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Joys and sorrows of FEM with strong discontinuities for the variational approximation of free-disc. problems

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Introduction

- Objective: FEM code for Variational Fracture
- Basic Ref's on VF: Braides 92, Del Piero 97, Francfort & Marigo 98 (ask Ref's to speaker)
- What is Variational Fracture?

… Energy is the sum of bulk and interface terms: Energy convenience determines not only the when (early idea of Griffith) but also the how.

- Energy depends not only on displacements but also on the "jump set" K:

$$F(K;\mathbf{u}) = \int_{\Omega\setminus K} \varphi(\varepsilon)d\mathbf{x} + \int_{K} \vartheta([\![\mathbf{u}]\!])ds\ ,$$

- Complex mathematical question: regularity of K
- Solved with the Direct Method of the Calculus of Variations -> Global Minima:

…K is regular if the interface energy has nice properties…

necessary ingredient:

Concavity

Introduction

Difficulties in using Variational Fracture

- Non interpenetration: fracture has a definite sign … inequalities.
- Permanence of fracture ….. Irreversibility → evolutionary global minimization (Francfort & Marigo)
- In reality evolution follows local not global minima …… sensitivity to energy barriers.

Numerical implementation of Variational Fracture

- Standard numerical strategy (Bourdin et al, Del Piero et al): regularization.

  Ambrosio-Tortorelli approach (kind of damage: fractures appear smeared, by tuning the damage parameter ε get possible discontinuity in the limit).
- Other approaches (similar):

  phase fields (physics community), eigenstrains (Ortiz et al)…..

No strong discontinuities: Fractures are smeared over strips

Extremely fine meshes required to locate cracks

Introduction

Our code:

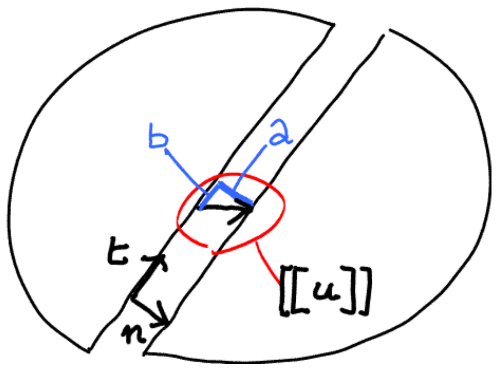
- Modelling quasi-static nucleation and propagation of cracks through FE with gaps …. Strong discontinuities

Numerical procedure:

- Descent Minimization …. local minima
- The VF model requires the ability to locate and approximate the crack. On adopting the “strong discontinuity” approach cracks cannot be restricted to the skeleton of a fixed FE mesh.
- … our mesh is variable: mesh nodes are taken as further unknowns (minimization over variable triangulations).
- With a Griffith type interface energy fracture nucleation is always brutal: descent directions for the energy do not exist in absence of singularities (such as a pre-existing crack).
- To reach more energetically convenient local minima the system must have the ability to surmount small energy barriers …

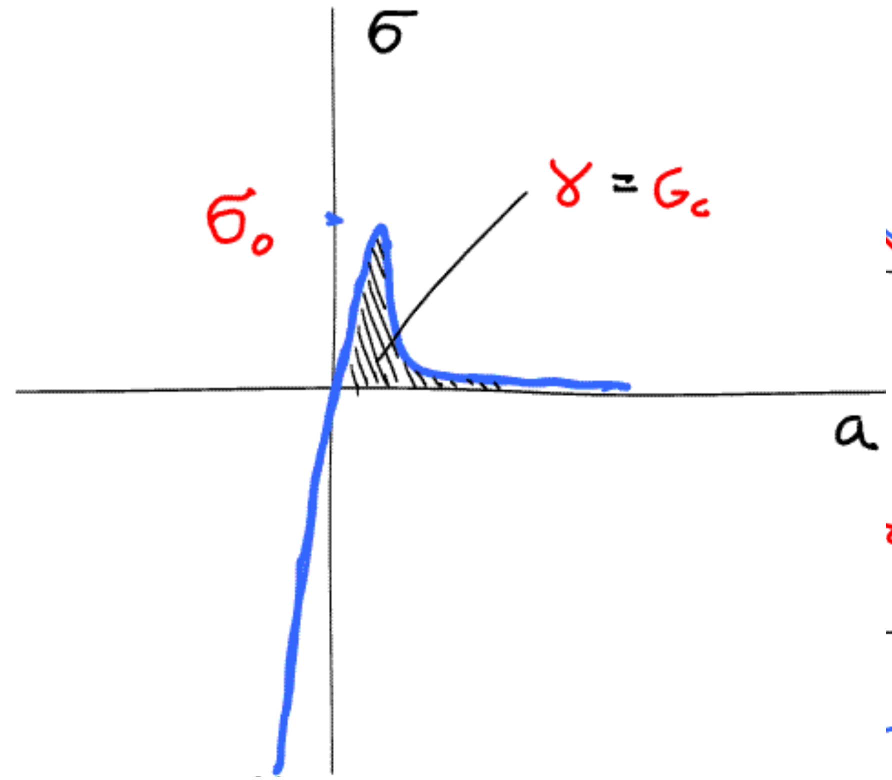
… energy relaxation (does not always work).

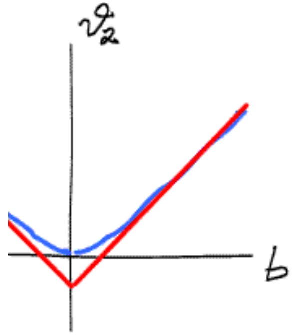
Relaxation of the interface energy

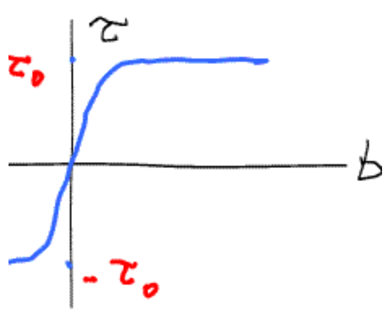


$$\vartheta_1(a) = \gamma + \frac{2\gamma}{\pi - 2}\left(e^{a\beta\frac{\sigma_0}{\gamma}} - 2\arctan\left[e^{a\beta\frac{\sigma_0}{\gamma}}\right]\right),$$

$$\vartheta_2(b) = \frac{2\tau_0}{k\pi^2}\left(h\pi b \arctan\left[\frac{h\pi b}{2\tau_0}\right] - \tau_0 \log\left[1 + \left(\frac{k\pi b}{2\tau 0}\right)^2\right]\right),$$

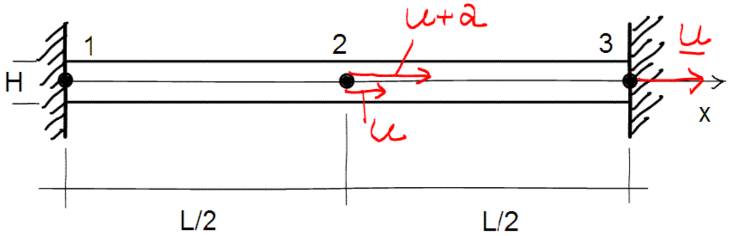






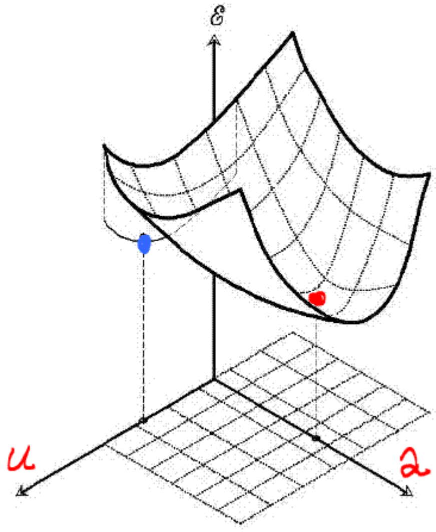
γ = G_c … fracture energy

Fracture Nucleation in 1d bar.



Fixed Mesh :

$$\mathcal{E}(u,a) = \begin{cases} \frac{1}{2}EA\frac{u^2}{L/2} + \frac{1}{2}EA\frac{(\underline{u}-u)^2}{L/2}, & a = 0, \\ \frac{1}{2}EA\frac{u^2}{L/2} + \frac{1}{2}EA\frac{(\underline{u}-u-a)^2}{L/2}, & a > 0, \\ +\infty, & a < 0. \end{cases}$$



Variable Mesh:

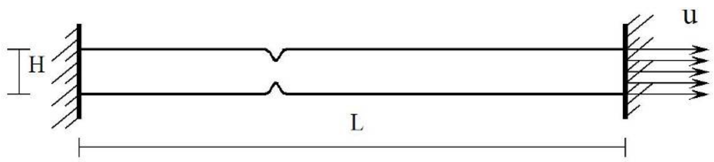
$$\mathcal{E}(u,a,x(2)) = \begin{cases} \frac{1}{2}EA\frac{u^2}{x(2)} + \frac{1}{2}EA\frac{(\underline{u}-u)^2}{L-x(2)}, & a = 0, \\ \frac{1}{2}EA\frac{u^2}{x(2)} + \frac{1}{2}EA\frac{(\underline{u}-u-a)^2}{L-x(2)}, & a > 0, \\ +\infty, & a < 0. \end{cases}$$

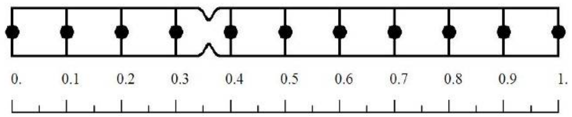
… but here changing mesh is useless. Preminimizing with respect to u …

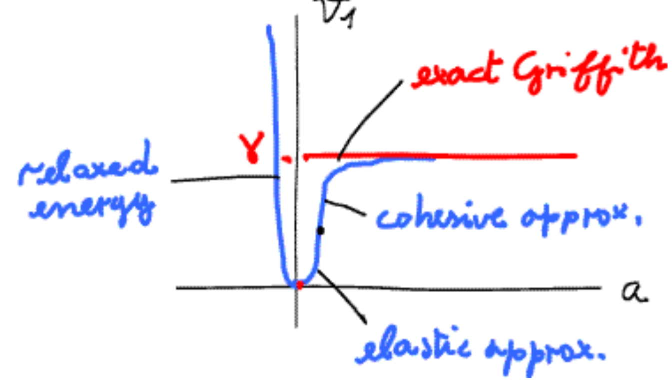
$$\mathcal{E}(u=u^\circ,a,x(2)) = \begin{cases} \frac{1}{2}EA\frac{\underline{u}^2}{L}, & a = 0, \\ \frac{1}{2}EA\frac{(\underline{u}-a)^2}{L} + \gamma, & a > 0, \\ +\infty, & a < 0. \end{cases}$$

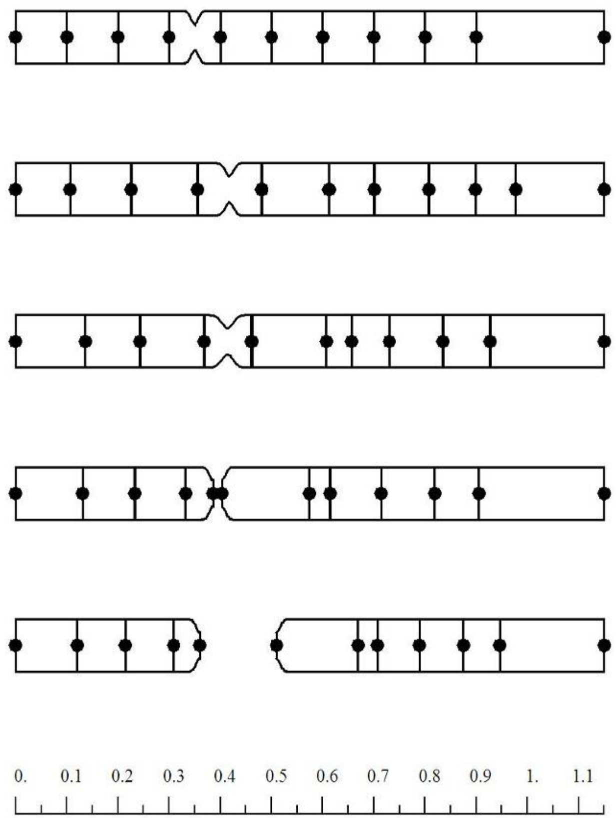
… independent of x(2)

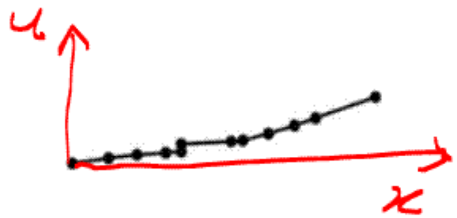
To make the movement of the interface “useful” …
… bar with weakened toughness γ at a given point …



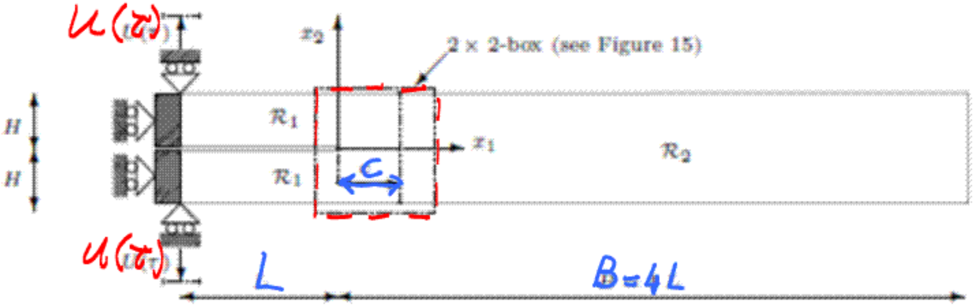






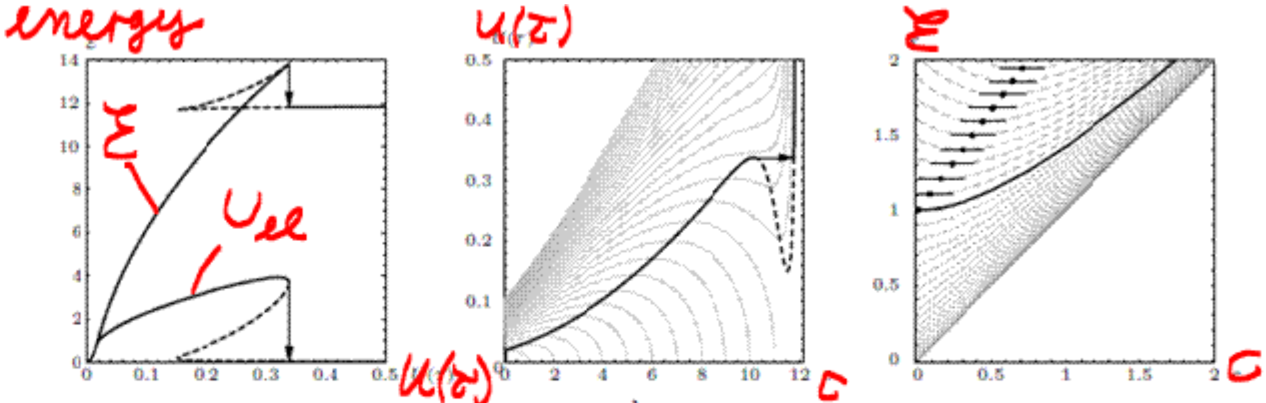


VALIDATION WITH CLASSICAL LINEAR FRACTURE MECHANICS (CLFM): PROPAGATION OF A STRAIGHT CRACK IN MODE I

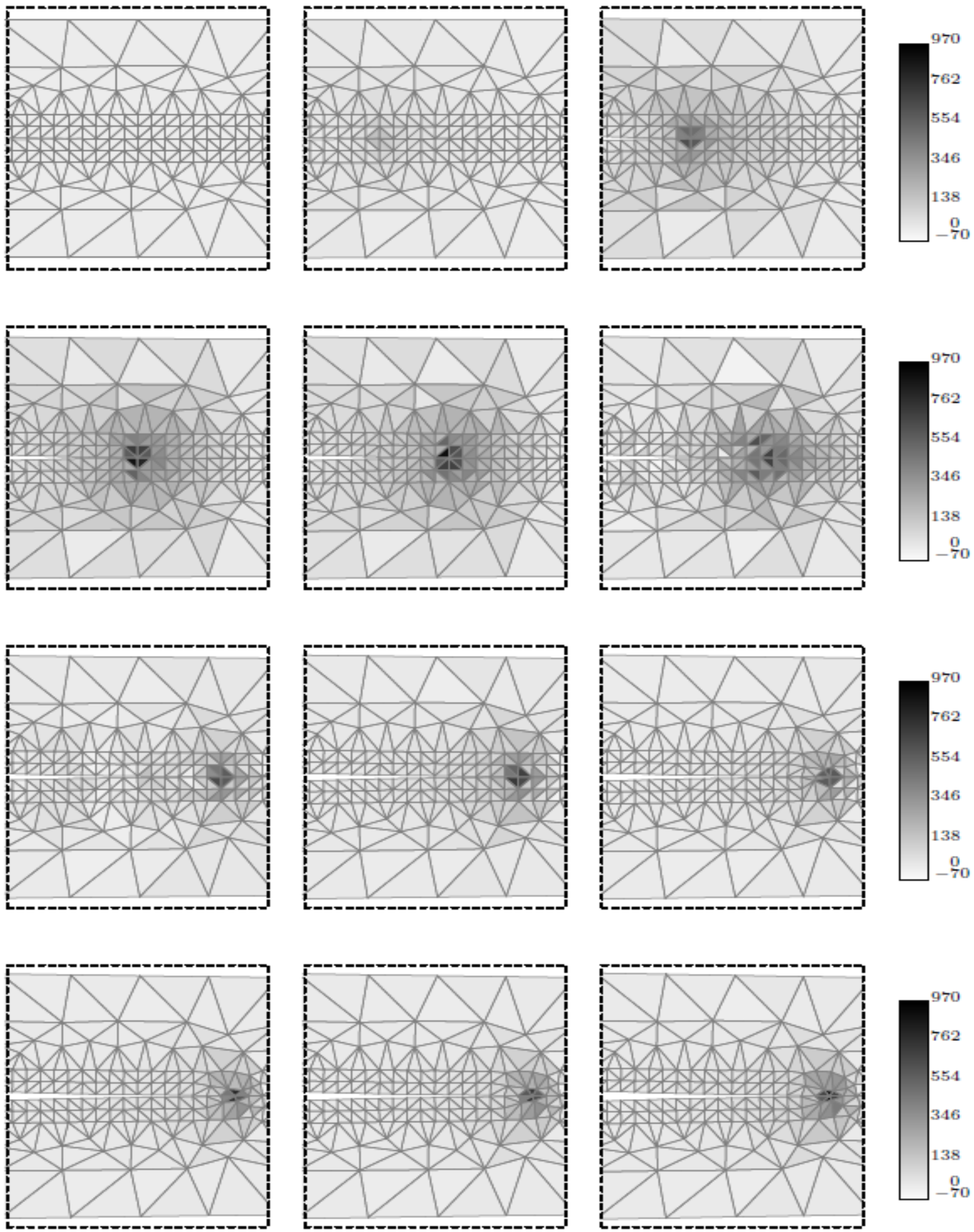


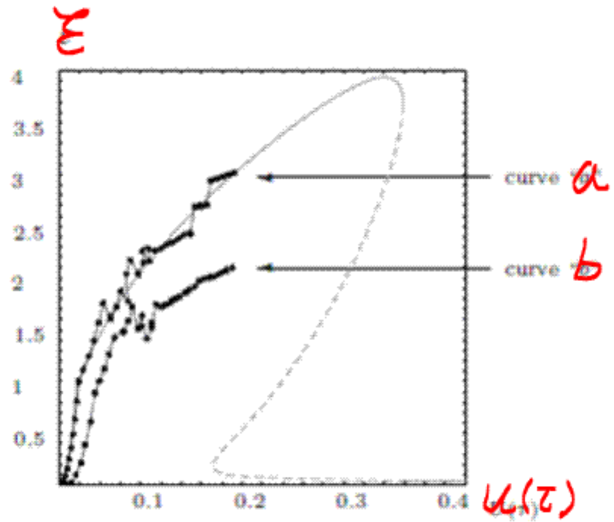
Height H (cm)	Length 5L (cm)	Thickness t (cm)	Young Mod. E (Ncm^{-2})	Poisson rat. ν	Thoughness G_c (Ncm^{-1})
1.0	15.0	1.0	$3.0\ 10^5$	0.0	1.0

Table1: Geometric and material constants.



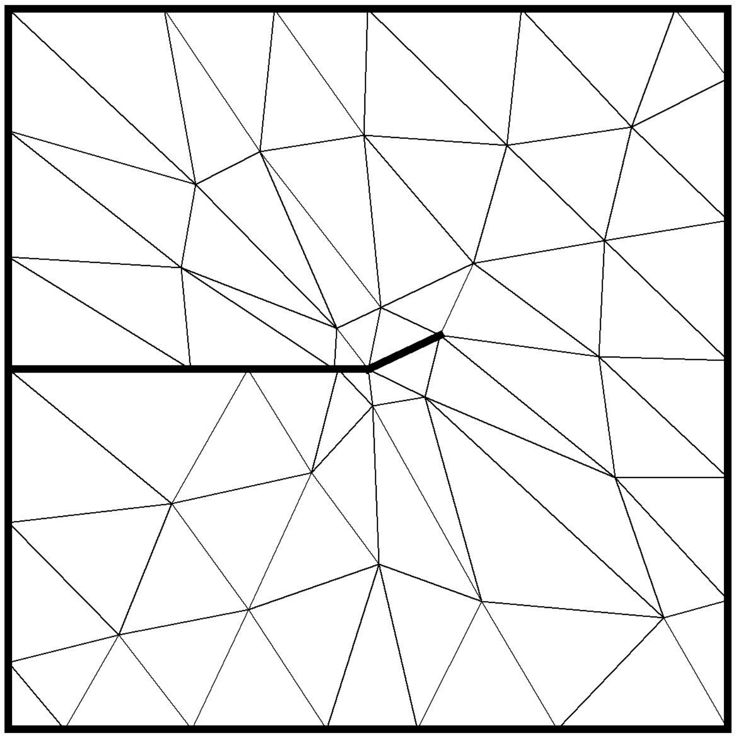
a. b. c.

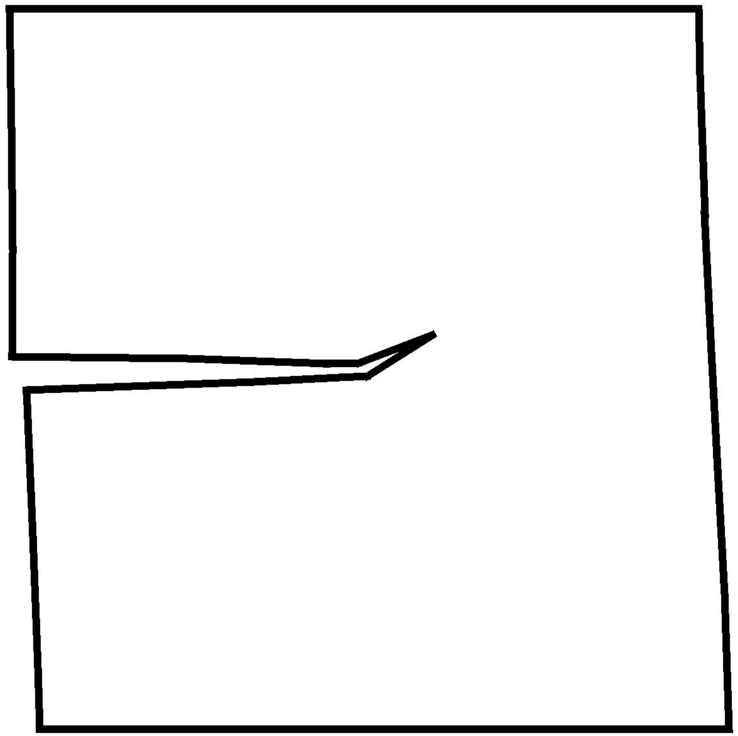

970
762
554
346
138
0
−70


ε
4
3.5
3
2.5
2
1.5
1
0.5
0.1
0.2
0.3
0.4
u(τ)
curve a
curve b

PROPAGATION: CONVERGENCE ANALYSIS IN MIXED MODE I&II.

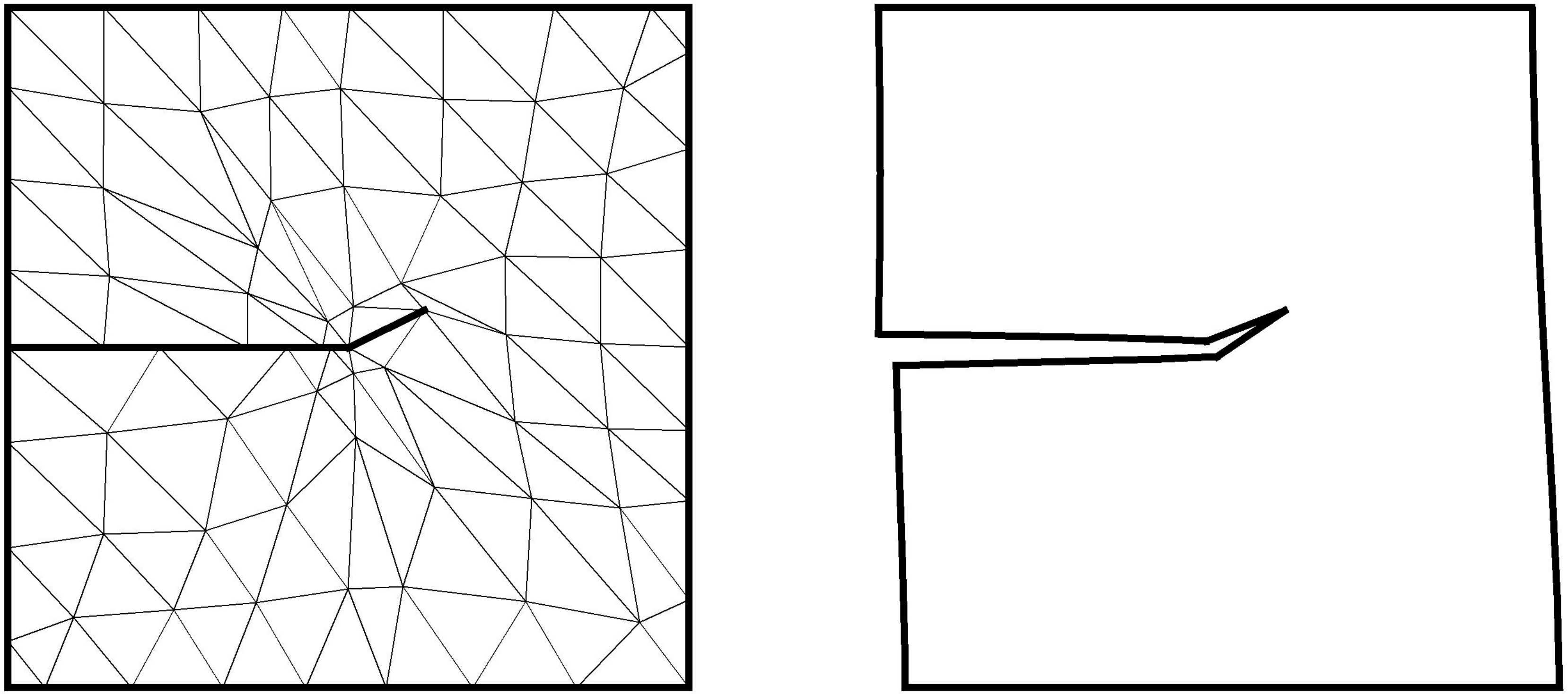
2 X 6 X 6 =72 elements, Structured mesh
3 X 72 = 216 nodes





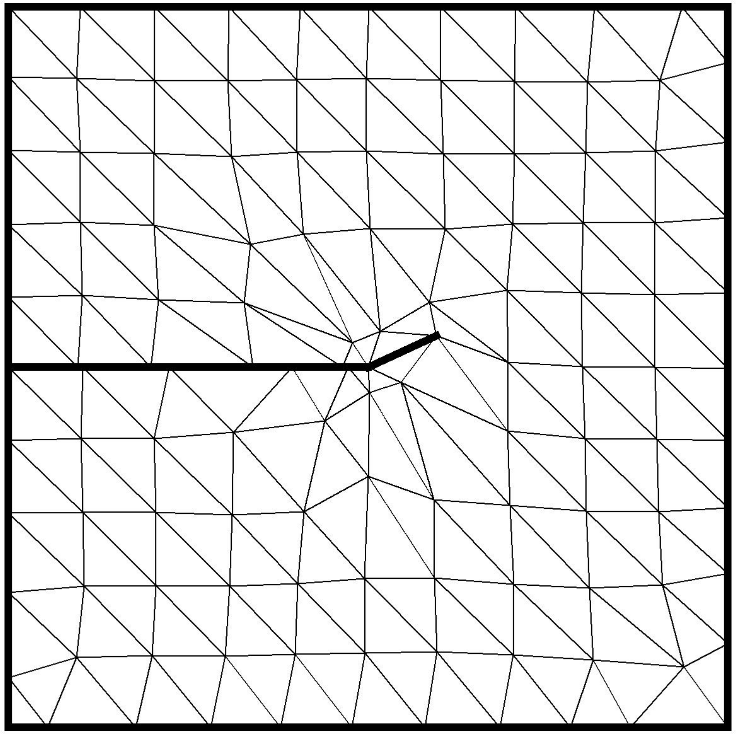
PROPAGATION: CONVERGENCE ANALYSIS IN MIXED MODE I&II.

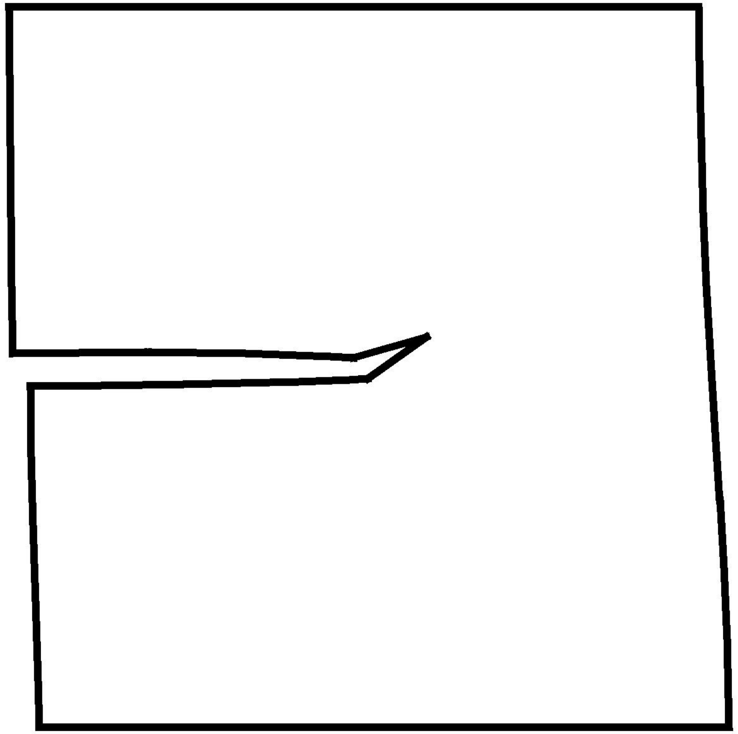
2 X 8 X 8 =128 elements, Structured mesh
3 X 128 = 384 nodes



PROPAGATION: CONVERGENCE ANALYSIS IN MIXED MODE I&II.

2 X 10 X 10 =200 elements, Structured mesh
3 X 200 = 600 nodes

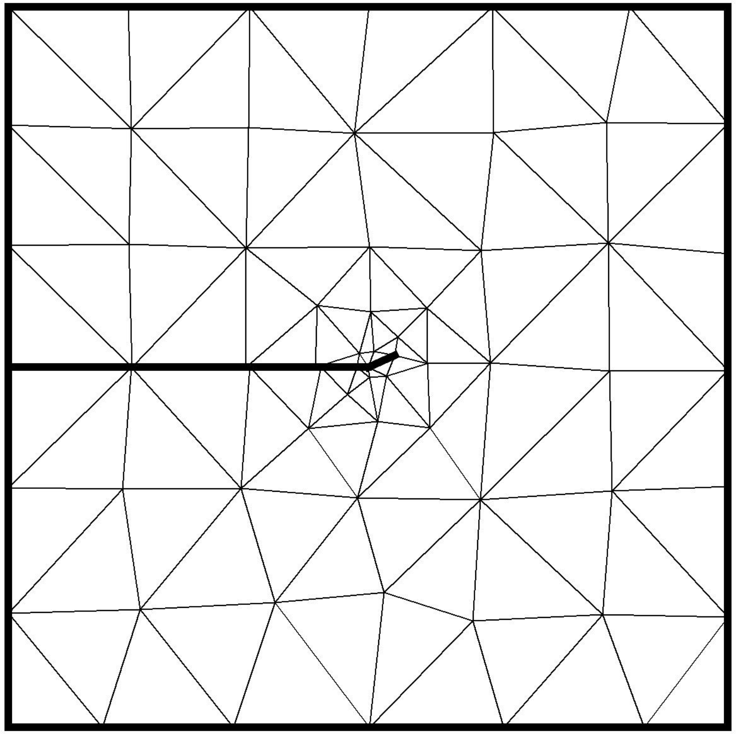


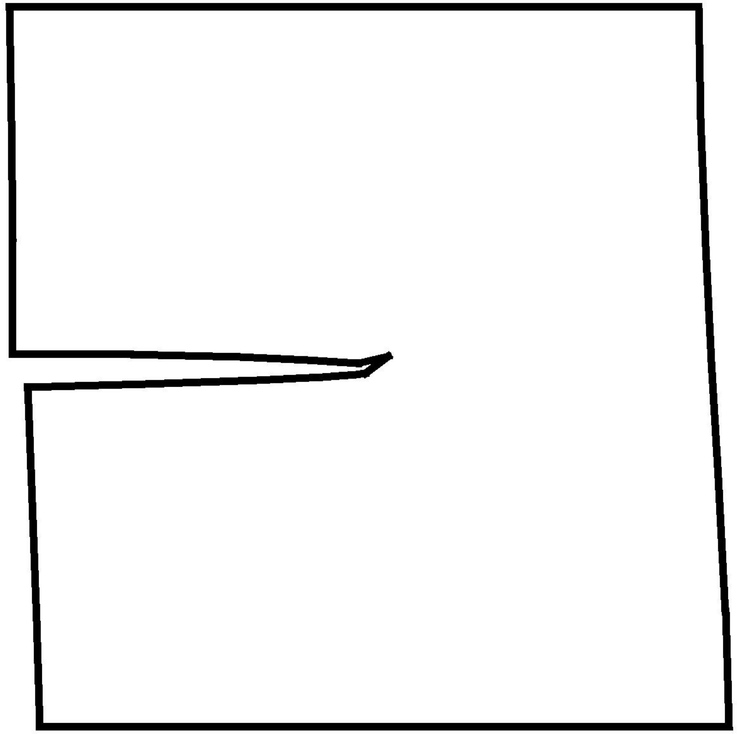


PROPAGATION: CONVERGENCE ANALYSIS IN MIXED MODE I&II.

Structured mesh, refined at the crack tip.

134 elements, 402 nodes





RADICAL MESH

Example 2.6.6 Radical meshes are meshes that are refined toward a boundary point can be constructed by mapping uniform meshes. This is illustrated in Fig. 2.6.12 where a radical mesh on $(0,1)^2$ is obtained as the image of a uniform mesh under the map $x \mapsto x\|x\|_\infty^{-1+1/(1-\mu)}$; the exponent μ is chosen as $\mu = 2/3$ in Fig. 2.6.12. ▪

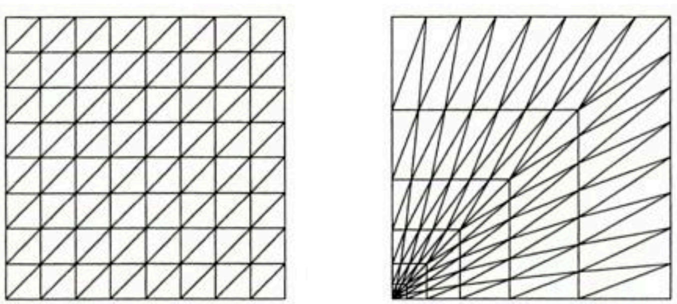
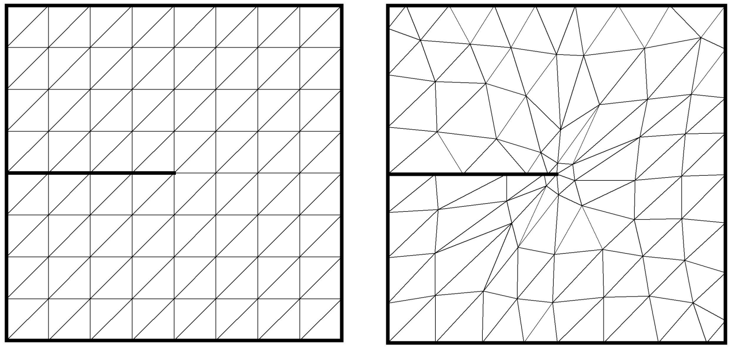


Fig. 2.6.12. (see Example 2.6.6) Radical meshes obtained by mapping uniform meshes: points of uniform mesh (left) are mapped under $x \mapsto x\|x\|_\infty^2$ (right).



Rate of convergence at reentrant corners

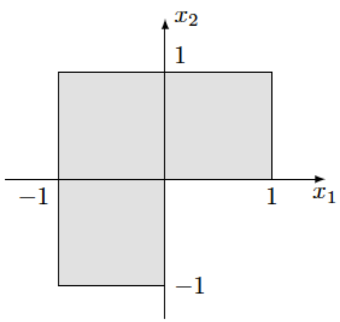


FIGURE 1. The L-shaped domain.

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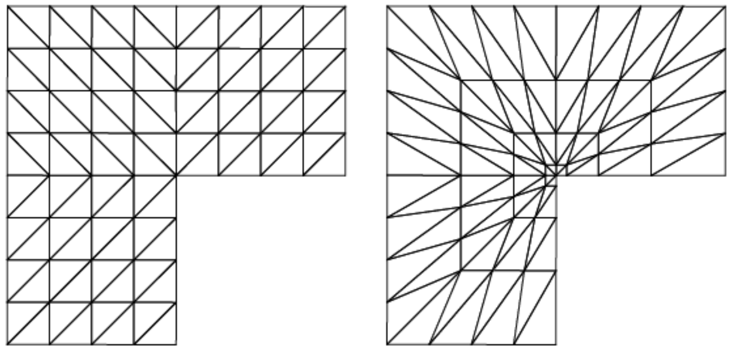
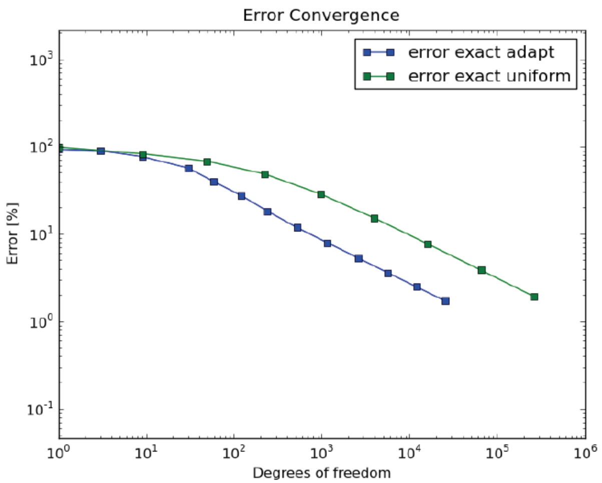


(a) Quasiuniform mesh: $\delta = 1$. (b) Radical mesh: $\delta = 2$.

h	$\|u-u_h\|_{L_\infty}$	$\|u-u_h\|_{H^1}$	$\|u-u_h\|_{V_1^2}$
1/8	1.971099e-01	3.863963e-01	8.896923e-01
1/16	1.259240e-01	2.471421e-01	5.768890e-01
1/32	7.984653e-02	1.572064e-01	3.704272e-01
1/64	5.045257e-02	9.906064e-02	2.361241e-01

TABLE 1. Errors: quasiuniform mesh.



h	$\|u-u_h\|_{L_\infty}$	$\|u-u_h\|_{H^1}$	$\|u-u_h\|_{V_1^2}$
1/8	1.232961e-02	2.758312e-02	6.814436e-02
1/16	3.083882e-03	6.907536e-03	1.709652e-02
1/32	7.710008e-04	1.727198e-03	4.275973e-03
1/64	1.927508e-04	4.318093e-04	1.069047e-03

TABLE 3. Errors: radical mesh, $\delta = 3$.

h	$\|u-u_h\|_{L_\infty}$	$\|u-u_h\|_{H^1}$	$\|u-u_h\|_{V_1^2}$
1/8	6.147282e-03	1.402917e-02	3.958289e-02
1/16	7.713058e-04	1.765369e-03	5.183194e-03
1/32	9.641922e-05	2.207031e-04	6.562081e-04
1/64	1.205262e-05	2.759403e-05	8.231771e-05

TABLE 4. Errors: radical mesh, $\delta = 4.5$.

h	$\|u-u_h\|_{L_\infty}$	$\|u-u_h\|_{H^1}$	$\|u-u_h\|_{V_1^2}$
1/8	1.042903e-02	2.480339e-02	7.415254e-02
1/16	7.010578e-04	1.704769e-03	6.410035e-03
1/32	4.384548e-05	1.067936e-04	5.021341e-04
1/64	2.740699e-06	6.838039e-06	3.744369e-05

TABLE 5. Errors: radical mesh, $\delta = 6$.

Error estimates

h	$\log_2 \frac{\|u-u_h\|_{L_\infty}}{\|u-u_{h/2}\|_{L_\infty}}$	$\log_2 \frac{\|u-u_h\|_{H^1}}{\|u-u_{h/2}\|_{H^1}}$	$\log_2 \frac{\|u-u_h\|_{V_1^2}}{\|u-u_{h/2}\|_{V_1^2}}$
1/16	0.6464	0.6447	0.6251
1/32	0.6573	0.6527	0.6391
1/64	0.6623	0.6583	0.6496

TABLE 2. Convergence rates: quasiuniform mesh.

h	$\log_2 \frac{\|u-u_h\|_{L_\infty}}{\|u-u_{h/2}\|_{L_\infty}}$	$\log_2 \frac{\|u-u_h\|_{H^1}}{\|u-u_{h/2}\|_{H^1}}$	$\log_2 \frac{\|u-u_h\|_{V_1^2}}{\|u-u_{h/2}\|_{V_1^2}}$
1/16	2.9946	2.9904	2.9330
1/32	2.9999	2.9998	2.9816
1/64	3.0000	2.9997	2.9949

TABLE 7. Convergence rates: radical mesh, $\delta = 4.5$.

h	$\log_2 \frac{\|u-u_h\|_{L_\infty}}{\|u-u_{h/2}\|_{L_\infty}}$	$\log_2 \frac{\|u-u_h\|_{H^1}}{\|u-u_{h/2}\|_{H^1}}$	$\log_2 \frac{\|u-u_h\|_{V_1^2}}{\|u-u_{h/2}\|_{V_1^2}}$
1/16	1.9993	1.9975	1.9949
1/32	1.9999	1.9997	1.9994
1/64	2.0000	2.0000	1.9999

TABLE 6. Convergence rates: radical mesh, $\delta = 3$.

h	$\log_2 \frac{\|u-u_h\|_{L_\infty}}{\|u-u_{h/2}\|_{L_\infty}}$	$\log_2 \frac{\|u-u_h\|_{H^1}}{\|u-u_{h/2}\|_{H^1}}$	$\log_2 \frac{\|u-u_h\|_{V_1^2}}{\|u-u_{h/2}\|_{V_1^2}}$
1/16	3.8949	3.8629	3.5321
1/32	3.9991	3.9967	3.6742
1/64	3.9998	3.9651	3.7453

TABLE 8. Convergence rates: radical mesh, $\delta = 6$.

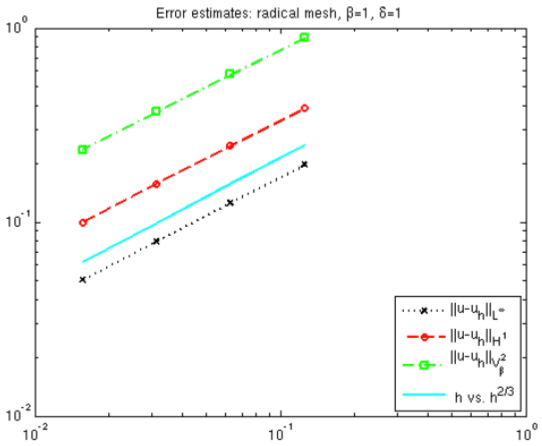


FIGURE 2. Convergence rates: quasiuniform mesh.

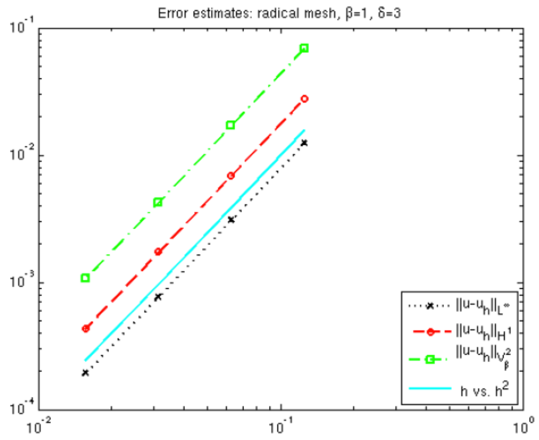


FIGURE 4. Convergence rates: radical mesh, $\delta = 3$.

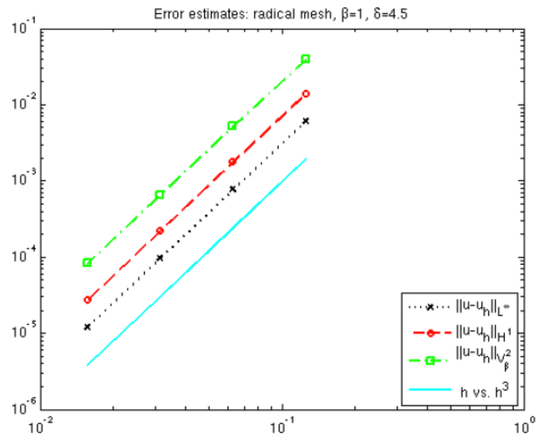
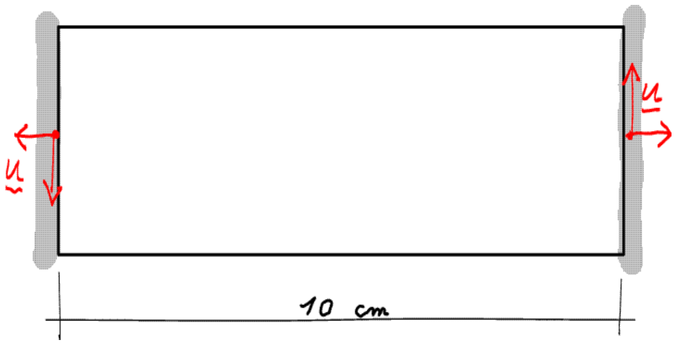


FIGURE 5. Convergence rates: radical mesh, $\delta = 4.5$.

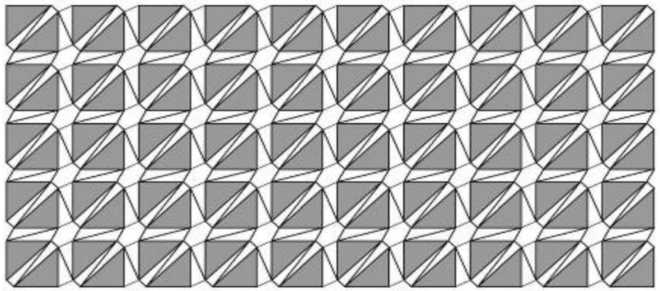
NUCLEATION IN 2d. RUPTURE OF A STRETCHED AND SHEARED STRIP.



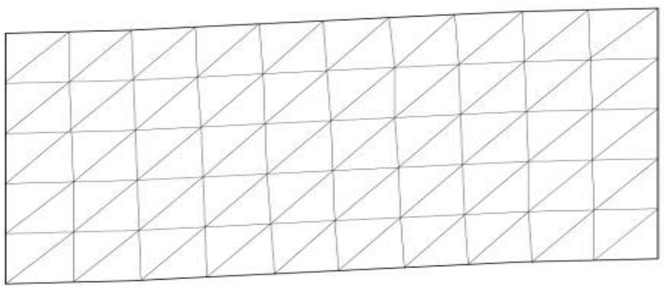
Given displacement: at right end:
$\underline{u}$ = 0.01 cm (**i** + **j**)

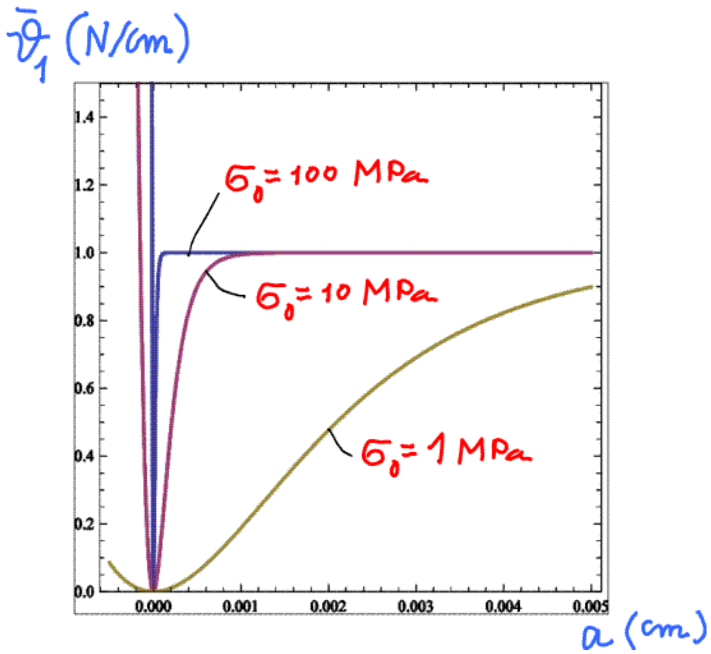
E=3000 MPa , γ = 1N/cm
(e.g. Polycarbonate)

Mesh topology



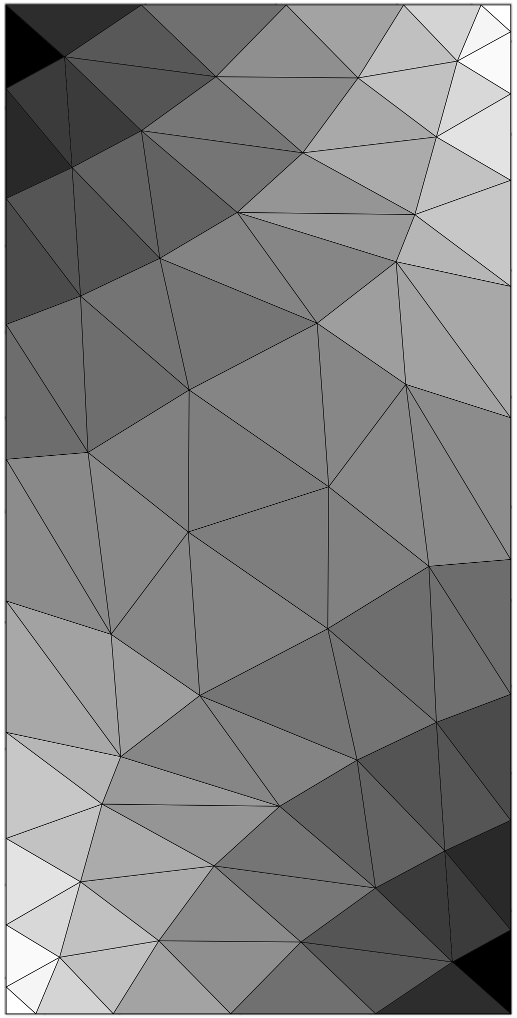
Early elastic evolution





NUCLEATION IN 2d. RUPTURE OF A STRETCHED AND SHEARED STRIP.

“Elastic evolution of the mesh”

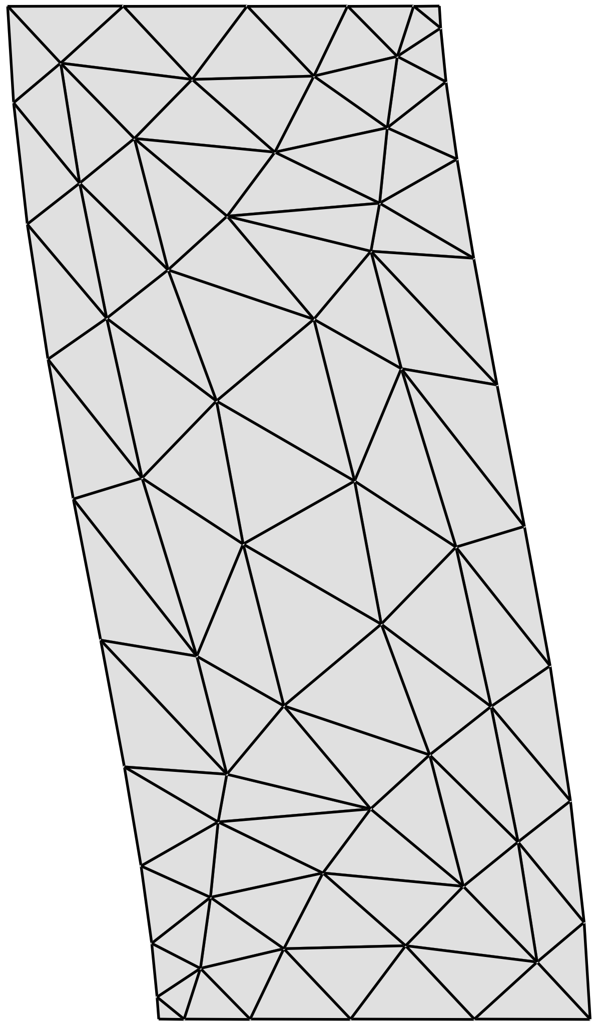


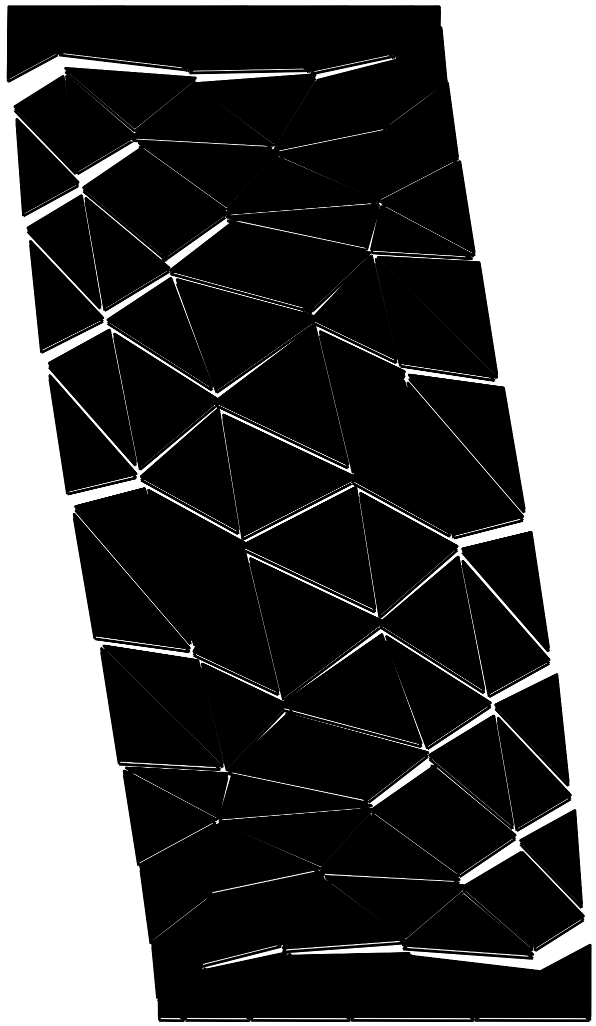
Minimal
fractured state

For $\sigma°$=100 MPa
$\tau°$=10 MPa

NUCLEATION IN 2d. RUPTURE OF A STRETCHED AND SHEARED STRIP.

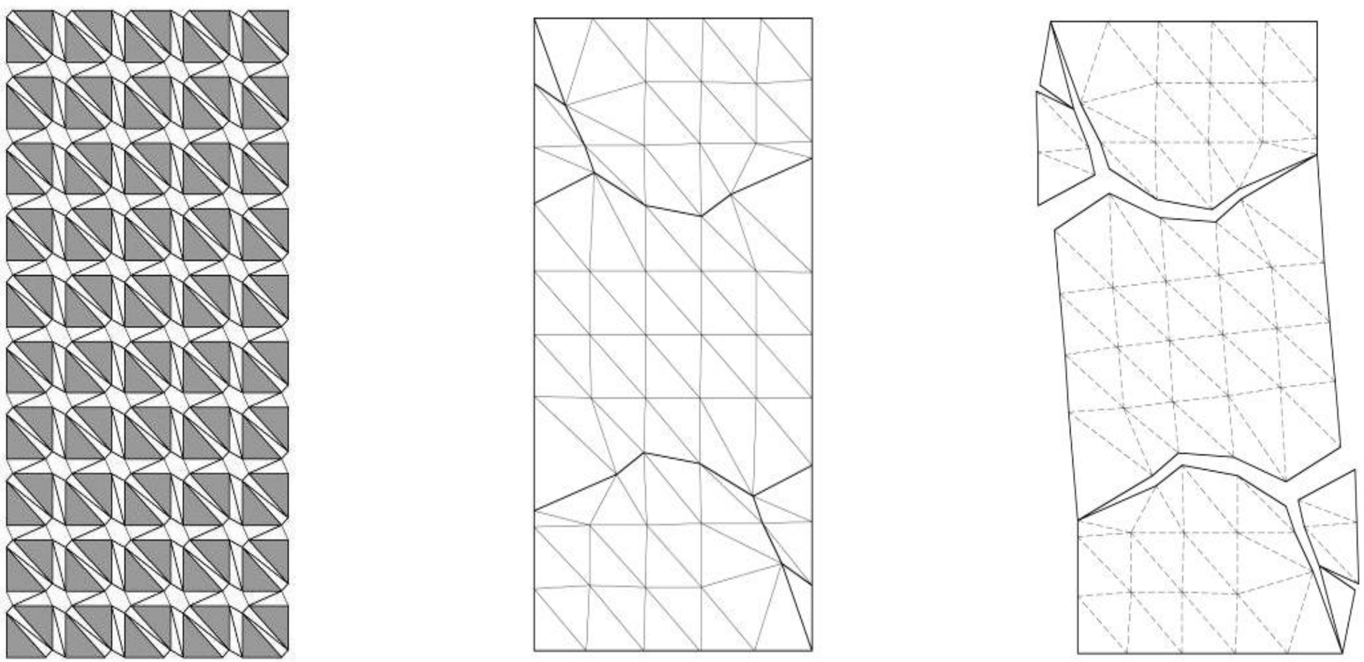
Same value of $\tau°$, γ, low value of $\sigma°$ (1MPa).





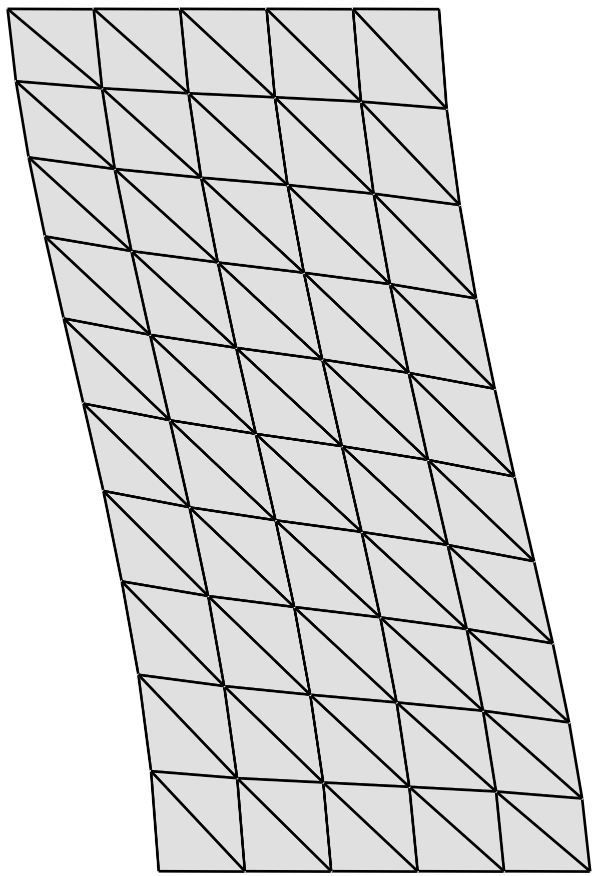
NUCLEATION IN 2d. RUPTURE OF A STRIP IN SIMPLE SHEARING.

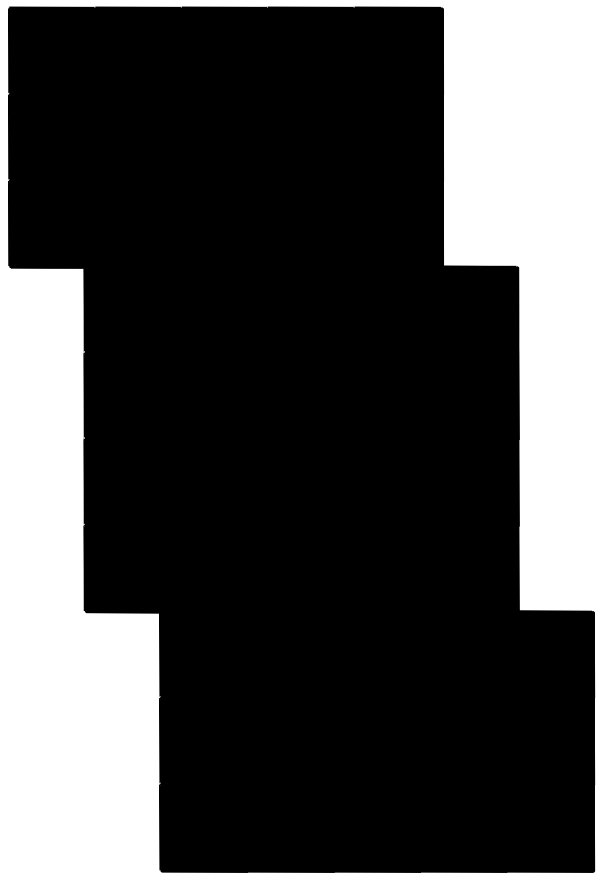
$\sigma°$=100MPa, γ = 1 N/cm, high value of $\tau°$ (100 MPa)



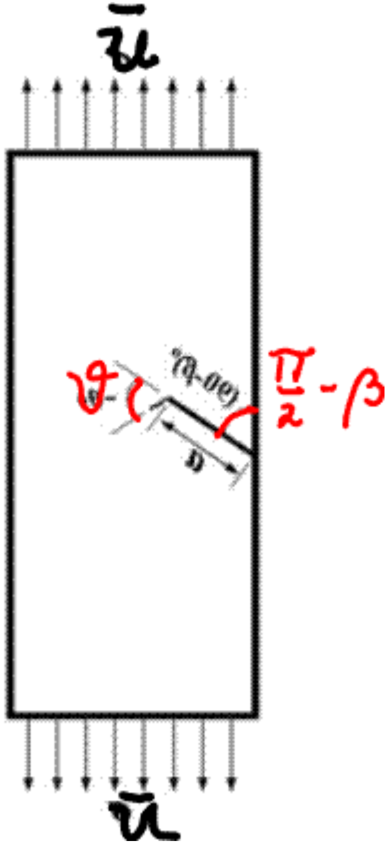
NUCLEATION IN 2d. RUPTURE OF A STRIP IN SIMPLE SHEARING.

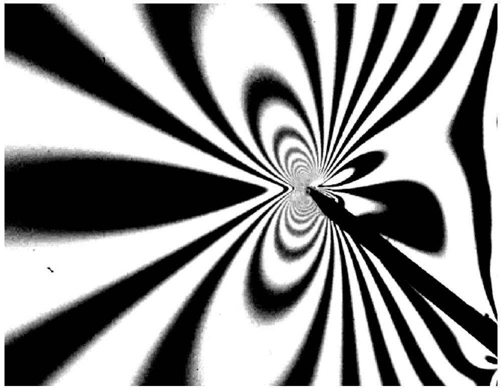
…. lowering $\tau°$ (0.1 MPa)

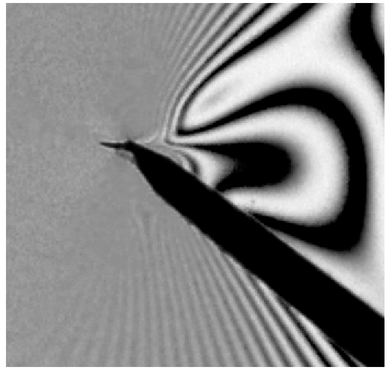


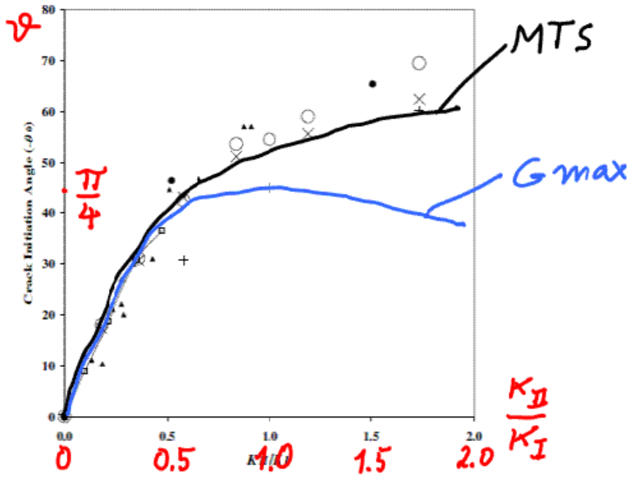


PROPAGATION: KINKING OF A STRAIGHT CRACK IN MIXED MODE I&II.
Experiment and predictions of CLFM

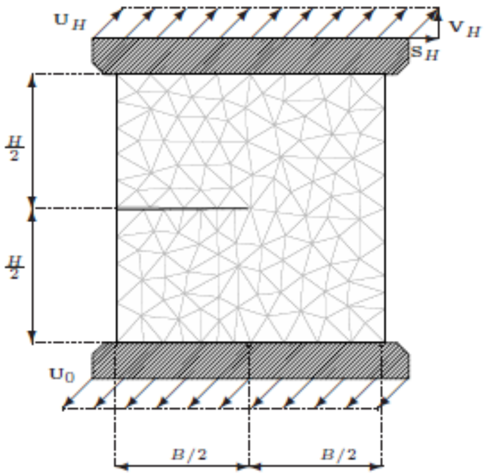




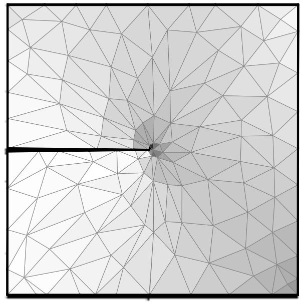




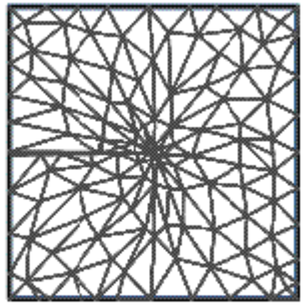
PROPAGATION: KINKING OF A STRAIGHT CRACK IN MIXED MODE I&II.
Results of numerical simulation



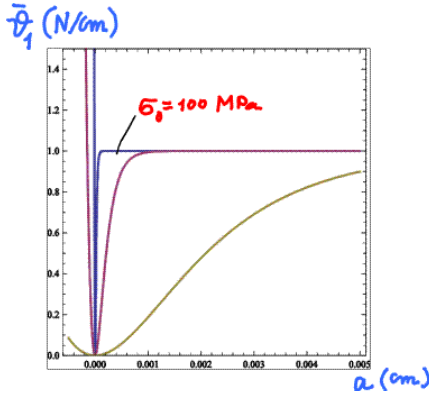
$K_I = 1.3176$, $K_{II} = 0.4288$.

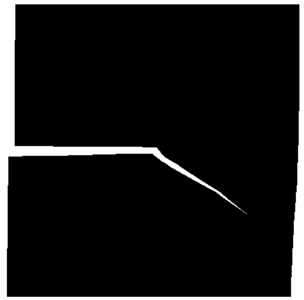


Original mesh



Final (optimized) mesh

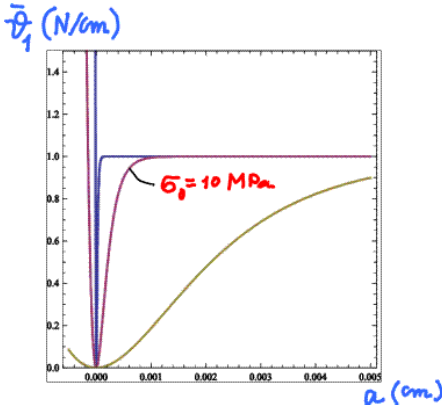


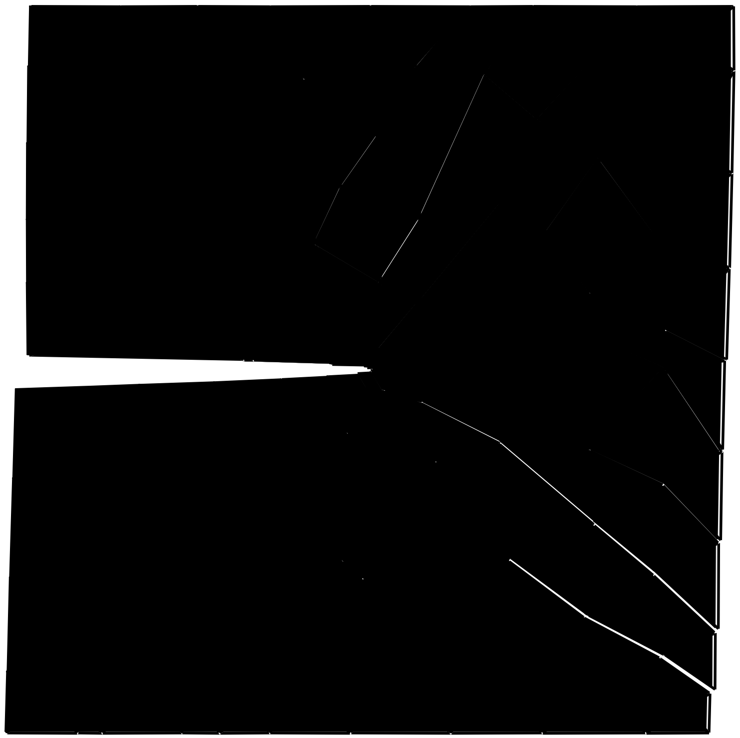


MTS	GMax	Numerical
30.84°	30.48°	30.46°

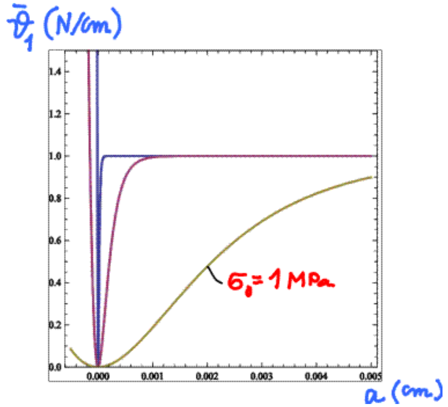
Table 2. Kinking angle with MTS, GMax criteria and from FEM analysis.

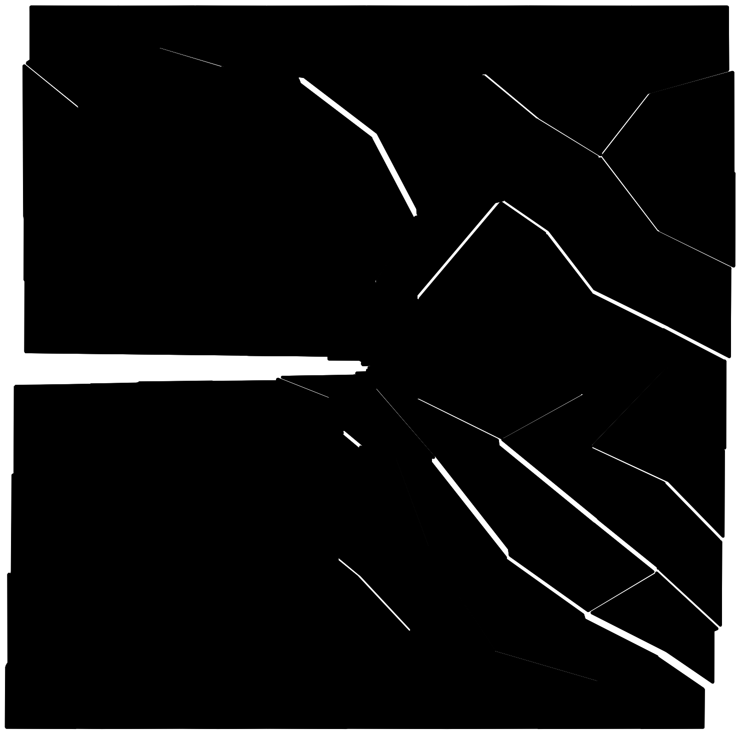
PROPAGATION: KINKING OF A STRAIGHT CRACK IN MIXED MODE I&II.
Results of numerical simulation





PROPAGATION: KINKING OF A STRAIGHT CRACK IN MIXED MODE I&II.
Results of numerical simulation





CONCLUDING REMARKS

- A FE approximation of Griffith type fracture in 2d VF has been presented.
- The way we propose to approximate cracks in variational fracture is different from regularization methods : the displ. jumps occur on the skeleton of the mesh … mesh is variable.
- Local minima of the energy with respect both to displacements and jump sets are searched through descent methods.
- To overcome small energy barriers (either physical or artifacts of the FE approximation) the interface energy is relaxed.
- To follow the evolution of cracks the real quasi-static trajectory is approximated with a sequence of states (step by step approx.)
- The closeness of the approximate trajectory to the real one depends on a sensible choise of the parameters ($\sigma°$, $\tau°$, ..)

CONCLUDING REMARKS

Joys: The results we show are encouraging (but much has to be done)

- We could reproduce the results of CL Fracture Mechanics
- We got good indications of convergence as the mesh is refined
- The mesh is self adapting to singularities and curved crack paths
- Though the approach is necessarily restricted to triangular elements, mesh minimality produces efficient geometries (radical meshes)
- Variable mesh seems to have the potentiality to describe branching with a coarse mesh → dynamics

Sorrows:

- Parameters are “mesh dependent” and “path dependent”
- A part from very rough bounds on the parameters we were not able to identify any simple rule to set $\sigma°$, $\tau°$, … case by case … look for more efficient ways to get out of energy wells.
- Reproducing kinking in mixed mode required a great effort.
- The descent method is still numerically too inefficient to be pushed at the refinement levels required by the singularities (reduce unknowns)