IEEE POSTER WEEK

Numerical Methods: "Mesh vs Particles"
a.k.a. "Weak vs Strong"

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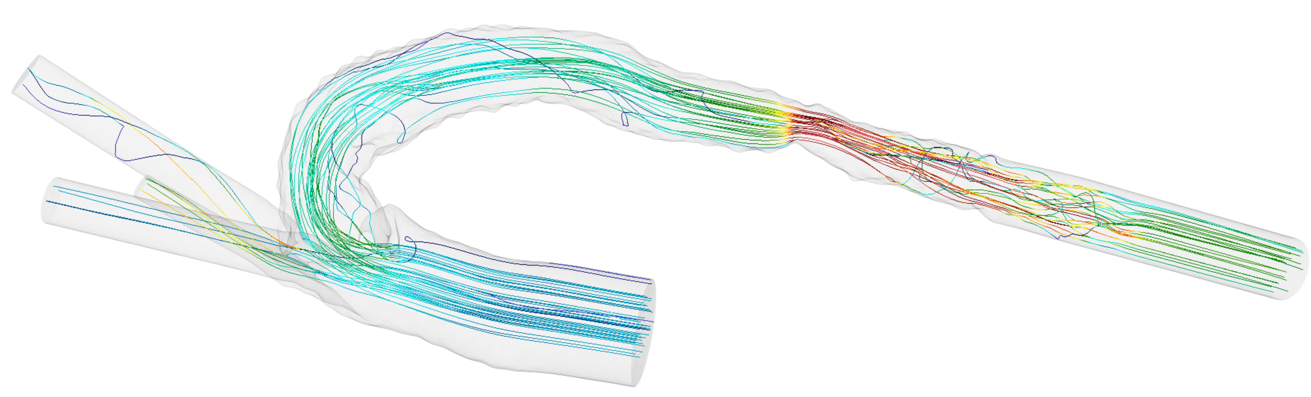
Introduction: Modeling

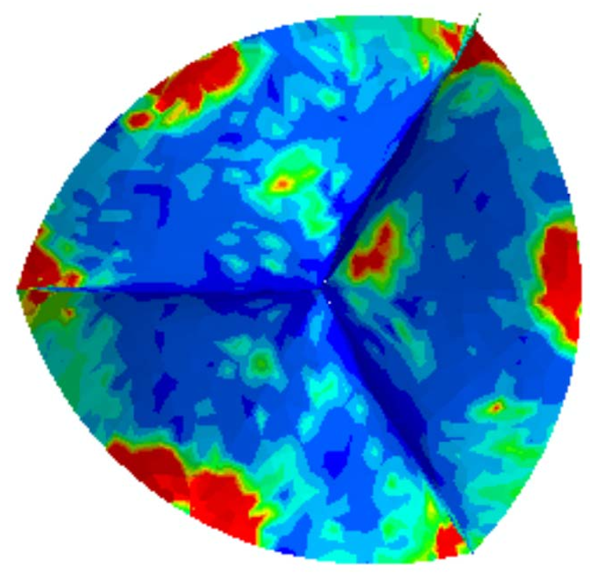
Many important modeling of physical phenomena are formulated in terms of **differential equations**. Among them, one of the most famous is the Newton's second law: $m\dot{u}(t) = F(t)$, where *u(t)* is the velocity, *m* the mass, and *F(t)* the total forces applying on the body during the time. This equation may be solved easily by hand, or not. Therefore, we may seek to solve it with a computer by discretizing time.

However, partial differential equations are also assigned continuously in space, and it is the main focus of this poster.

We focus on two important methods for space discretization: the finite-element method that discretizes the spatial domain in **a continuous subdivision** while the particle method discretizes the domain **pointwise**.



Simulation in an aorta of the blood flow with the Navier-Stokes equations performed by LifeV with 24 millions of tetrahedra. Streamlines postprocessing is realized with Paraview.



Von Mises stress distribution on leaflets during diastolic phase (SPH model)

Finite Elements

The finite-element method is based on integral formulations of partial differential equations, called weak-forms. It is also equivalent in physics to the **variational principle** which states that a solution minimize an energy functional.

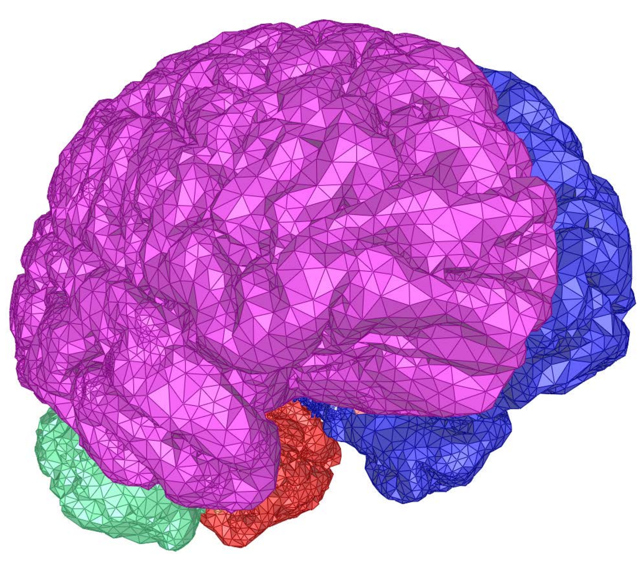
Consider the Laplacian problem

$$\begin{cases} -\Delta u(x,y) = f(x,y) & (x,y) \in \Omega \\ u(x,y) = 0 & (x,y) \in \partial\Omega \end{cases}$$

Its weak form (which is equivalent, under specific conditions, to the strong form) is

Find a function u *vanishing on the boundary such that for all test functions* v *vanishing on the boundary, we have*

$$\int_\Omega \nabla u(x,y) \cdot \nabla v(x,y) dxdy = \int_\Omega f(x,y) v(x,y) dxdy$$



3D Finite Element discretization of a brain with the CGAL library

Now, given a triangulation of Ω, i.e. $\Omega = \bigcup_k \triangle_k$

where *k* is a **finite** number, we write the Laplacian problem as

$$\sum_k \int_{\triangle_k} \nabla u(x,y) \cdot \nabla v(x,y) dxdy = \sum_k \int_{\triangle_k} f(x,y) v(x,y) dxdy$$

Of course it remains to discretize *u(x,y)* and *v(x,y)* but we may already see that computations are going to be performed on $\triangle_k$, which we consider as **elements**.

Finite Particles

Particle methods are a class of numerical methods particularly suited to describe problems characterized by large deformations and fast dynamics, since the state of a continuum is represented by a set of **particles**, which do not have any "rigid" connection between themselves.

In particular, the Modified Finite Particle Method (MFPM) is a recent methodology of approximation of differential operators, based on the projection of the Taylor series of a function ***u(x)*** on a set of *projection functions*.

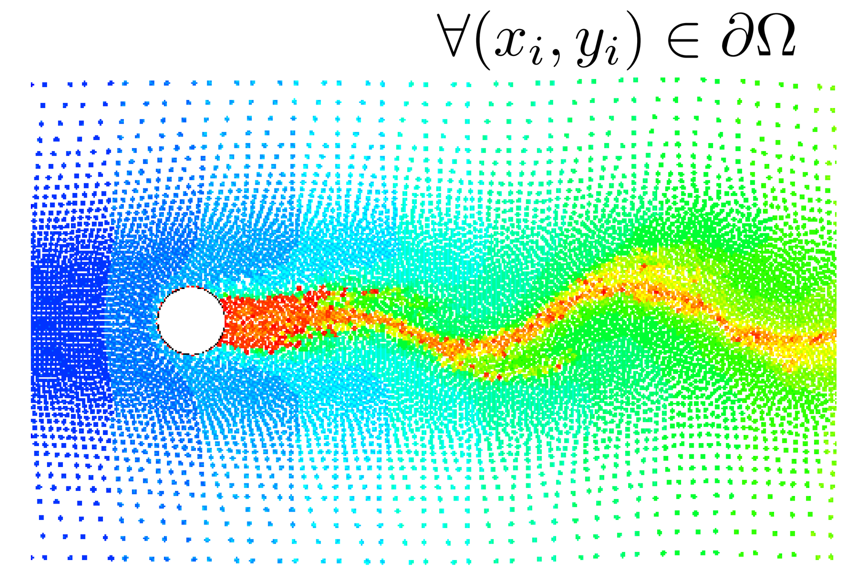
Consider the Laplacian problem in **strong-form**

$$\begin{cases} -\Delta u(x,y) = f(x,y) & (x,y) \in \Omega \\ u(x,y) = 0 & (x,y) \in \partial\Omega \end{cases}$$

It becomes, in the discrete MFPM form

$$\begin{cases} -\sum_j \left[D_{xx,ij} + D_{yy,ij}\right] u(x_j,y_j) = f(x_i,y_i) & \forall (x_i,y_i) \in \Omega \\ u(x_i,y_i) = 0 & \forall (x_i,y_i) \in \partial\Omega \end{cases}$$

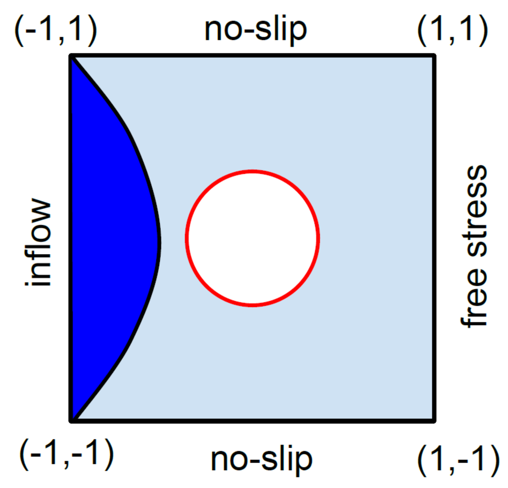
where $D_{xx,ij}$ and $D_{yy,ij}$ are discrete differential operators which computation is based on the projection of *u(x,y)* on some known **projection functions**. The summation operates on a set of particles in the neighborhood of (x_i, y_i).



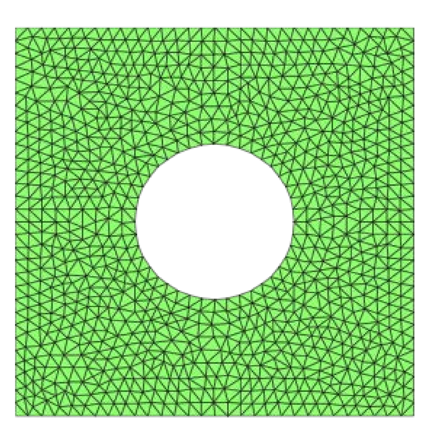
Particle representation of the Von Kármán vortex sheet

Applications

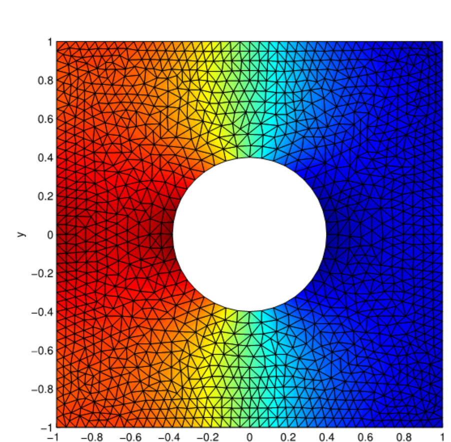
Fluid mechanics problem: creeping flow around a disk



Continous problem

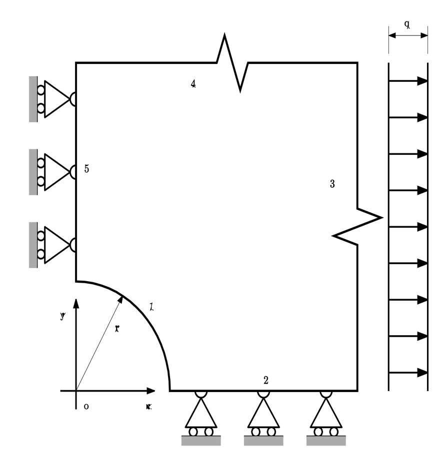


Discretized Domain

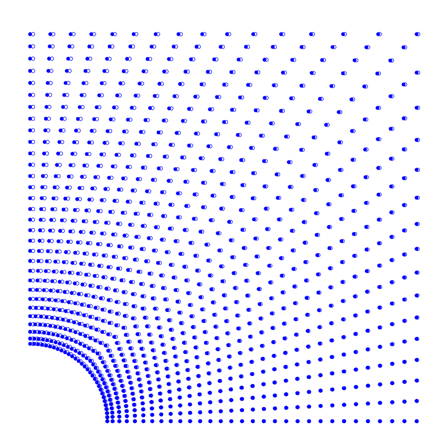


Discrete solution

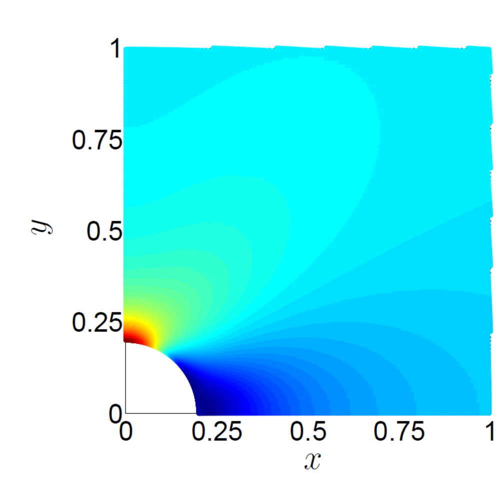
Elasticity problem: a quarter of perforated plate



Continous problem



Discretized Domain



Discrete solution

REFERENCES

D. Elman, H. Silvester and A. Wathen. *Finite Elements and Fast Iterative Solver with application in incompressible fluid dynamics.* Oxford Press, 2005.

D.Asprone, F.Auricchio, A.Reali, *Modified Finite Particle Method: applications to elasticity and plasticity problems.* International Journal of Computational Methods, 2012