Self-contact, self-collisions and large deformations

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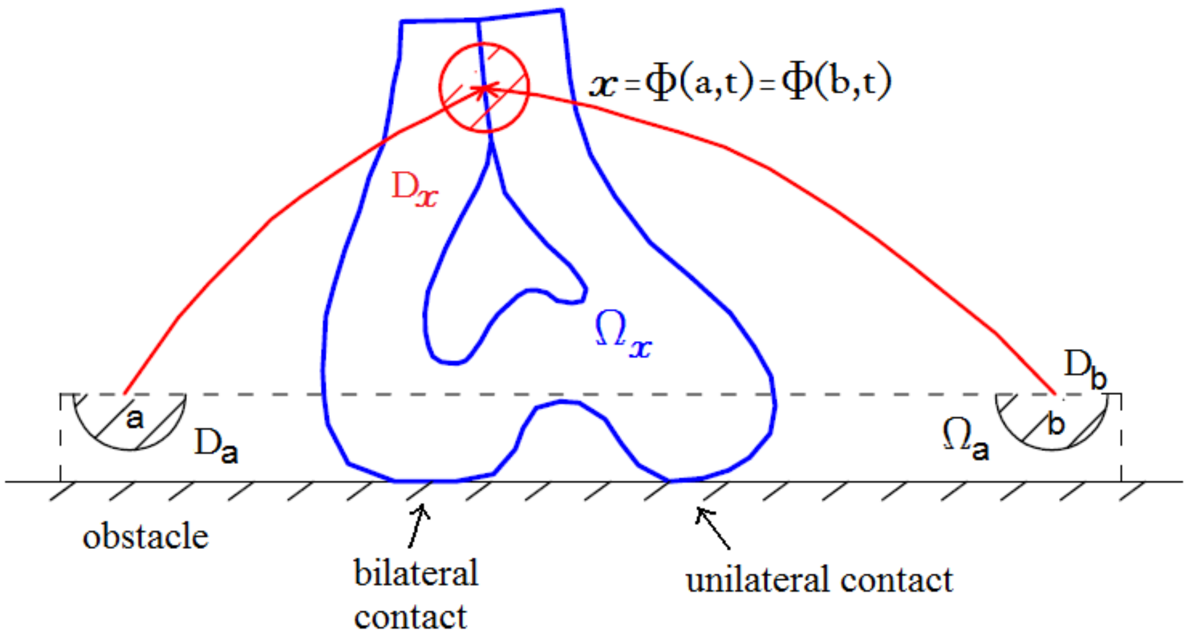


$$x = \Phi(a,t) \qquad x = (x_i) \qquad a = (a_\alpha)$$

$$\vec{U} = \frac{\partial \Phi}{\partial t}$$ velocity

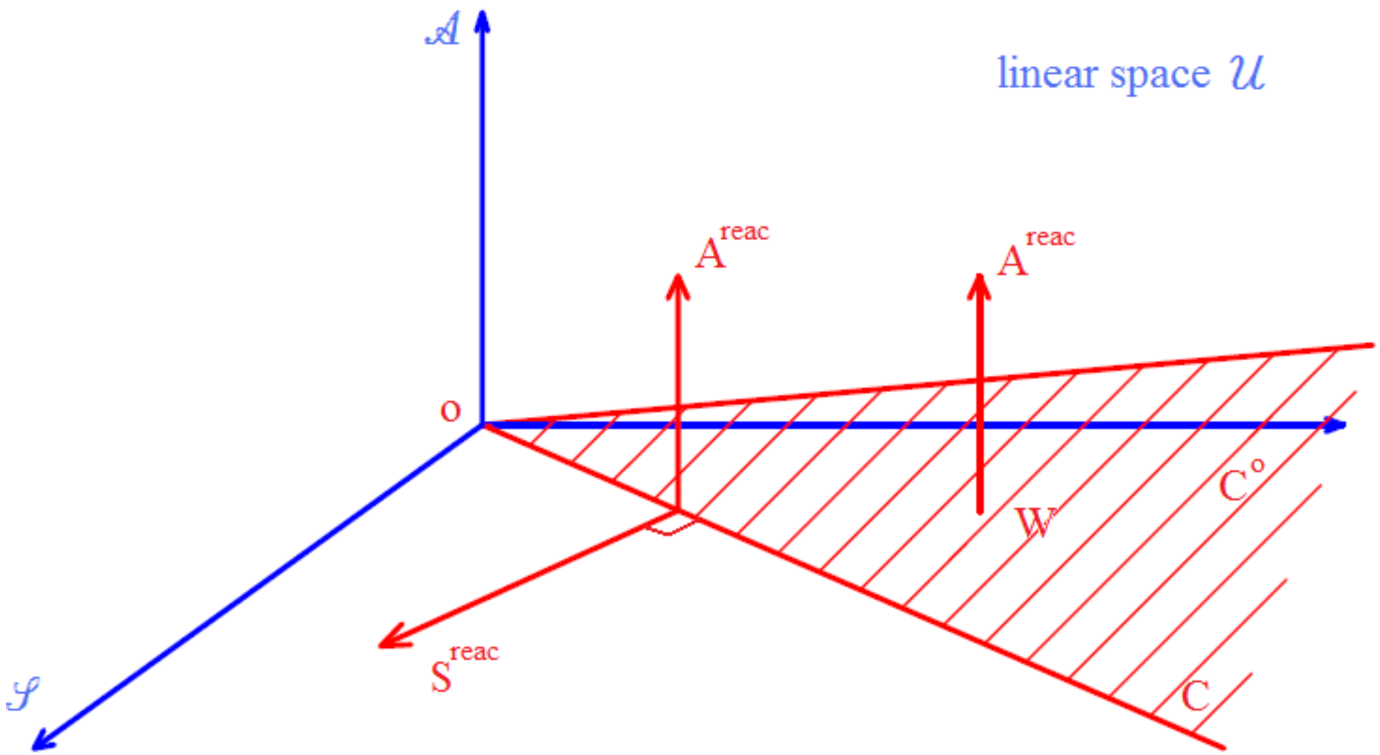
$$F = grad\Phi = (F_{i\alpha}) = \frac{\partial \Phi_i}{\partial a_\alpha}$$ gradient matrix

3x3 Matrices

$$A : B = A_{ij} B_{ij}$$

$$\begin{cases} \mathsf{S} & \text{symmetric matrices} \\ A & \text{antisymmetric matrices} \end{cases} \qquad \mathbf{S} \perp \mathbf{A}$$

$C = \{W | W \in \mathbf{S};\ W \ \textit{is semidefinite positive}\}$ is a closed convex cone



Classical theory (Elastic behaviour)

$$\Pi = \frac{\partial \psi}{\partial F}, \qquad F = grad\Phi$$

Π Piola Boussinesq stress, ψ the free energy, $\psi = \psi(F)$

$$\det F > 0$$

non interpenetration condition.

Rotation matrix R does not intervene; it is impossible that ψ is convex function of F,

but

there exist mathematical results.

How to take R into account?

VELOCITY OF DEFORMATION

$$grad\vec{U}$$

$$\Omega = \frac{\partial R}{\partial t} R^T$$

$$grad\Omega = \Omega_\alpha$$

INTERNAL FORCE

$$\Pi$$

$$M \in A$$

$$\Lambda = (\Lambda_\alpha)$$

EQUATIONS OF MOTION

$$\rho \frac{\partial \vec{U}}{\partial t} = div\Pi \qquad in\ \Omega_a$$

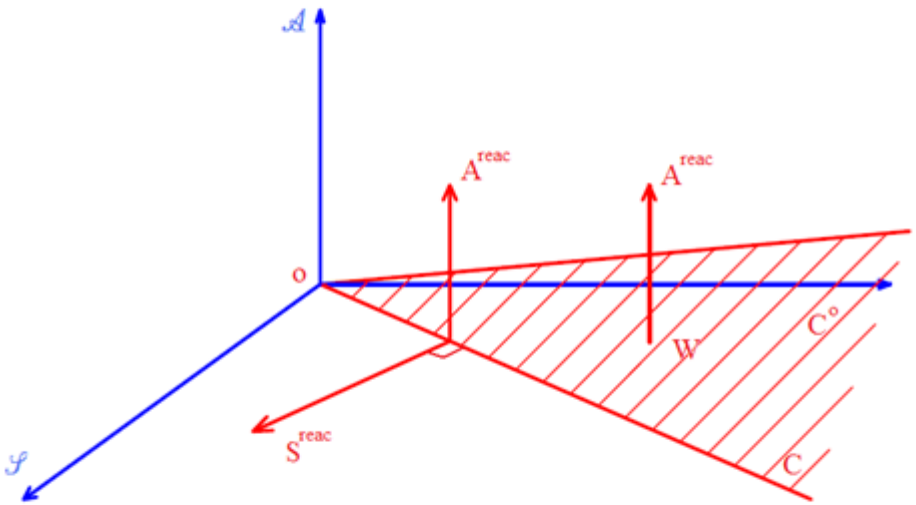
$$div\Lambda + M = 0 \quad in\ \Omega_a$$

+ B.C. + I.C.

W is the physical quantity which describe the elongation.

Impenetrability condition

$$W \in C$$



$rank W = 3$ ⟷ no flattening

$rank W = 2$ ⟷ flattening into a surface

$rank W = 1$ ⟷ flattening into a curve

$rank W = 0$ ⟷ flattening into a point

Elastic constitutive laws

grad R describes the spacial variation of the rotation matrix.

Free energy

$$\overline{\Psi}(W, gradR) = \Psi(W, \|gradR\|^2) + I_C(W)$$

Constitutive laws

$$\begin{cases} \Pi = R(\dfrac{\partial \Psi}{\partial W} + S^{reac} + A^{reac}) \\ S^{reac} \in \mathsf{S}, \quad A^{reac} \in \mathsf{A}, \quad S^{reac} + A^{reac} \in \partial I_c(W) \end{cases}$$

$$\Lambda = 4\left(\frac{\partial \Psi}{\partial \|gradR\|^2}\right)(gradR)R^T$$

$$M = \Pi W R^T - R W \Pi^T$$

In this theory

- $(W, gradR) \rightarrow \Psi(W, gradR)$ may be convex .

  The effect may be proportional to the cause.

- There exist non unique equilibrium positions in $W^{1,p}(\Omega), \quad p > 3.$

- If $\Psi(W) \rightarrow +\infty$ when $\det W \rightarrow 0$ there is no flattening. Reaction A^{reac} is present.

- If $\Psi(W) < +\infty$ for $\det W = 0$ flattening is possible. Reaction A^{reac} and S^{reac} are present.

- Constitutive law

$$\Pi = \frac{\partial \Psi}{\partial F} = R \frac{\partial \Psi}{\partial W}$$ is not always valid.

$$\Pi \in R\left(\frac{\partial \Psi}{\partial W} + \partial I_C(W) \right)$$ is always valid.

Self contact

VELOCITY OF DEFORMATION

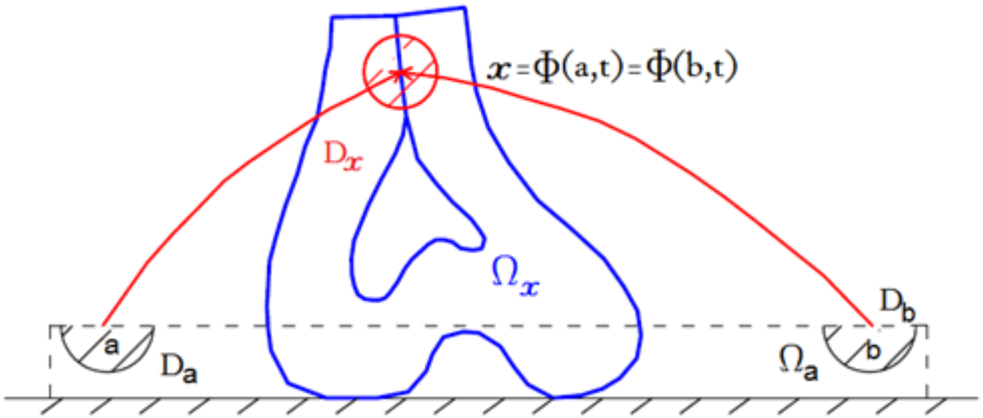
$$\vec{U}_l(a) \quad \vec{U}_l(b)$$

INTERNAL FORCE

$$\vec{R}(a) \quad \vec{R}(b)$$

EQUATIONS OF MOTION

$$\frac{\vec{R}(a)}{\|cof\ W(a)\|} + \frac{\vec{R}(b)}{\|cof\ W(b)\|} = 0$$



CONSTITUTIVE LAW

$$\vec{R} \in \partial\Phi(\vec{U}(a) - \vec{U}(b))$$

Collisions

Self collisions

Interior forces become percussions

Constitutive laws involve

$$\frac{\vec{U}^{+}(a)-\vec{U}^{+}(b)+\vec{U}^{-}(a)-\vec{U}^{-}(b)}{2}$$

Collisions when flattening

Piola Boussinesq stress becomes Piola Boussinesq percussion stress

No difficulty: velocity $\vec{U}^{+}$ is uniquely given by velocity $\vec{U}^{-}$

Numerics (Francesca Nerilli)

$$rank W = 3$$

$$Unknowns\ \Phi(a,t), R(a,t), A(a,t)$$

$$\rho \frac{\partial^2 \Phi}{\partial t^2} = div\left\{ R\left(\frac{\partial \Psi}{\partial W} + A \right) \right\}$$

$$grad \Phi = R\, W(\Phi)$$

$$div\left(\frac{\partial \Psi}{\partial grad R} R^T \right) + R(AW(\Phi) + W(\Phi)A)R^T = 0$$

+ B.C. + I.C.

A is the antisymmetric reaction matrix

$$A \in \partial I_c(W)$$

A schematic example

$$\Psi(W, gradR) = \frac{k}{2}(W - I)^2 + \frac{\hat{k}}{2}\|gradR\|^2 + I_C(W)$$

$$\rho \frac{\partial^2 \Phi}{\partial t^2} = div\{R(k(W(\Phi) - I) + RA\}$$

$$grad\Phi = R\, W(\Phi)$$

$$div\left(4\hat{k}(gradR)R^T\right) + R(AW(\Phi) + W(\Phi)A)R^T = 0$$

$$\Pi\vec{N} = \vec{g}, \qquad \Lambda\vec{N} = \vec{m} \qquad on\ \Gamma_1$$

$$\Phi(a,t) = a, \qquad R = I \qquad on\ \Gamma_0$$

$$\Phi(a,0) = a, \qquad \frac{\partial\Phi}{\partial t}(a,0) = 0$$

A schematic example

Iterative method: $\Phi^n, \quad R^n, \quad A^n$ are known at time $n\Delta t.$

Step 1. $$\rho \frac{\partial^2 \Phi}{\partial t^2} = div\left\{R^n (k(W(\Phi^n) - I) + R^n A^n\right\}$$

$\longrightarrow$ give $\boxed{\Phi^{n+1}}$

Step 2. $$grad\Phi^{n+1} = RW(\Phi^{n+1})$$

$\longrightarrow$ give $\boxed{R^{n+1}}$

Step 3. $$div\left(4\hat{k}(gradR^{n+1})R^{n+1^T}\right) + R^{n+1}(AW(\Phi^{n+1}) + W(\Phi^{n+1})A) = 0$$

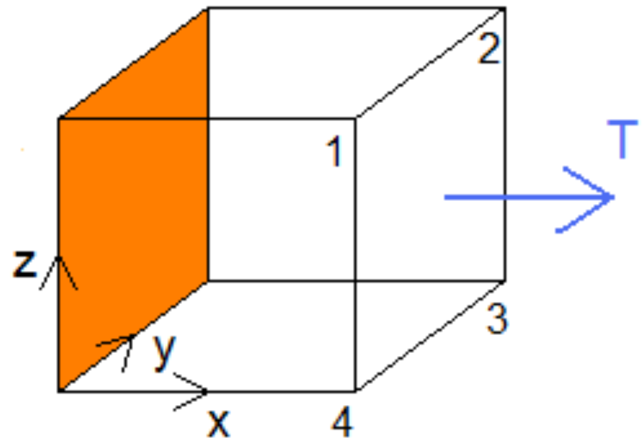
$\longrightarrow$ give $\boxed{A^{n+1}}$

Weak equations and finite elements

$\Phi, \ R$ are piecewise continous

A is either piecewise continous or piecewise constant

Application 1 - Traction



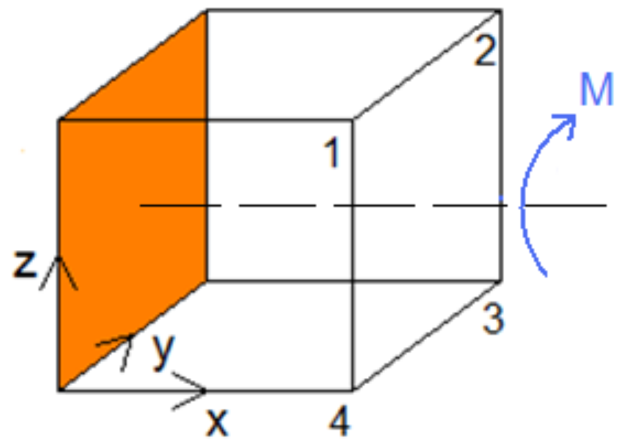
Only elongation $\longrightarrow$ $A = 0$ at each time step

$R = I$ at each time step

k is the elongation rigidity

Periodic motion

Application 2 - Torque



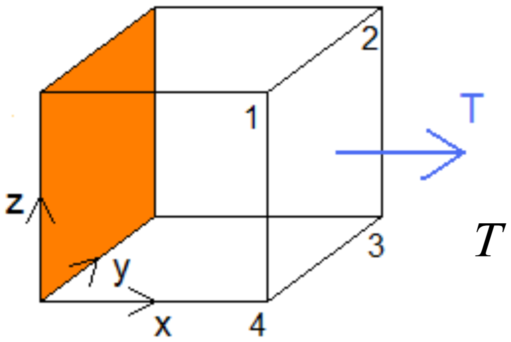
Only torsion $\longrightarrow$ $A \neq 0$ at each time step

$R \neq I$ at each time step

$\hat{k}$ is the rotation rigidity.

Periodic motion

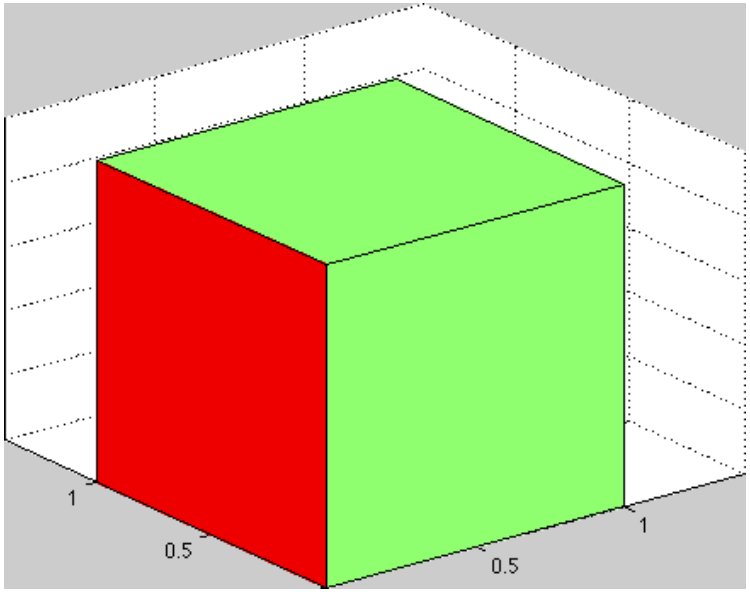
In case $\hat{k} = 0$ (the usual theory) rotation does not have limitation.

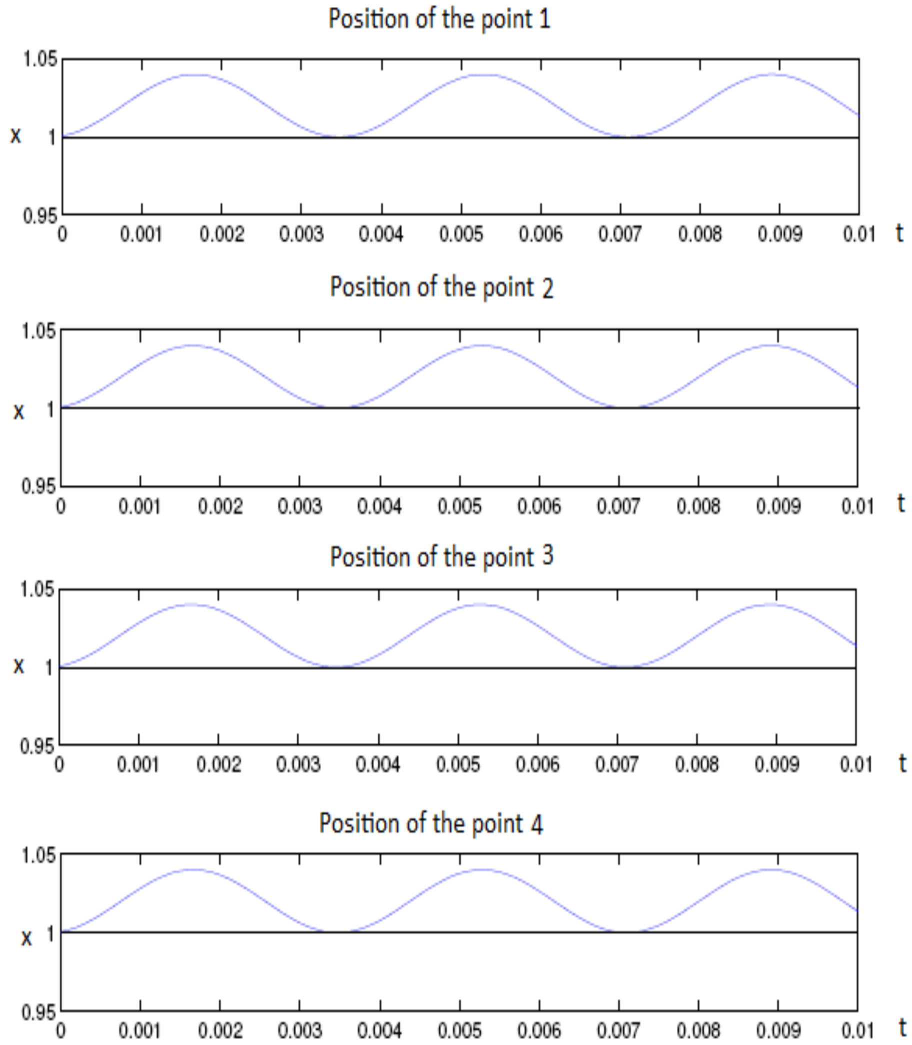


$$T = 2 \cdot 10^8 \, N$$

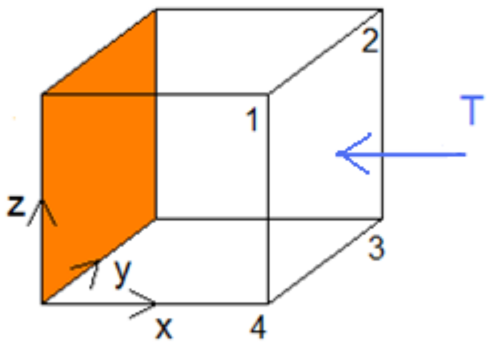
$$k = 1 \cdot 10^{10} \, \frac{N}{m^2}$$

$$\rho = 1 \cdot 10^3 \, \frac{kg}{m^3}$$



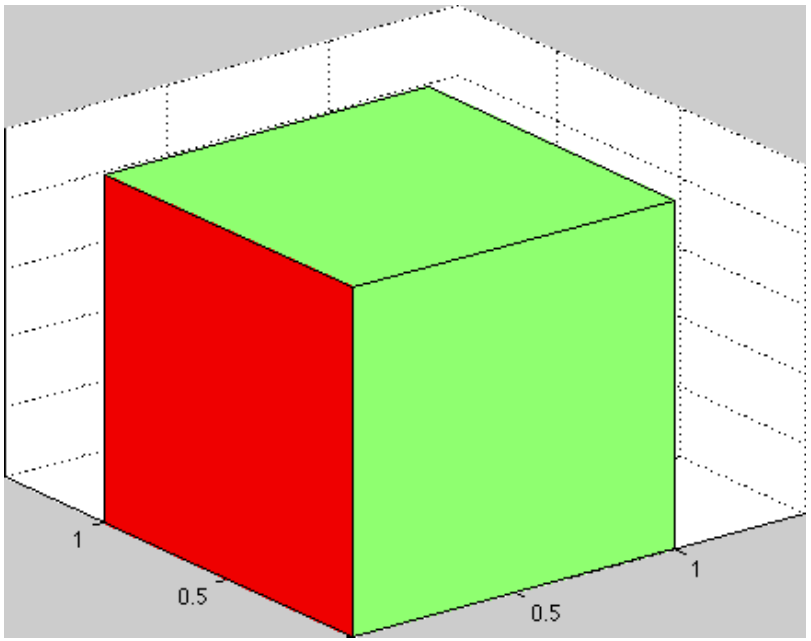


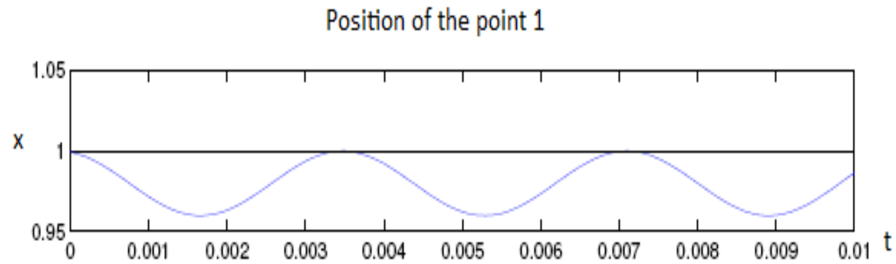
Results

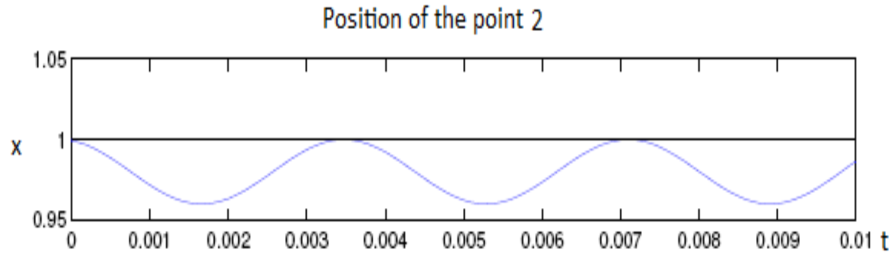


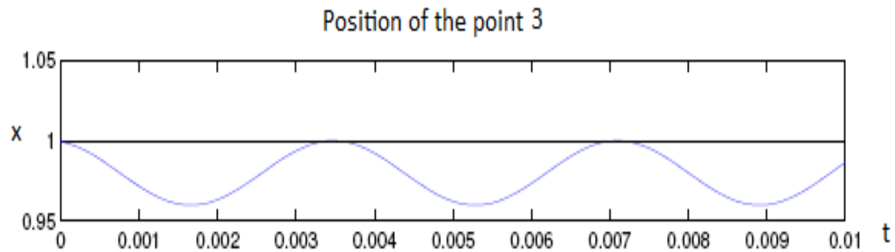
$$T = -2 \cdot 10^{8}\, N$$

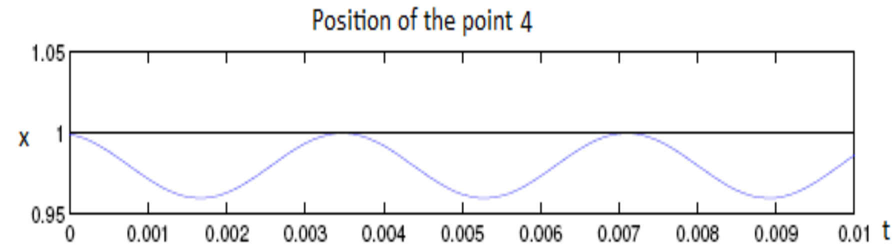
$$k = 1 \cdot 10^{10}\, \frac{N}{m^{2}} \qquad \rho = 1 \cdot 10^{3}\, \frac{kg}{m^{3}}$$



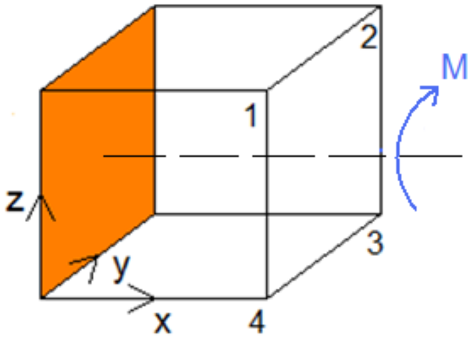






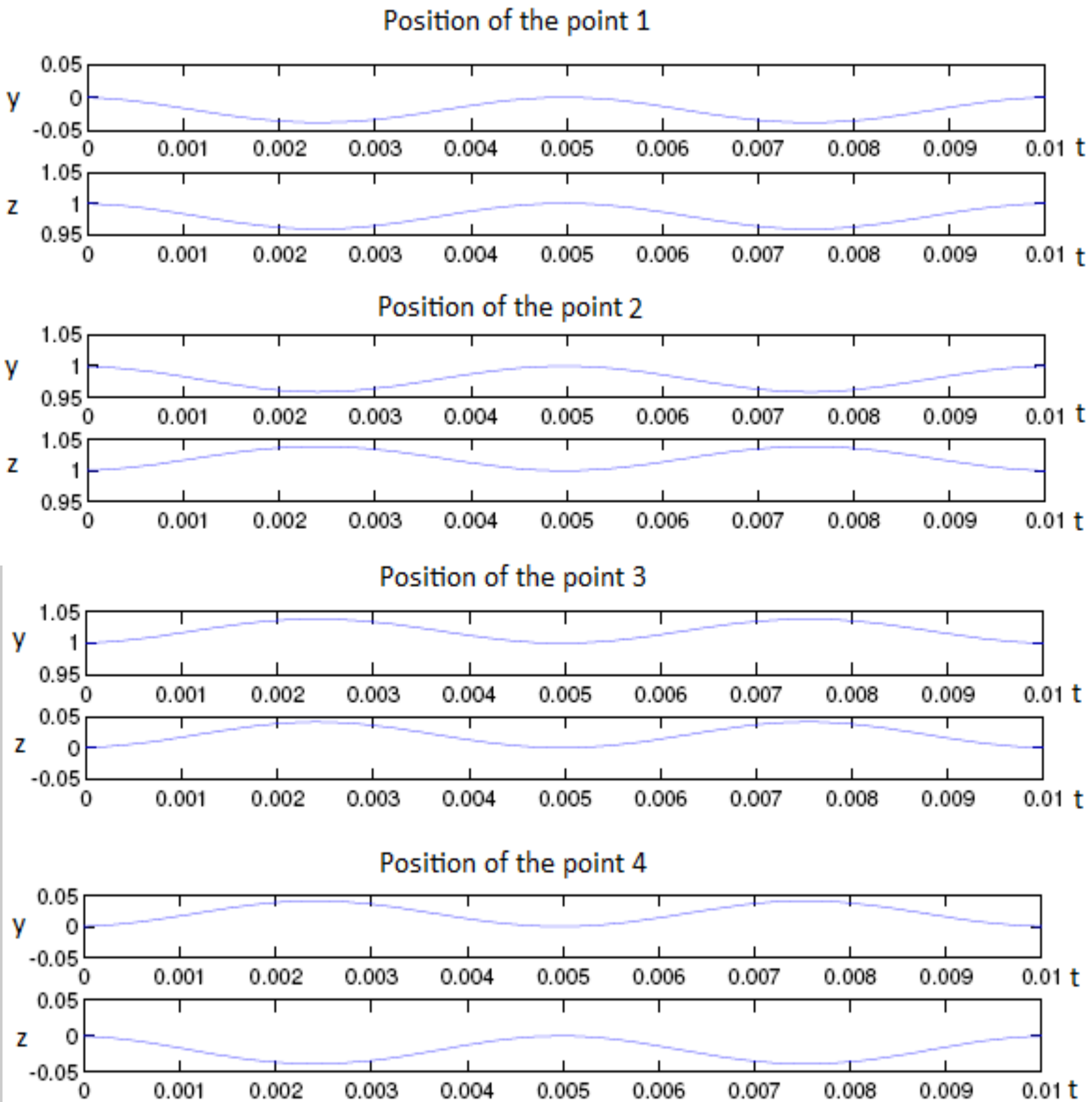


Results

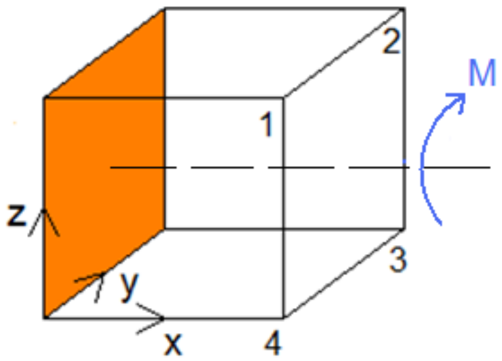


$$\hat{k} = 100$$

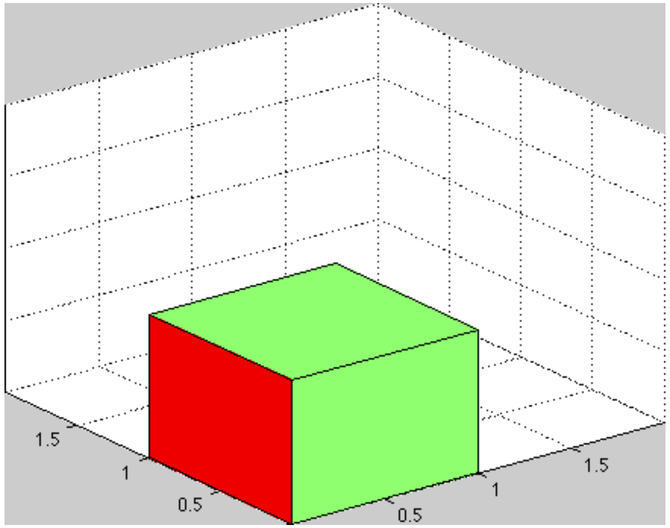
$$k = 1\cdot 10^{10}\,\frac{N}{m^2} \qquad \rho = 1\cdot 10^{3}\,\frac{N}{m^3}$$



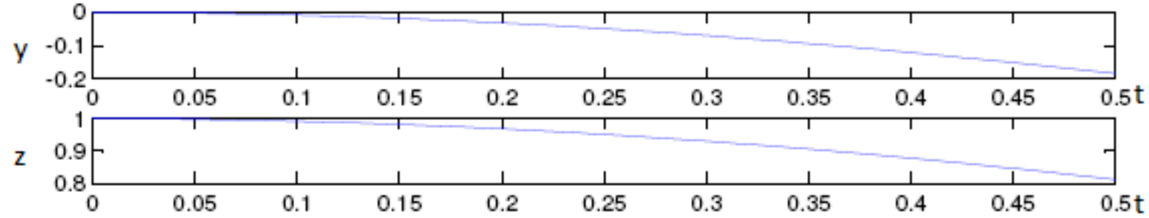
Results



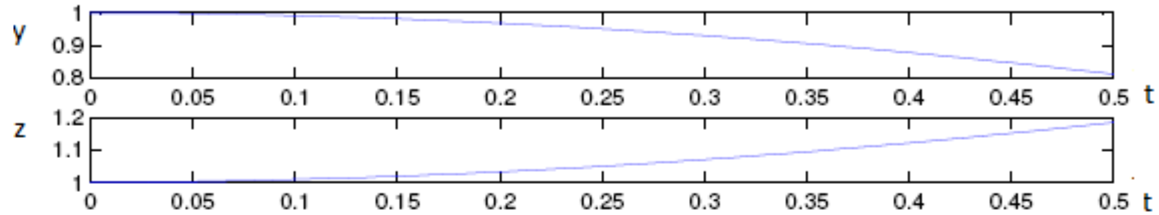
$$\hat{k} = 0$$



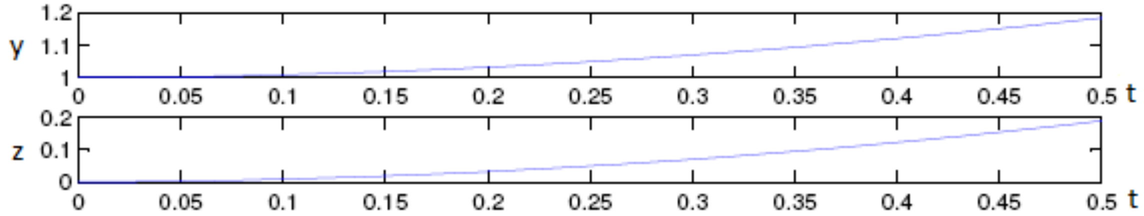
Position of the point 1



Position of the point 2



Position of the point 3



Position of the point 4