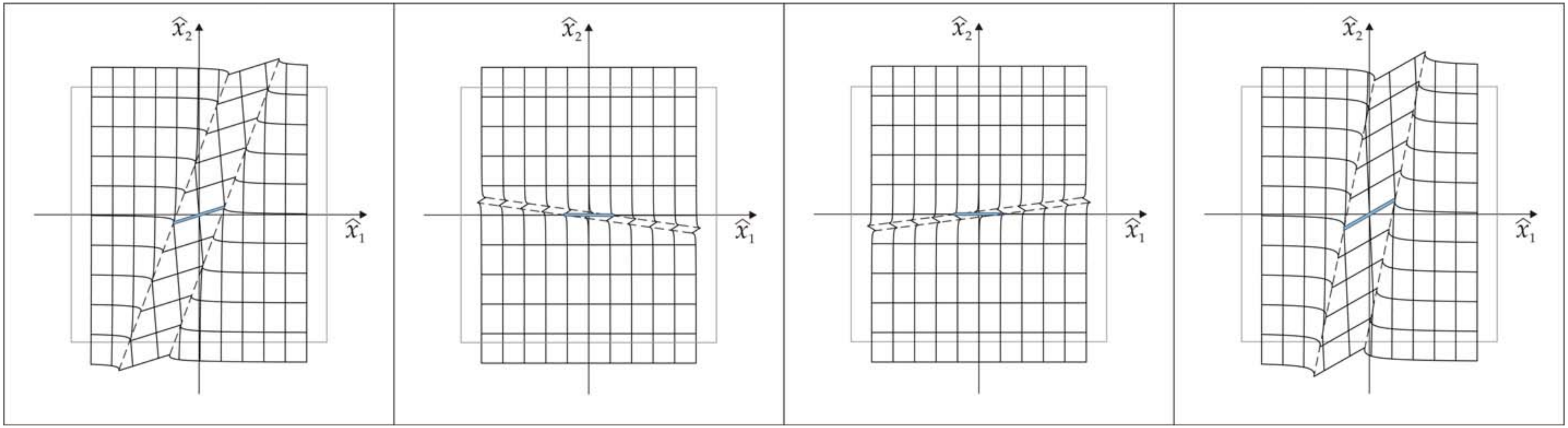
Localization of a single shear band
at the tip of a quartz vein
(Misra and Mandal, 2007)

Being analytical, our solution permits investigation of interactions between stiffener singularity and shear band formation and growth, a problem where numerical techniques can hardly have the necessary resolution...

MOREOVER, OUR ANALYTICAL SOLUTION CAN BE EXTENDED BEYOND THE ELLIPTIC BOUNDARY, INTO THE HYPERBOLIC REGION

(does it make sense? At least we can see what happens...)

SOLUTIONS OUTSIDE ELLIPTICITY: HYPERBOLIC REGIME



- an infinite number of solutions is possible
- these solutions DO NOT decay at infinity
- they correspond to infinite incremental strain/stress along shear bands
- these shear bands emanate from the tips of the stiffener

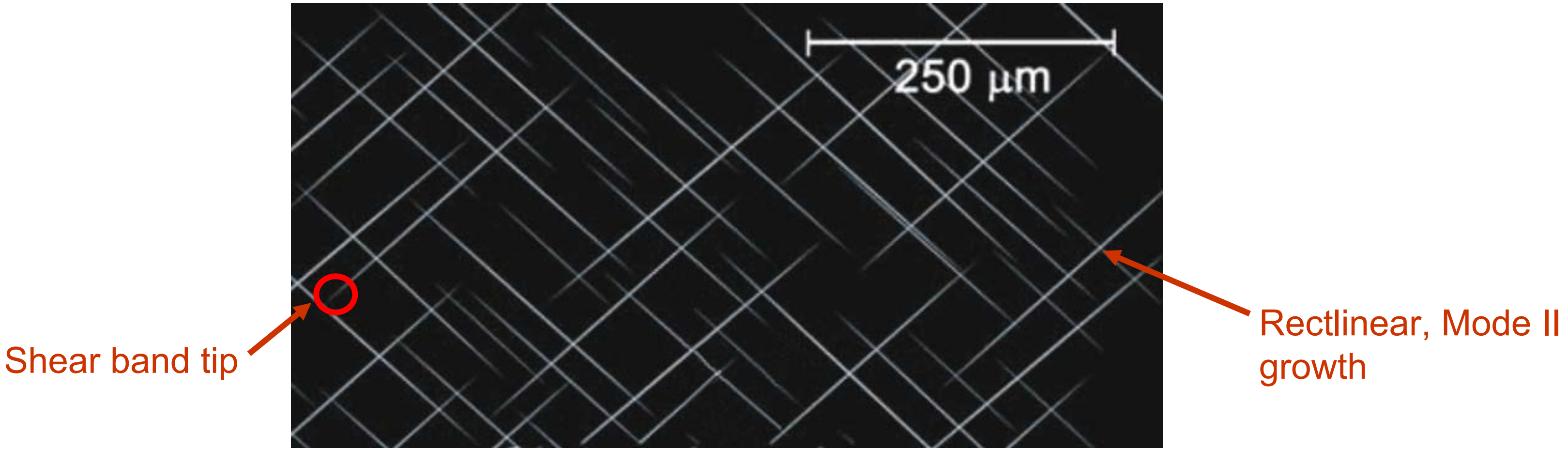
THIS JUSTIFIES –ANALYTICALLY– THE WELL-KNOWN PROBLEMS RELATED TO ILL-POSEDNESS OF A B.V.P.

Is the perturbative approach important in other circumstances ?

YES!! 

The unrestrainable growth of a shear band

OPEN PROBLEMS ON SHEAR BAND



Shear bands in polystyrene (Anand & Spitzig, 1982)

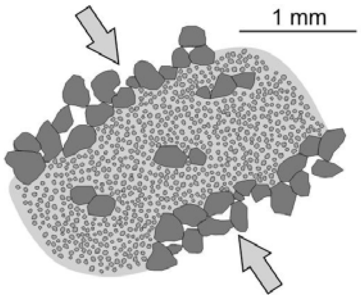
(after tirthy years of research...) there are still *many, important* <u>open</u> problems on shear bands:

1. how is the highly inhomogeneous stress state near a shear band tip ?
2. does it involve a stress concentration ('singularity') ?
3. why, unlike cracks, do shear bands grow rectilinearly and for 'long distances' under Mode II loading ?
4. why are shear bands a preferred failure mode for ductile materials ?

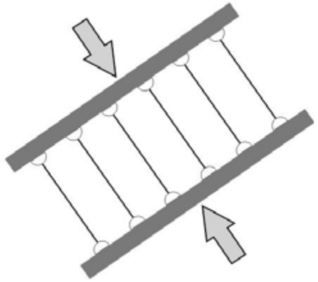
.... let's employ the perturbative approach ...

SHEAR BAND MODEL

deformation band observed
in dry sandstone
(Sulem and Ouffroukh, 2006)



zero thickness hinged
quadrilateral model



- Null incremental nominal shearing tractions:

$$\hat{t}_{21}(\hat{x}_1, 0^{\pm}) = 0, \quad \forall |\hat{x}_1| < l$$

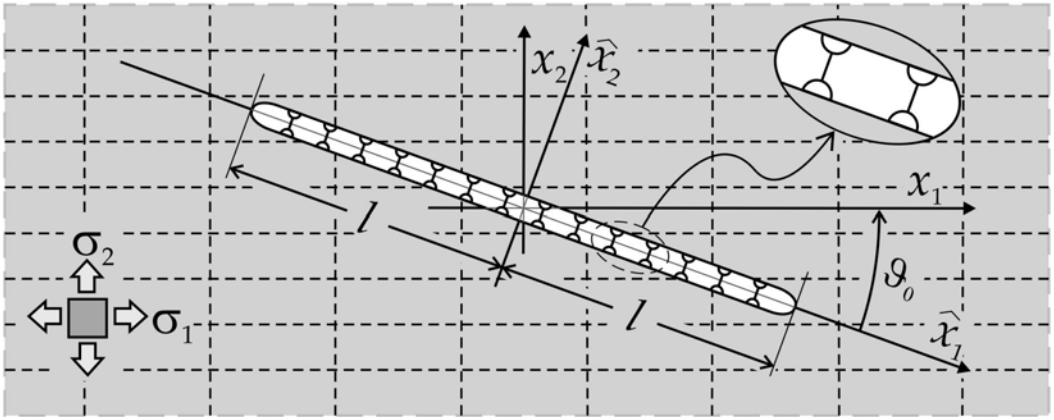
- Continuity of the incremental nominal normal traction:

$$[[\hat{t}_{22}(\hat{x}_1, 0)]] = 0, \quad \forall |\hat{x}_1| < l$$

- Continuity of normal incremental displacement:

$$[[\hat{v}_2(\hat{x}_1, 0)]] = 0, \quad \forall |\hat{x}_1| < l$$

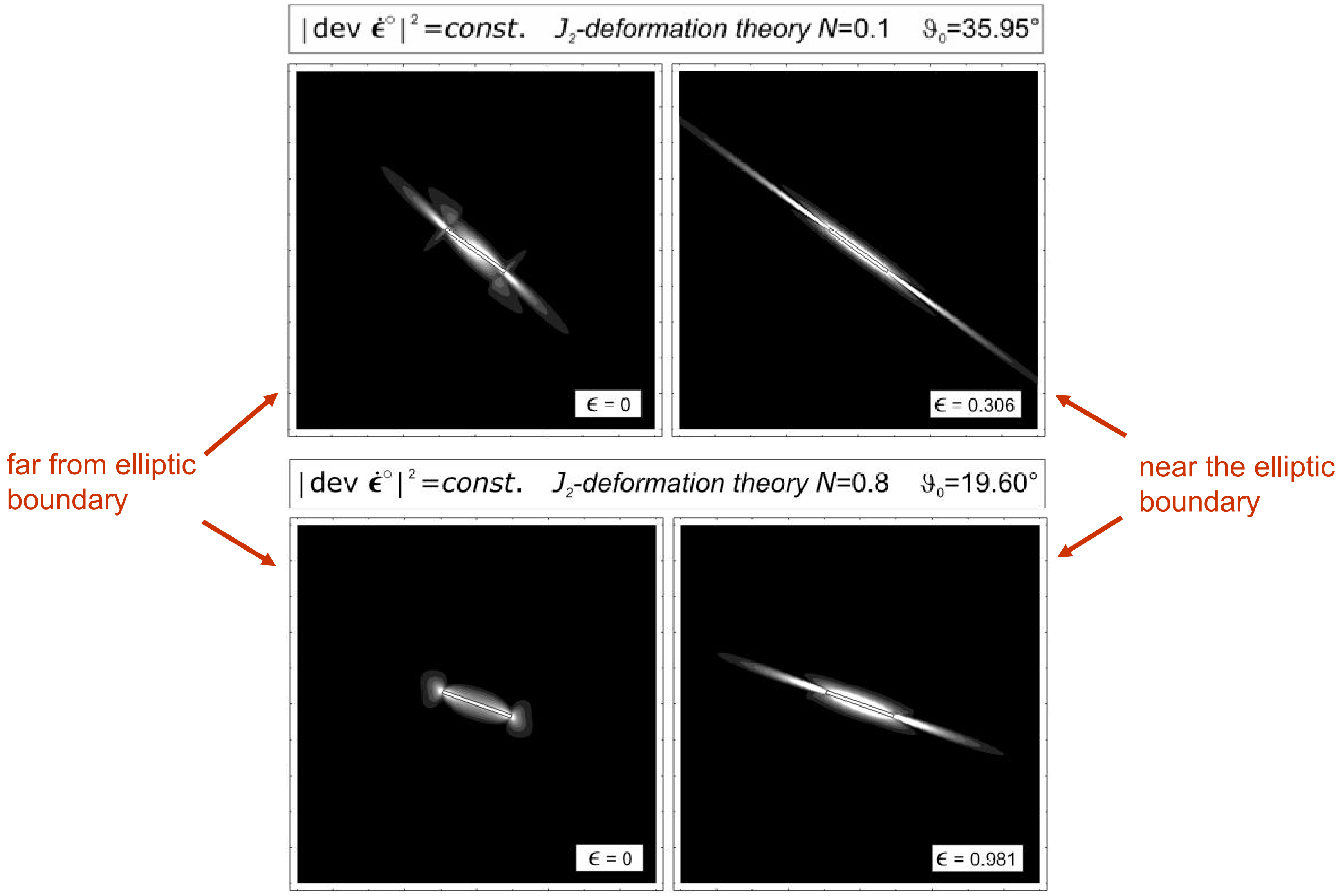
SHEAR BAND GROWTH



$$\hat{\psi}^{\circ}(\hat{x}_1, \hat{x}_2) = \frac{\hat{t}_{21}^{\infty}}{2\mu} \sum_{j=1}^{2} \mathrm{Re}\left[B_j^{II} f(\hat{z}_j)\right]$$

$$\begin{bmatrix} -c_{21} & c_{11} & -c_{22} & c_{12} \\ c_{31} & c_{41} & c_{32} & c_{42} \\ -c_{41} & c_{31} & -c_{42} & c_{32} \\ 0 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathrm{Re}[B_1^{II}] \\ \mathrm{Im}[B_1^{II}] \\ \mathrm{Re}[B_2^{II}] \\ \mathrm{Im}[B_2^{II}] \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

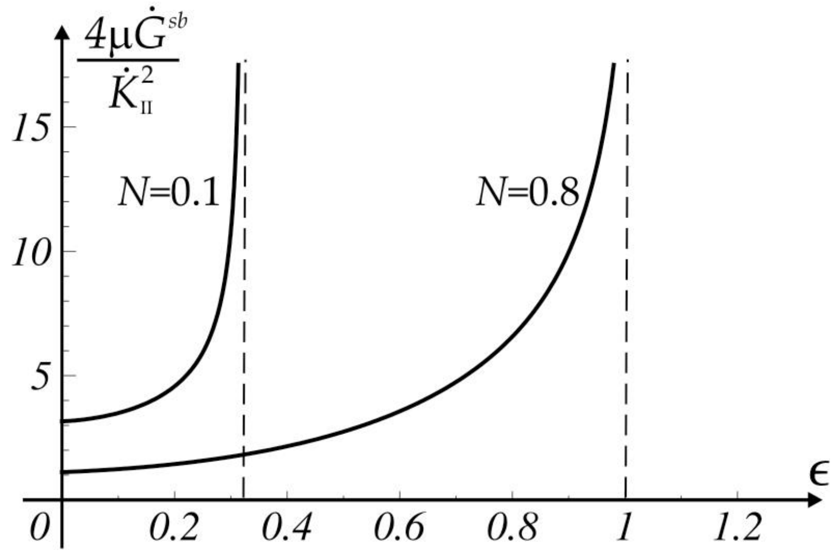
INCREMENTAL DEFORMATION FOCUSSED ALONG SHEAR BAND



does this explain the tendency toward Mode II propagation?

ENERGY RELEASE RATE FOR SHEAR BAND GROWTH

for a J_2 deformation theory material

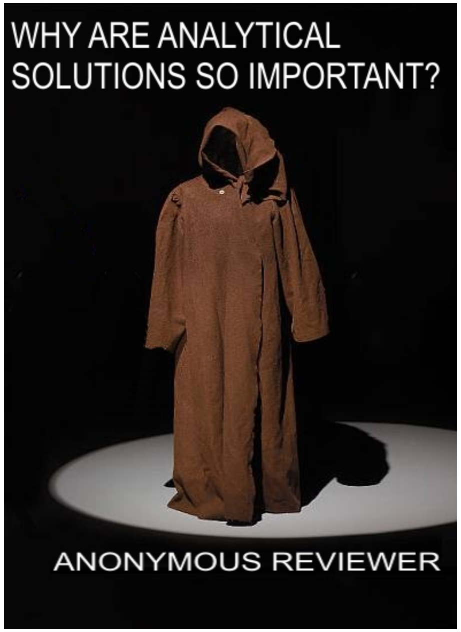


THE GROWTH BECOMES UNRESTRAINABLE!!

does this explain the fact that shear bands are preferred failure modes ?

CONCLUSIONS

ANALYTICAL SOLUTIONS TO FAILURE PROBLEMS INVOLVING INTERACTIONS BETWEEN SHEAR BANDS AND SINGULARITIES ARE POSSIBLE USING THE PERTURBATIVE APPROACH PROPOSED BY BIGONI & CAPUANI (*)



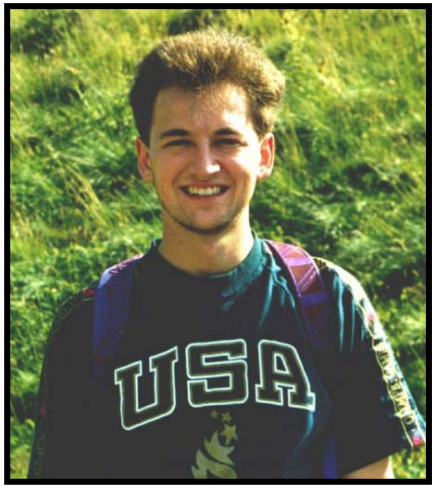
DIFFERENTLY FROM NUMERICAL SOLUTIONS, ANALYTICAL SOLUTIONS HAVE THE NECESSARY RESOLUTION TO DETAIL NEAR-FAILURE MECHANICAL FIELDS

(*) **D. Bigoni and D. Capuani, "Green's function for incremental nonlinear elasticity..." JMPS, 2002, 50, 471-500.**
D. Bigoni and D. Capuani, "Time-harmonic Green's function ..." JMPS, 2005, 53, 1163-1187.
A. Piccolroaz, D. Bigoni, and J.R. Willis, "A dynamical interpretation of flutter instability..."JMPS, 2006, 54, 2391-2417.

THANKS TO

Francesco Dal Corso

Massimiliano Gei



D. Bigoni and F. Dal Corso, The unrestrainable growth of a shear band in a prestressed material.
Proceedings of the Royal Society A 2008, in Press.

F. Dal Corso, D. Bigoni and M. Gei, The stress concentration near a rigid line inclusion in a prestressed, elastic material. Part I Full-field solution and asymptotics. Journal of the Mechanics and Physics of Solids, 2008, 56, 815–838.

D. Bigoni, F. Dal Corso and M. Gei, The stress concentration near a rigid line inclusion in a prestressed, elastic material. Part II Implications on shear band nucleation, growth and energy release rate. Journal of the Mechanics and Physics of Solids, 2008, 56, 839–857.

F. Dal Corso and D. Bigoni, Interactions between shear bands and rigid lamellar inclusions in a ductile metal matrix.
Proceedings of the Royal Society A 2008, in Press.

... and thank you for the attention!